Doubled Field Theory and the Geometry of Duality

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arXiv:0904.4664,
0908.1792

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arXiv:1003.5027, 1006.4664

Appraches to string theory

- Supergravity limit - misses stringy features
- Perturbative string - misses non-perturbative
- Full theory exotic and complex
- Winding modes, T-duality, cocycles, algebraic structure not Lie algebra, non-polynomial
- String field theory: interactions, T-duality
- Seek subsector capturing exotic structure & duality, simple enough to analyse explicitly

Strings on d-Torus

- Supergravity limit: symmetry O(d,d)
- String: Perturbative T-duality O(d,d;Z)
- Kaluza-Klein theory: includes momentum modes on torus
- Include string winding or brane wrapping modes? Duality symmetry?

String Field Theory

- Strings on torus, coordinates $\{x^a\}$ plus extra dual coordinates $\{\tilde{x}_a\}$ conjugate to winding
- String field theory gives infinite set of fields on doubled torus $\psi(x, \tilde{x})$
- General solution of SFT: double fields $\psi(x, \tilde{x})$
- *Real* dependence on *full* doubled geometry, dual dimensions not auxiliary or gauge artifact. Double geom. *physical* and *dynamical*

String Field Theory on Torus

- Construct a subsector of SFT, "massless" fields $g_{ab}(x^a, \tilde{x}_a),\ b_{ab}(x^a, \tilde{x}_a),\ \phi(x^a, \tilde{x}_a)$
- Double field theory on doubled torus
- Novel symmetry, reduces to diffeos + B-field trans. in *any* half-dimensional subtorus
- Backgrounds depending on $\{x^a\}$ seen by particles, on $\{\tilde{x}_a\}$ seen by winding modes. Backgrounds with both: unfamiliar.

Earlier versions: Siegel, Tseytlin

- Restriction to “massless” fields NOT a low-energy limit
- Lowest terms in level expansion
- Does a full gauge-invariant theory exist for just these degrees of freedom?
- Is it T-dual? Background independent?

Strings on a Torus

$$\mathbb{R}^{n-1,1} \times T^d$$

- Coordinates $x^i = (y^\mu, x^a)$ $x^a \sim x^a + 2\pi$
- Momentum $p_i = (k_\mu, p_a)$
- Winding w^a $(p_a, w^a) \in \mathbb{Z}^{2d}$
- Fourier transform $(k_\mu, p_a, w^a) \to (y^\mu, x^a, \tilde{x}_a)$
- Doubled Torus $\mathbb{R}^{n-1,1} \times T^{2d}$ $\tilde{x}_a \sim \tilde{x}_a + 2\pi$
- String Field Theory gives infinite set of fields $\phi(y^\mu, x^a, \tilde{x}_a)$

$$n + d = D = 26 \ or \ 10$$

T-Duality

- Interchanges momentum and winding
- Equivalence of string theories on dual backgrounds with very different geometries
- String field theory symmetry, provided fields depend on both $x, \tilde{x}$ Kugo, Zwiebach
- For fields $\psi(y)$ not $\psi(y, x, \tilde{x})$ Buscher
- Aim: generalise to fields $\psi(y, x, \tilde{x})$

Generalised T-duality Dabholkar & CMH

Free field equn, M mass in D dimensions

$$M^2 \equiv -(k^2 + p^2 + w^2) = \frac{2}{\alpha'}(N + \bar{N} - 2)$$

Constraint

$$L_0 - \bar{L}_0 = N - \bar{N} - p_a w^a = 0$$

Massless states $N = \tilde{N} = 1 \qquad M^2 = 0 \qquad p_a w^a = 0$

Constrained fields $\phi(y, x, \tilde{x})$

$$\Delta\phi = 0 \qquad\qquad \Delta \equiv -\frac{2}{\alpha'}\frac{\partial}{\partial x^a}\frac{\partial}{\partial \tilde{x}_a}$$

$$h_{ij}(y^\mu, x^a, \tilde{x}_a),\ \ b_{ij}(y^\mu, x^a, \tilde{x}_a),\ \ d\,(y^\mu, x^a, \tilde{x}_a)$$

$$h_{ij} \rightarrow \{h_{\mu\nu}, h_{\mu a}, h_{ab}\}$$

Torus Backgrounds $\alpha' = 1$

Constant

$$G_{ij} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & G_{ab} \end{pmatrix}, \quad B_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & B_{ab} \end{pmatrix} \quad E_{ij} \equiv G_{ij} + B_{ij}$$

$$x^i = \{y^\mu, x^a\} \qquad \tilde{x}_i = \{\tilde{y}_\mu, \tilde{x}_a\} = \{0, \tilde{x}_a\}$$

Left and Right Derivatives

$$D_i = \frac{\partial}{\partial x^i} - E_{ik}\frac{\partial}{\partial \tilde{x}_k}, \quad \bar{D}_i = \frac{\partial}{\partial x^i} + E_{ki}\frac{\partial}{\partial \tilde{x}_k}$$

$$\Delta = \frac{1}{2}(D^2 - \bar{D}^2) = -2\frac{\partial}{\partial \tilde{x}_i}\frac{\partial}{\partial x^i}$$

$$\Box = \frac{1}{2}(D^2 + \bar{D}^2) \qquad D^2 = G^{ij} D_i D_j$$

Quadratic Action

$$S^{(2)} = \int [dxd\tilde{x}] \Big[\frac{1}{2} e_{ij} \Box e^{ij} + \frac{1}{4} (\bar{D}^j e_{ij})^2 + \frac{1}{4} (D^i e_{ij})^2 \\ -2\, d\, D^i \bar{D}^j e_{ij} \; - \; 4\, d\, \Box\, d \Big]$$

Invariant under

$$\delta e_{ij} = \; \bar{D}_j \lambda_i + D_i \bar{\lambda}_j \; ,$$

$$\delta d \; = -\frac{1}{4} D \cdot \lambda - \frac{1}{4} \bar{D} \cdot \bar{\lambda}$$

using constraint $\Delta \lambda = \Delta \bar{\lambda} = 0$

Discrete Symmetry

$$e_{ij} \rightarrow e_{ji} \; , \;\; D_i \rightarrow \bar{D}_i \; , \;\; \bar{D}_i \rightarrow D_i \; , \;\; d \rightarrow d$$

Comparison with Conventional Actions

Take $\quad B_{ij} = 0 \qquad \tilde{\partial}_i \equiv G_{ik} \frac{\partial}{\partial \tilde{x}_k}$

$$D_i = \partial_i - \tilde{\partial}_i \,, \quad \bar{D}_i = \partial_i + \tilde{\partial}_i$$

$$\Box = \partial^2 + \tilde{\partial}^2 \quad \Delta = -2\, \partial_i \tilde{\partial}^i$$

$$e_{ij} = h_{ij} + b_{ij}$$

Usual action

$$\int dx \sqrt{-g} e^{-2\phi} \Big[R + 4(\partial\phi)^2 - \frac{1}{12} H^2 \Big]$$

quadratic part $\quad \int dx \;\, L[\, h, b, d; \partial \,]$

Double Field Theory Action

$$S^{(2)} = \int [dx d\tilde{x}] \Big[L[h, b, d; \partial] + L[-h, -b, d; \tilde{\partial}]$$

$$+ (\partial_k h^{ik})(\tilde{\partial}^j b_{ij}) + (\tilde{\partial}^k h_{ik})(\partial_j b^{ij}) - 4\, d\, \partial^i \tilde{\partial}^j b_{ij} \Big]$$

Action + dual action + strange mixing terms

$$\delta h_{ij} = \partial_i \epsilon_j + \partial_j \epsilon_i + \tilde{\partial}_i \tilde{\epsilon}_j + \tilde{\partial}_j \tilde{\epsilon}_i \,,$$

$$\delta b_{ij} = -(\tilde{\partial}_i \epsilon_j - \tilde{\partial}_j \epsilon_i) - (\partial_i \tilde{\epsilon}_j - \partial_j \tilde{\epsilon}_i) \,,$$

$$\delta d = -\partial \cdot \epsilon + \tilde{\partial} \cdot \tilde{\epsilon} \,.$$

Diffeos and B-field transformations mixed.
Cubic action found

General fields $\psi(x, \tilde{x})$

Fields on Spacetime M $\psi(x)$

Restricted Fields on N, T-dual to M $\psi(x')$

M,N null wrt O(D,D) metric $ds^2 = 2dx^i dx_i$

Subsector with fields and parameters all restricted to M or N

- Constraint satisfied on all fields and products of fields
- No projectors or cocycles
- T-duality covariant: independent of choice of N
- Can find full non-linear form of gauge transformations
- Full gauge algebra, full non-linear action

Background Independent Action

Fields and parameters restricted to null space N

Background independent field:

$$\mathcal{E}_{ij} \equiv E_{ij} + e_{ij} + \frac{1}{2} e_i{}^k e_{kj} + \mathcal{O}(e^3)$$

$$\mathcal{E} \equiv E + \left(1 - \frac{1}{2}\, e\right)^{-1} e$$

- Write action and transformations in terms of $\mathcal{E}_{ij}, \ d$

- and derivatives
$$\mathcal{D}_i = \partial_i - \mathcal{E}_{ik}\tilde{\partial}^k$$
$$\bar{\mathcal{D}}_i = \partial_i + \mathcal{E}_{ki}\tilde{\partial}^k$$

- Find manifestly background independent forms that agree with action and transformations to lowest order

- Unique BI terms that agree with lowest order results. Complete non-linear structure!

BI Action and Transformations

$$
\begin{aligned}
S \;=\; \int dx d\tilde{x}\; e^{-2d} \Big[& -\frac{1}{4}\, g^{ik} g^{jl}\, \mathcal{D}^p \mathcal{E}_{kl}\, \mathcal{D}_p \mathcal{E}_{ij} \\
& + \frac{1}{4} g^{kl} \Big(\mathcal{D}^j \mathcal{E}_{ik} \mathcal{D}^i \mathcal{E}_{jl} + \bar{\mathcal{D}}^j \mathcal{E}_{ki}\, \bar{\mathcal{D}}^i \mathcal{E}_{lj} \Big) \\
& + \Big(\mathcal{D}^i d\; \bar{\mathcal{D}}^j \mathcal{E}_{ij} + \bar{\mathcal{D}}^i d\; \mathcal{D}^j \mathcal{E}_{ji} \Big) + 4 \mathcal{D}^i d\, \mathcal{D}_i d \;\Big]
\end{aligned}
$$

BI Action and Transformations

$$S = \int dx d\tilde{x}\; e^{-2d}\Big[-\frac{1}{4}\, g^{ik} g^{jl}\, \mathcal{D}^p \mathcal{E}_{kl}\, \mathcal{D}_p \mathcal{E}_{ij}$$

$$+ \frac{1}{4} g^{kl} \big(\mathcal{D}^j \mathcal{E}_{ik} \mathcal{D}^i \mathcal{E}_{jl} + \bar{\mathcal{D}}^j \mathcal{E}_{ki}\, \bar{\mathcal{D}}^i \mathcal{E}_{lj} \big)$$

$$+ \big(\mathcal{D}^i d\; \bar{\mathcal{D}}^j \mathcal{E}_{ij} + \bar{\mathcal{D}}^i d\; \mathcal{D}^j \mathcal{E}_{ji} \big) + 4 \mathcal{D}^i d\, \mathcal{D}_i d \;\Big]$$

$$\delta \mathcal{E}_{ij} = \xi^M \partial_M \mathcal{E}_{ij}$$

$$+ \mathcal{D}_i \tilde{\xi}_j - \bar{\mathcal{D}}_j \tilde{\xi}_i + \mathcal{D}_i \xi^k \mathcal{E}_{kj} + \bar{\mathcal{D}}_j \xi^k \mathcal{E}_{ik}$$

$$\delta_\lambda d = -\frac{1}{2} \partial_M \xi^M + \xi^M \partial_M\, d$$

- Remarkable action, reduces to familiar ones
- uses stringy combination $\mathcal{E}_{ij} = g_{ij} + b_{ij}$
- checked gauge invariant
- Invariant under SO(D,D)

2-derivative action

$$S \; = \; S^{(0)}(\partial, \partial) + S^{(1)}(\partial, \tilde{\partial}) + S^{(2)}(\tilde{\partial}, \tilde{\partial})$$

Write $S^{(0)}$ in terms of usual fields

$$\mathcal{E}_{ij} = g_{ij} + b_{ij} \qquad\qquad e^{-2d} = \sqrt{-g}e^{-2\phi}$$

2-derivative action

$$S \;=\; S^{(0)}(\partial,\partial) + S^{(1)}(\partial,\tilde{\partial}) + S^{(2)}(\tilde{\partial},\tilde{\partial})$$

Write $S^{(0)}$ in terms of usual fields

$$\mathcal{E}_{ij} = g_{ij} + b_{ij} \qquad\qquad e^{-2d} = \sqrt{-g}e^{-2\phi}$$

Gives usual action (+ surface term)

$$\int dx\sqrt{-g}e^{-2\phi}\left[R + 4(\partial\phi)^2 - \frac{1}{12}H^2\right]$$

$$S^{(0)} = S(\mathcal{E}, d, \partial)$$

2-derivative action

$$S \;=\; S^{(0)}(\partial,\partial) + S^{(1)}(\partial,\tilde\partial) + S^{(2)}(\tilde\partial,\tilde\partial)$$

Write $S^{(0)}$ in terms of usual fields

$$\mathcal{E}_{ij} = g_{ij} + b_{ij} \qquad\qquad e^{-2d} = \sqrt{-g}e^{-2\phi}$$

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$$S^{(0)} = S(\mathcal{E}, d, \partial) \qquad\qquad S^{(2)} = S(\mathcal{E}^{-1}, d, \tilde\partial)$$

T-dual!

2-derivative action

$$S \;=\; S^{(0)}(\partial,\partial) + S^{(1)}(\partial,\tilde\partial) + S^{(2)}(\tilde\partial,\tilde\partial)$$

Write $S^{(0)}$ in terms of usual fields

$$\mathcal{E}_{ij} = g_{ij} + b_{ij} \qquad\qquad e^{-2d} = \sqrt{-g}e^{-2\phi}$$

Gives usual action (+ surface term)

$$\int dx\sqrt{-g}e^{-2\phi}\left[R + 4(\partial\phi)^2 - \frac{1}{12}H^2\right]$$

$$S^{(0)} = S(\mathcal{E}, d, \partial) \qquad\qquad S^{(2)} = S(\mathcal{E}^{-1}, d, \tilde\partial)$$

T-dual! $\quad S^{(1)}$ strange mixed terms

Generalised T-duality transformations:

$$X'^M \equiv \begin{pmatrix} \tilde{x}'_i \\ x'^i \end{pmatrix} = hX^M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix}$$

h in O(d,d) acts on toroidal coordinates only

$$\mathcal{E}'(X') = (a\mathcal{E}(X) + b)(c\mathcal{E}(X) + d)^{-1}$$

$$d'(X') = d(X)$$

Buscher if fields independent of toroidal coordinates
Generalisation to case without isometries

O(D,D) Covariant Notation

$$X^M \equiv \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix} \qquad \partial_M \equiv \begin{pmatrix} \partial^i \\ \partial_i \end{pmatrix}$$

$$\eta_{MN} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad M = 1, ..., 2D$$

Parameters $\quad (\lambda, \bar{\lambda}) \to \Sigma^M$

Gauge Algebra $\quad [\delta_{\Sigma_1}, \delta_{\Sigma_2}] = \delta_{[\Sigma_1, \Sigma_2]_C}$

C-Bracket:

$$[\Sigma_1, \Sigma_2]_C \equiv \; [\Sigma_1, \Sigma_2] - \frac{1}{2}\,\eta^{MN}\eta_{PQ}\,\Sigma^P_{[1}\,\partial_N\,\Sigma^Q_{2]}$$

Lie bracket + metric term

Parameters $\Sigma^M(X)$ restricted to N
Decompose into vector + 1-form on N
C-bracket reduces to Courant bracket on N

Same covariant form of gauge algebra found in similar context by Siegel

Symmetry is Reducible

Parameters of the form $\Sigma^M = \eta^{MN}\partial_N \chi$
do not act

Gauge algebra determined up to such transformations

cf 2-form gauge field $\delta B = d\alpha$
Parameters of the form $\alpha = d\beta$
do not act

Jacobi Identities not satisfied!

$$J(\Sigma_1, \Sigma_2, \Sigma_3) \equiv [\,[\Sigma_1, \Sigma_2]\,, \Sigma_3\,] + \text{cyclic} \neq 0$$

for both C-bracket and Courant-bracket

How can bracket be realised as a symmetry algebra?

$$[\,[\delta_{\Sigma_1}, \delta_{\Sigma_2}]\,, \delta_{\Sigma_3}\,] + \text{cyclic} = \delta_{J(\Sigma_1, \Sigma_2, \Sigma_3)}$$

Jacobi Identities not satisfied!

$$J(\Sigma_1, \Sigma_2, \Sigma_3) \equiv [\,[\Sigma_1, \Sigma_2]\,, \Sigma_3\,] + \text{cyclic} \neq 0$$

for both C-bracket and Courant-bracket

How can bracket be realised as a symmetry algebra?

$$[\,[\delta_{\Sigma_1}, \delta_{\Sigma_2}]\,, \delta_{\Sigma_3}\,] + \text{cyclic} = \delta_{J(\Sigma_1, \Sigma_2, \Sigma_3)}$$

Resolution:

$$J(\Sigma_1, \Sigma_2, \Sigma_3)^M = \eta^{MN} \partial_N \chi$$

$\delta_{J(\Sigma_1, \Sigma_2, \Sigma_3)}$ does not act on fields

Rewrite in terms of Generalised Metric

$$\mathcal{H}_{MN} \quad = \begin{pmatrix} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij} - b_{ik}g^{kl}b_{lj} \end{pmatrix} .$$

Gauge transformation as "Generalised Lie Derivative"

$$\delta_\xi \mathcal{H}^{MN} = \widehat{\mathcal{L}}_\xi \mathcal{H}^{MN}$$

Action in terms of Generalized scalar curvature

$$S = \int dx\, d\tilde{x}\, e^{-2d}\, \mathcal{R}$$

Generalised Metric Formulation

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij} - b_{ik}g^{kl}b_{lj} \end{pmatrix} .$$

2 Metrics on double space $\mathcal{H}_{MN},\ \eta_{MN}$

Generalised Metric Formulation

$$\mathcal{H}_{MN} \;=\; \begin{pmatrix} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij} - b_{ik}g^{kl}b_{lj} \end{pmatrix} .$$

2 Metrics on double space $\quad \mathcal{H}_{MN}, \; \eta_{MN}$

$$\mathcal{H}^{MN} \equiv \eta^{MP}\mathcal{H}_{PQ}\eta^{QN}$$

Constrained metric $\quad \mathcal{H}^{MP}\mathcal{H}_{PN} \;=\; \delta^{M}{}_{N}$

Generalised Metric Formulation

$$\mathcal{H}_{MN} \quad = \begin{pmatrix} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij} - b_{ik}g^{kl}b_{lj} \end{pmatrix} .$$

2 Metrics on double space $\quad \mathcal{H}_{MN}, \ \eta_{MN}$

$$\mathcal{H}^{MN} \equiv \eta^{MP}\mathcal{H}_{PQ}\eta^{QN}$$

Constrained metric $\quad \mathcal{H}^{MP}\mathcal{H}_{PN} \ = \ \delta^M{}_N$

Covariant Transformation

$$h^P{}_M h^Q{}_N \mathcal{H}'_{PQ}(X') \ = \ \mathcal{H}_{MN}(X)$$

$$X' = hX \qquad\qquad h \in O(D,D)$$

O(D,D) covariant action

$$S = \int dx d\tilde{x} \, e^{-2d} \, L$$

$$L = \frac{1}{8} \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \, \partial_N \mathcal{H}_{KL} - \frac{1}{2} \mathcal{H}^{MN} \partial_N \mathcal{H}^{KL} \, \partial_L \mathcal{H}_{MK} - 2 \, \partial_M d \, \partial_N \mathcal{H}^{MN} + 4 \mathcal{H}^{MN} \, \partial_M d \, \partial_N d$$

O(D,D) covariant action

$$S = \int dx d\tilde{x}\, e^{-2d}\, L$$

$$L = \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\,\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_N\mathcal{H}^{KL}\,\partial_L\mathcal{H}_{MK}$$
$$- 2\,\partial_M d\,\partial_N\mathcal{H}^{MN} + 4\mathcal{H}^{MN}\,\partial_M d\,\partial_N d$$

Gauge Transformation

$$\delta_\xi\mathcal{H}^{MN} = \xi^P\partial_P\mathcal{H}^{MN}$$
$$+ (\partial^M\xi_P - \partial_P\xi^M)\,\mathcal{H}^{PN} + (\partial^N\xi_P - \partial_P\xi^N)\,\mathcal{H}^{MP}$$

O(D,D) covariant action

$$S = \int dx d\tilde{x}\, e^{-2d}\; L$$

$$L = \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\,\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_N\mathcal{H}^{KL}\,\partial_L\mathcal{H}_{MK}$$
$$-\,2\,\partial_M d\,\partial_N\mathcal{H}^{MN} + 4\mathcal{H}^{MN}\,\partial_M d\,\partial_N d$$

Gauge Transformation

$$\delta_\xi\mathcal{H}^{MN} = \xi^P\partial_P\mathcal{H}^{MN}$$
$$+\,(\partial^M\xi_P - \partial_P\xi^M)\,\mathcal{H}^{PN} + (\partial^N\xi_P - \partial_P\xi^N)\,\mathcal{H}^{MP}$$

Rewrite as “Generalised Lie Derivative”

$$\delta_\xi\mathcal{H}^{MN} = \widehat{\mathcal{L}}_\xi\mathcal{H}^{MN}$$

Generalised Lie Derivative

$$A_{N_1 \ldots}^{M_1 \ldots}$$

$$\widehat{\mathcal{L}}_\xi A_M{}^N \equiv \xi^P \partial_P A_M{}^N$$
$$+(\partial_M \xi^P - \partial^P \xi_M) A_P{}^N + (\partial^N \xi_P - \partial_P \xi^N) A_M{}^P$$

Generalised Lie Derivative

$$A^{M_1 \dots}_{N_1 \dots}$$

$$\widehat{\mathcal{L}}_\xi A_M{}^N \equiv \xi^P \partial_P A_M{}^N$$
$$+(\partial_M \xi^P - \partial^P \xi_M) A_P{}^N + (\partial^N \xi_P - \partial_P \xi^N) A_M{}^P$$

$$\widehat{\mathcal{L}}_\xi A_M{}^N = \mathcal{L}_\xi A_M{}^N - \eta^{PQ} \eta_{MR}\, \partial_Q \xi^R\, A_P{}^N$$
$$+\, \eta_{PQ} \eta^{NR}\, \partial_R \xi^Q\, A_M{}^P$$

Generalised Lie Derivative

$$A_{N_1\dots}^{M_1\dots}$$

$$\widehat{\mathcal{L}}_\xi A_M{}^N \equiv \xi^P \partial_P A_M{}^N$$
$$+(\partial_M \xi^P - \partial^P \xi_M) A_P{}^N + (\partial^N \xi_P - \partial_P \xi^N) A_M{}^P$$

$$\widehat{\mathcal{L}}_\xi A_M{}^N = \mathcal{L}_\xi A_M{}^N - \eta^{PQ} \eta_{MR}\, \partial_Q \xi^R\, A_P{}^N$$
$$+ \eta_{PQ} \eta^{NR}\, \partial_R \xi^Q\, A_M{}^P$$

Algebra given by C-bracket

$$\left[\widehat{\mathcal{L}}_{\xi_1}, \widehat{\mathcal{L}}_{\xi_2} \right] = -\widehat{\mathcal{L}}_{[\xi_1, \xi_2]_C}$$

D-Bracket

$$\left[\,A, B\,\right]_{\mathrm{D}} \equiv \widehat{\mathcal{L}}_A B$$

$$\left[\,A, B\,\right]_{\mathrm{D}}^{M} = \left[\,A, B\,\right]_{\mathrm{C}}^{M} + \frac{1}{2}\partial^{M}\left(B^{N} A_{N}\right)$$

D-Bracket

$$[\,A, B\,]_{\mathrm{D}} \equiv \widehat{\mathcal{L}}_A B$$

$$[\,A, B\,]_{\mathrm{D}}^{M} = [\,A, B\,]_{\mathrm{C}}^{M} + \frac{1}{2}\partial^M (B^N A_N)$$

Not skew, but satisfies Jacobi-like identity

$$[A, [B, C]_{\mathrm{D}}]_{\mathrm{D}} = [[A, B]_{\mathrm{D}}], C]_{\mathrm{D}} + [B, [A, C]_{\mathrm{D}}]_{\mathrm{D}}$$

D-Bracket $$[\,A, B\,]_{\rm D} \equiv \widehat{\mathcal{L}}_A B$$

$$[\,A, B\,]_{\rm D}^M = [\,A, B\,]_{\rm C}^M + \frac{1}{2}\partial^M (B^N A_N)$$

Not skew, but satisfies Jacobi-like identity

$$[A, [B, C]_{\rm D}]_{\rm D} = [[A, B]_{\rm D}], C]_{\rm D} + [B, [A, C]_{\rm D}]_{\rm D}$$

On restricting to null subspace N
C-bracket → Courant bracket
D-bracket → Dorfman bracket
Gen Lie Derivative → GLD of Grana, Minasian, Petrini and Waldram

Generalized scalar curvature

$$
\begin{aligned}
\mathcal{R} \equiv \; & 4\,\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} \\
& - 4\,\mathcal{H}^{MN}\partial_M d\,\partial_N d + 4\partial_M\mathcal{H}^{MN}\,\partial_N d \\
& + \frac{1}{8}\,\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\,\partial_K\mathcal{H}_{NL}
\end{aligned}
$$

Generalized scalar curvature

$$\mathcal{R} \equiv \; 4\,\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN}$$
$$- 4\,\mathcal{H}^{MN}\partial_M d\,\partial_N d + 4\partial_M\mathcal{H}^{MN}\,\partial_N d$$
$$+ \frac{1}{8}\,\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\,\partial_K\mathcal{H}_{NL}$$

$$S = \int dx\,d\tilde{x}\,e^{-2d}\,\mathcal{R}$$

Generalized scalar curvature

$$\mathcal{R} \equiv 4\,\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} - 4\,\mathcal{H}^{MN}\partial_M d\,\partial_N d + 4\partial_M\mathcal{H}^{MN}\,\partial_N d + \frac{1}{8}\,\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\,\partial_K\mathcal{H}_{NL}$$

$$S = \int dx\,d\tilde{x}\,e^{-2d}\,\mathcal{R}$$

Gauge Symmetry

$$\delta_\xi\mathcal{R} = \widehat{\mathcal{L}}_\xi\mathcal{R} = \xi^M\partial_M\mathcal{R}$$

$$\delta_\xi\,e^{-2d} = \partial_M(\xi^M e^{-2d})$$

Generalized scalar curvature

$$\mathcal{R} \equiv \; 4\,\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} - 4\,\mathcal{H}^{MN}\partial_M d\,\partial_N d + 4\partial_M\mathcal{H}^{MN}\,\partial_N d + \frac{1}{8}\,\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\,\partial_K\mathcal{H}_{NL}$$

$$S = \int dx\, d\tilde{x}\, e^{-2d}\,\mathcal{R}$$

Gauge Symmetry

$$\delta_\xi\mathcal{R} \;= \widehat{\mathcal{L}}_\xi\mathcal{R} = \; \xi^M\partial_M\mathcal{R}$$

$$\delta_\xi\, e^{-2d} = \partial_M(\xi^M e^{-2d})$$

Field equations give gen. Ricci tensor

Double Field Theory

- Captures some of the magic of string theory
- Constructed cubic action, quartic should have new stringy features
- T-duality symmetry, cocycles, homotopy Lie, constraints
- For fields restricted to null subspace, have full non-linear action and gauge transformations.
- Background independent, duality covariant
- Courant bracket gauge algebra

- Stringy issues in simpler setting than SFT
- Geometry? Meaning of curvature?
- Use for non-geometric backgrounds?
- Generalised Geometry doubles Tangent space, DFT doubles coordinates.
- Full theory without restriction? Does it close on a geometric action with just these fields?
- Doubled geometry *physical* and *dynamical*