Field Theory on Schrödinger Space-times

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[Blau, J.H., Rollier, 2009]

[Blau, J.H., Rollier, 2010]

Introduction

- Many systems in nature exhibit critical points with non-relativistic scale invariance $z > 1$:

$$D_z \; : \qquad \vec{x} \to \lambda \vec{x} \qquad t \to \lambda^z t \, .$$

  Such systems have Lifshitz symmetries: translations, rotations and NR dilatations (and possibly more).

- Aim: to construct holographic techniques for (strongly coupled) systems with NR symmetries.

- An ultracold dilute gas of point-like interacting spin-1/2 fermions fine-tuned to infinite scattering length is a Schrödinger invariant system [Mehen, Stewart, Wise, 1999] [Bloch, Dalibard, Zwerger, 2007].
- Schrödinger holography: [Son, 2008] [Balasubramanian, McGreevy, 2008].
- (Asymptotically) Schrödinger space-times can be obtained via a solution generating technique known as TsT (or Melvin twist).

TsT

- Deformations of AAdS$_5 \times S^5$ that change the AdS$_5$ asymptotics can be obtained by TsT.
- Write $ds^2_{\text{AAdS}_5 \times S^5} = g_{\mu\nu} dx^\mu dx^\nu + ds^2_{\mathbb{CP}^2} + (d\xi + \mathcal{A})^2$
- Apply TsT [Lunin, Maldacena, 2005], [Imeroni, 2008]:
  - T-dualize along ξ
  - shift $V \to V + \beta\xi$ where V is a coordinate that parametrizes an isometry of $g_{\mu\nu}$
  - T-dualize along ξ again.
- Reduce the resulting solution of type IIB sugra (over a squashed 5-sphere) down to 5 dimensions. This gives the metric [Maldacena, Martelli, Tachikawa, 2008]

$$g^{\text{TsT}}_{\mu\rho} = (1 + \beta^2 g_{VV})^{1/3} g_{\mu\rho} - \beta^2 (1 + \beta^2 g_{VV})^{-2/3} g_{V\mu} g_{V\rho} \,.$$

If we take $g_{\mu\nu}$ to be Poincaré ($\omega = 0$) or plane wave AdS ($\omega \neq 0$), i.e.

$$g_{\mu\nu}dx^\mu dx^\nu = \frac{1}{R^2}\left[-\omega^2(R^2 + \vec{X}^2)dT^2 - 2dTdV + d\vec{X}^2 + dR^2\right]$$

then out come Poincaré and global Schrödinger:

$$ds^2_{\mathsf{TsT}} = \frac{1}{R^2}\left[-\left(\frac{\beta^2}{R^2} + \omega^2\left(R^2 + \vec{X}^2\right)\right)dT^2 - 2dTdV + d\vec{X}^2 + dR^2\right]$$

- All isometries that commute with $\partial_V g_{\mu\nu} = 0$ are preserved under TsT. Here these constitute the Schrödinger algebra.

- Sch algebra: $\{H, C, D_{z=2}, N, P_i, V_i, M_{ij}\}$, $i = 1, \ldots, d$.
  - $D_{z=2}$: NR dilatations $r \to \lambda r$ and $t \to \lambda^2 t$
  - C : special conformal generator
  - $H, C, D_{z=2}$ form an $\mathtt{sl}(2, \mathbb{R})$ algebra
  - Heisenberg algebra $[P_i, V_j] = \delta_{ij} N$ with P_i momenta, V_i Galilean boosts and N a central element.
  - M_{ij} rotations
- $\mathtt{sch}(d+3) \subset \mathtt{so}(2, d+2)$ and consists of all those elements that commute with some lightcone momentum (∂_V in TsT picture).

NRCFT

- Some basic facts about NRCFT [Nishida, Son, 2007]:
  - Theory of second quantized Schrödinger fields.
  - Local operators form irreps of $\mathtt{sch}(d+3)$.
  - Local operators are build upon primary operators.
  - $\mathcal{O}$ is primary if $[C, \mathcal{O}] = [V_i, \mathcal{O}] = 0$.
  - Each primary operator corresponds to an energy eigenstate of a system in a harmonic potential with Hamiltonian $H + C$.

Harmonic trapping

$$ds^2 = \frac{1}{R^2}\left[-\left(\frac{\beta^2}{R^2} + \omega^2\left(R^2 + \vec{X}^2\right)\right)dT^2 - 2dTdV + d\vec{X}^2 + dR^2\right]$$

- $\partial_T = H + \omega^2 C$ is a global timelike Killing vector

$$ds^2\,|_{R,V=\text{cst}} = -\left(1 + \omega^2\rho^2\right)dT^2 + d\rho^2 + \rho^2 d\Omega^2_{d-1}$$

- takes the form of a Newtonian limit with isotropic harmonic oscillator potential.
- In a NRCFT primary operators correspond to energy eigenstates of a system in a harmonic potential.

$$ds^2 = \frac{1}{R^2}\left[-\left(\frac{\beta^2}{R^2}+\omega^2\left(R^2+\vec{X}^2\right)\right)dT^2 - 2dTdV + d\vec{X}^2 + dR^2\right]$$

- For $\omega \neq 0$ this metric provides the unique global and time-independent metric for Sch$(d+3)$.
- There exists a co-dimension two embedding of Sch$(d+3)$ in $\mathcal{M}^{2,d+3}$:

$$\begin{aligned}-(X^0)^2+(X^1)^2+(X^{1+i})^2+(X^{d+2})^2-(X^{d+3})^2 &= -1-\tfrac{4}{3}(X^{d+4})^2\\ \beta\left[\left(X^0-X^1\right)^2+(X^{d+2}-X^{d+3})^2\right] &= \tfrac{2}{\sqrt{3}}X^{d+4}\end{aligned}$$

whose natural and global parametrization gives the above metric for $\omega \neq 0$.

- For the Schrödinger group one can define a NRCFT.
- The Schrödinger space-time can be thought of as a deformation of AdS.
- How far can we extend the analogy with AdS?
- How NR is physics on a Schrödinger space-time?
- I will discuss the causal structure in some detail and argue that although for point particle probes the space-time is (close to) pathological this is not so for scalar field probes.

Causal structure

- Sch does not admit a time function (there is no function that can distinctly label all causally related events).
- Global coordinate time T is nondecreasing on all future-directed causal curves.
- Sch is causal (no closed causal curves exist).
- Sch is nondistinguishing, i.e. different points have identical pasts and futures [Hubeny, Rangamani, Ross, 2005]. It is Galilean-like $I^+(P) = \{Q \in \text{Sch}(d+3) \mid T_Q > T_P\}$.
- Under small perturbations of the lightcone structure CTCs can form.
- The future domain of dependence of a $T =$ cst slice is empty.

- This casts doubt on there being a well-defined field theory on Sch, for what to think about:
  - Time ordering (there is no time function)
  - Time evolution or predictability (future domain of dependence of $T = \text{cst}$ is empty)
  - Causally unstable (CTSs form under small perturbations of the lightcone structure)

Scalar field theory

Consider a massive complex scalar ϕ whose action is

$$S = -\int d^{d+3}x\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi + m_0^2\phi^*\phi\right) \ .$$

Define the inner product:

$$\langle u_M \mid u_{M'}\rangle = \frac{i}{2}\int_{T=T_0} d\Sigma^\mu u_M^* \overleftrightarrow{\partial_\mu} u_{M'} \ ,$$

where the u_M are the modes in terms of which the field ϕ is decomposed

$$\phi = \sum_M a_M u_M \ .$$

UIRs : $\phi(T,V,\vec{X},R) = e^{-imV}\psi(T,\vec{X},R) = e^{-imV}\sum_M a_M v_M \ .$

We impose the conditions that u_M must be regular away from the boundary $R=0$ and that the inner product $\langle u_M \mid u_{M'} \rangle$ is conserved in time which implies

$$\lim_{\varepsilon\to 0} \int_{R=\varepsilon} R^{-(d+1)} u_M^* \overleftrightarrow{\partial_R} u_{M'} dV d\vec{X} = 0$$

and is the condition that the flux of the current $u_M^* \overleftrightarrow{\partial_\mu} u_{M'}$ through the boundary at $R=0$ vanishes. These conditions imply:

1. The Breitenlohner–Freedman bound: $\nu = \sqrt{\frac{(d+2)^2}{4} + m_0^2 + \beta^2 m^2} \in \mathbb{R}$
2. For $\nu > 1$ there is one set of normalizable modes
3. For $0 < \nu < 1$ there are two sets of norm. modes $(+/-)$

Time evolution

Despite the fact that the future domain of dependence of a constant T slice is empty there exists a well-posed initial value problem for both Hilbert spaces $(+/-)$. Given initial data for ϕ at some time $T = T_0$ the future development follows from the decomposition $\phi = e^{-imV} \sum_M a_M v_M$ where the coefficients follow from

$$\langle e^{-imV} v_{L,n,k} \mid \phi(T = T_0)\rangle = \text{sign}(m) a_{L,n,k} \,.$$

In order to have a well-defined time evolution we only need to specify the values of the field ϕ at time $T = T_0$ and not its time derivative.

- In Sch$(d+3)$ there are no timelike Killing vectors X^μ with the property that $X_{[\mu}\nabla_\nu X_{\rho]} = 0$ (Frobenius $\to X^\mu$ is hypersurface orthogonal)
- Let $X = \partial_\tau$ be some timelike Killing vector and choose a patch that is stationary w.r.t. this X. Then the hypersurfaces $\tau =$ cst must be null (it can also be timelike but this is not suitable as an initial data surface) and hence $g^{\tau\tau} = 0$.
- It follows that the KG equation

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = \ldots \qquad \text{has no 2nd order time derivatives}$$

Green's functions

Define the Wightman functions $G^{\pm}$:

$$
\begin{aligned}
G^{+}(x,x') &= \langle 0 \mid \phi(x)\phi^{\dagger}(x') \mid 0\rangle\,, \\
G^{-}(x,x') &= \langle 0 \mid \phi^{\dagger}(x')\phi(x) \mid 0\rangle\,.
\end{aligned}
$$

These can be computed (e.g. using symmetry arguments) to be

$$
\begin{aligned}
G^{+}(x,x') &= \theta(m)C(m\zeta_{-\epsilon})^{\frac{d+2}{2}}J_{\pm\nu}(m\zeta_{-\epsilon})e^{im\eta_{-\epsilon}}\,, \\
G^{-}(x,x') &= -\theta(-m)C^{*}(-m\zeta_{+\epsilon})^{\frac{d+2}{2}}J_{\pm\nu}(-m\zeta_{+\epsilon})e^{im\eta_{+\epsilon}}\,,
\end{aligned}
$$

where $\zeta_{\pm\epsilon}$ and $\eta_{\pm\epsilon}$ are (for $\epsilon = 0$) Schrödinger invariant two-point functions

$$
\begin{aligned}
\zeta_{\pm\epsilon} &= \frac{\omega R R'}{\sin\omega(T-T'\pm i\epsilon)}\,, \\
\eta_{\pm\epsilon} &= -(V-V') + \frac{\omega(\vec{X}^2+\vec{X}'^2+R^2+R'^2)}{2\tan\omega(T-T'\pm i\epsilon)} - \frac{\omega\vec{X}\cdot\vec{X}'}{\sin\omega(T-T'\pm i\epsilon)}\,.
\end{aligned}
$$

- The bulk-to-bulk propagator and the retarded Green function

$$\begin{aligned} G_F(x,x') &= \theta(T-T')G^+(x,x') + \theta(T'-T)G^-(x,x')\,, \\ G_R(x,x') &= \theta(T-T')\left(G^+(x,x') - G^-(x,x')\right)\,. \end{aligned}$$

- For $m > 0$ we have $G_F = G_R \to$ Galilean-like.
- Even though T is not a time function, time ordering is well-defined due to $\theta(\pm m)$ inside $G^\pm$.

Properties of $G^+(x,x') - G^-(x,x')$:

1. It vanishes on the lightcone of the point $x = x'$ which overlaps with the set of spacelike separated points.
2. It vanishes when there does not exist a classical path connecting x and x', i.e. whenever $\sin\omega(T-T') = 0$.

Open questions

- How do scalars behave under small perturbations of the metric which for point particle probes lead to CTCs? Do these perturbations have finite energy?
- Here only free theory. Can we define a perturbatively well defined interacting field theory on Sch$(d+3)$?

Black holes in Schrödinger space-time (work in progress)

- Apply TsT to AdS black holes to construct black holes in Schrödinger spacetime.
- There are various (time-independent) AdS black hole solutions that are asymptotically either: global AdS, Poincaré AdS or plane wave AdS.
- We know that TsT of Poincaré AdS and plane wave AdS lead to Poincaré and global Schrödinger, respectively.
- TsT transformations of (asymptotically) global AdS spacetimes that are time-independent do not give rise to asymptotically Schrödinger geometries.

- TsT of AdS black holes that are time-independent and asymptotically Poincaré AdS (black branes) have been studied [Herzog, Rangamani, Ross, 2008], [Adams, Balasubramanian, McGreevy, 2008].
- A black hole solution that is asymptotically plane wave AdS is given in [Maldacena, Martelli, Tachikawa, 2008]. We have added angular momentum to this black hole and performed TsT on it.
- Which (thermodynamic) properties are generally preserved under TsT? Those that only depend on the horizon:
  - Temperature (up to a normalization constant)
  - Entropy
  - Chemical potentials

- Properties of thermodynamic quantities that depend on the asymptotics such as energy and grand potential are still under construction.
- The renormalized on-shell action before TsT changes sign at a specific location of the horizon. Is this property preserved under TsT and is there an associated phase transition?
- Particle number is spontaneously broken in systems of ultracold trapped atoms. How to address this in a holographic context [Balasubramanian, McGreevy, 2010]?