Arithmetic Quantum Gravity

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Based on work with

Axel Kleinschmidt and Michael Koehn

[arXiv:0907.3048][arXiv:0912.0854]

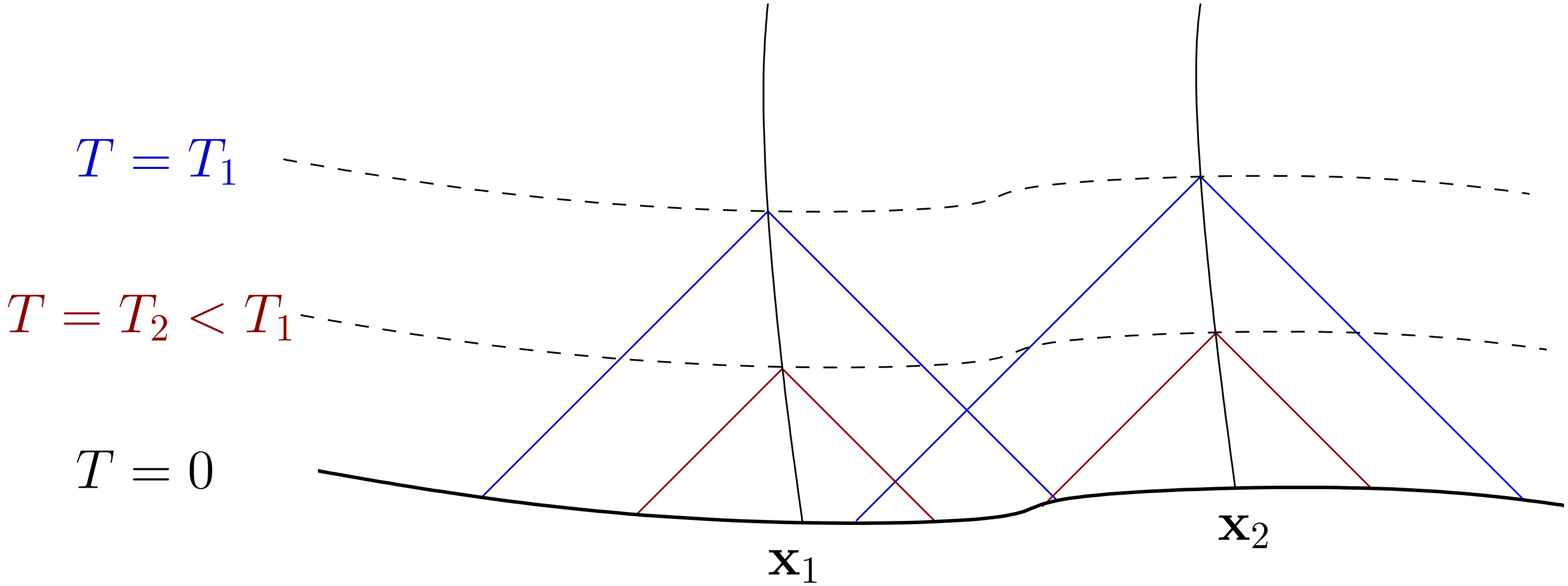
Context and Plan

- BKL analysis of spacelike singularities
- Hidden symmetries in supergravity
- Cosmological billiards and Kac–Moody algebras
- Minisuperspace models for quantum gravity
  $\rightarrow$ Quantum cosmological billiards
- Octonions, octavians and E_8
- $PSL_2(\mathtt{O})$ and arithmetic structure of wavefunctions
- Generalization: beyond the ultralocal approximation
- A Lie algebra mechanism for emergent space(-time)?

Cosmological billards: BKL

Gravitational dynamics near space-like singularity:

[Belinskii, Khalatnikov, Lifshitz 1970; Misner 1969; Chitre 1972]



Spatial points decouple $\Rightarrow$ dynamics becomes ultra-local.

Reduction of degress of freedom to spatial scale factors β^a

$$ds^2 = -N^2 dt^2 + \sum_{a=1}^{d} e^{-2\beta^a} dx_a^2 \qquad (t \sim -\log T)$$

Cosmological billiards: Dynamics

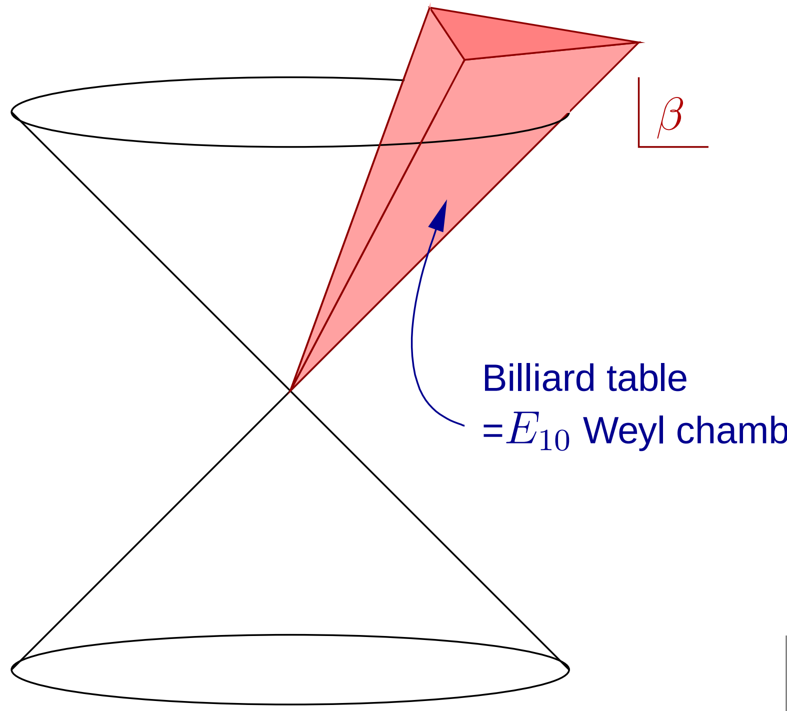
Effective Lagrangian for $\beta^a(t)$ ($a = 1, \ldots, d$)

$$\mathcal{L} = \frac{1}{2} n^{-1} \sum_{a,b=1}^{d} G_{ab} \dot{\beta}^a \dot{\beta}^b + V_{\text{eff}}(\beta)$$

[G_{ab}: DeWitt metric (Lorentzian signature)]

Close to the singularity V_{eff} consists of infinite potentials walls, obstructing free null motion of β^a.

Chaotic oscillations of the metric (*i.e.* ∞ many Kasner bounces) if billiard table inside lightcone.

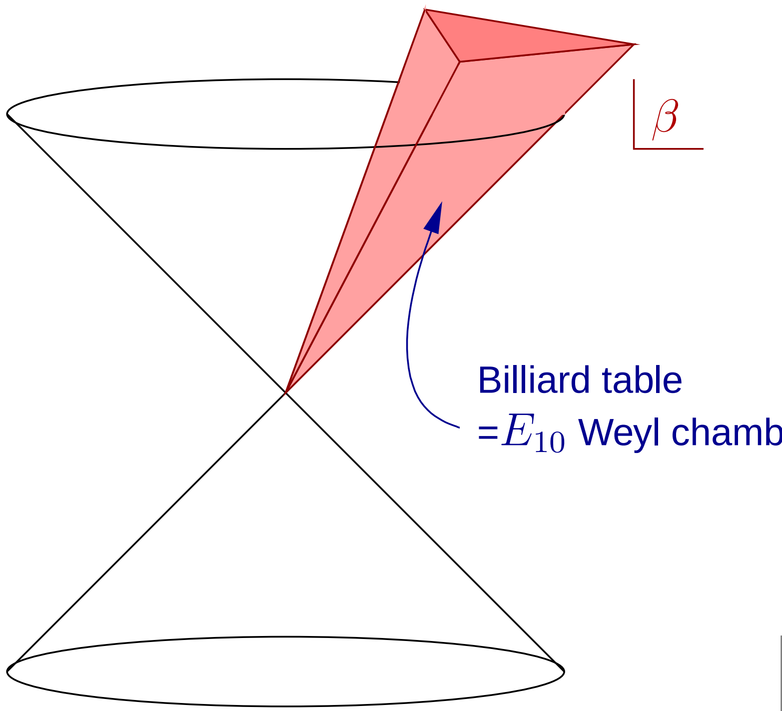


The Lie algebra connection

For maximal $D = 11$ supergravity space of logarithmic scale factors $\{\beta^a(t)\}$ (for $a = 1, \ldots, 10$) can be identified with the Cartan subalgebra of the maximally extended hyperbolic Kac–Moody algebra E_{10}.

Billiard 'wedge' is Weyl chamber of E_{10} [Damour,Henneaux,2000]

Chaotic oscillations if Kac-Moody algebra is *hyperbolic*, otherwise *AVD* near singularity [Damour,Henneaux,Julia,HN]



Quantum cosmological billiards

Setting $n = 1$ one has to quantize $\mathcal{L} = \frac{1}{2}\dot{\beta}^a G_{ab}\dot{\beta}^b$ with null constraint $\dot{\beta}^a G_{ab}\dot{\beta}^b = 0$ on billiard domain.

Canonical momenta: $\pi_a = G_{ab}\dot{\beta}^b \quad \Rightarrow \quad \mathcal{H}_0 = \frac{1}{2}\pi_a G^{ab}\pi_b$.

Wheeler–DeWitt (WDW) equation in canonical quantization

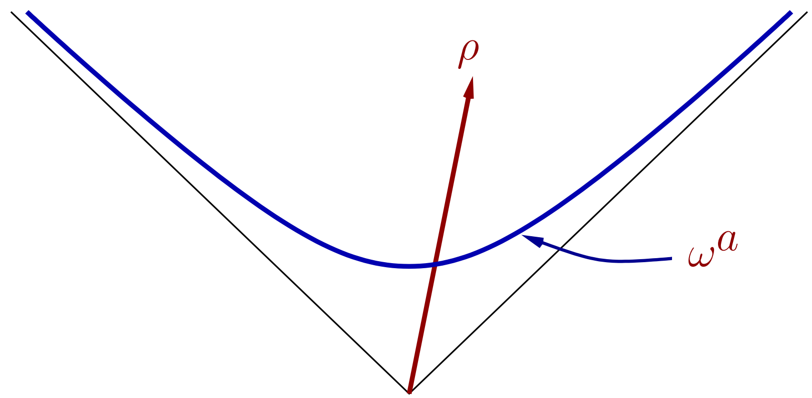
$$\mathcal{H}_0\Psi(\beta) = -\frac{1}{2}G^{ab}\partial_a\partial_b\Psi(\beta) = 0$$

with Klein–Gordon inner product

$$(\Psi_1|\Psi_2) = -i\int d\Sigma^a \Psi_1^* \overleftrightarrow{\partial_a} \Psi_2$$

Quantum cosmological billiards (II)

Introduce new coordinates ρ and $\omega^a(z)$ from 'radius' and co-ordinates z on unit hyperboloid



$$\beta^a = \rho\omega^a , \quad \omega^a G_{ab}\omega^b = -1$$
$$\rho^2 = -\beta^a G_{ab}\beta^b$$

WDW equation in these variables (*no* ordering ambiguities!)

$$\left[-\rho^{1-d}\frac{\partial}{\partial\rho}\left(\rho^{d-1}\frac{\partial}{\partial\rho}\right) + \rho^{-2}\Delta_{\mathsf{LB}}\right]\Psi(\rho, z) = 0$$

Laplace–Beltrami operator on unit hyperboloid

Solving the WDW equation

$$\left[-\rho^{1-d}\frac{\partial}{\partial\rho}\left(\rho^{d-1}\frac{\partial}{\partial\rho}\right)+\rho^{-2}\Delta_{\mathsf{LB}}\right]\Psi(\rho,z)=0$$

Separation of variables: $\Psi(\rho,z)=R(\rho)F(z)$. Thus

$$-\Delta_{\mathsf{LB}}F(z)=EF(z) \quad\Rightarrow\quad R_{\pm}(\rho)=\rho^{-\frac{d-2}{2}\pm i\sqrt{E-\left(\frac{d-2}{2}\right)^2}}$$

$\rightarrow$ must solve spectral problem on hyperbolic space.

- For $E>0$ we have $R_{\pm}(\rho)\rightarrow 0$ for $\rho\rightarrow\infty$.
- For $E>\frac{(d-2)^2}{4}$ wavefunction is *necessarily complex*.
- *Positive norm* wavefunctions with $R_{+}(\rho)$.

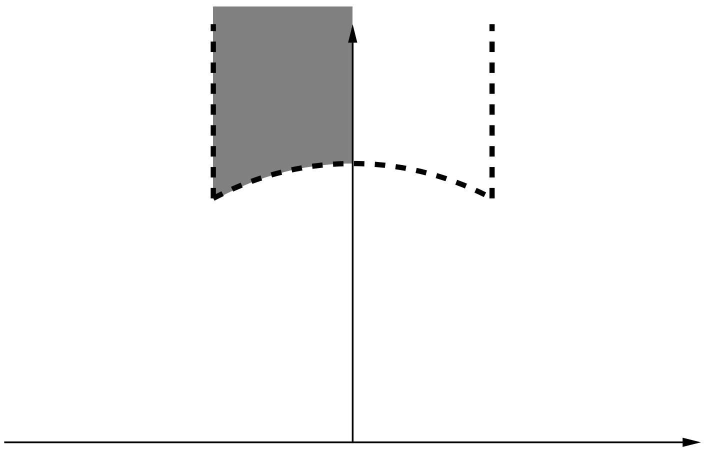
$\triangle_{\rm LB}$ and boundary conditions

The classical billiard ball is constrained to Weyl chamber with infinite potentials $\Rightarrow$ Dirichlet boundary conditions

Use upper half plane model

$$z = (\vec{u}, v)\,, \quad \vec{u} \in \mathbb{R}^{d-2}, v \in \mathbb{R}_{>0}$$

$$\Rightarrow \quad \Delta_{\sf LB} = v^{d-1}\partial_v(v^{3-d}\partial_v) + v^2\partial^2_{\vec{u}}$$



With *Dirichlet boundary conditions* (for $d = 3$ see [Iwaniec])

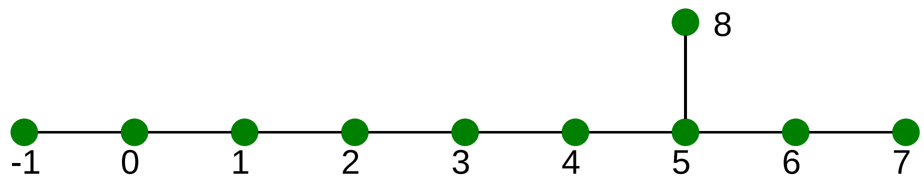
$$-\Delta_{\sf LB}F(z) = EF(z) \quad \Rightarrow \quad E \geq \left(\frac{d-2}{2}\right)^2$$

Octavians and E_8

Octonions $\mathbb{O}$ = largest division algebra $(i, j = 1, \dots, 7)$

$$e_i e_j + e_j e_i = -2\delta_{ij} \;\; , \quad e_j e_{j+1} e_{j+3} = -1 \quad (j \equiv j+7)$$

Simple roots of $E_8 \subset E_{10}$:



$$\varepsilon_1 = \frac{1}{2}(1 - e_1 - e_5 - e_6)\, , \;\; \varepsilon_2 = e_1\, , \;\; \varepsilon_3 = \frac{1}{2}(-e_1 - e_2 + e_6 + e_7)\, , \;\; \varepsilon_4 = e_2$$

$$\varepsilon_5 = \frac{1}{2}(-e_2 - e_3 - e_4 - e_7)\, , \;\; \varepsilon_6 = e_3\, , \;\; \varepsilon_7 = \frac{1}{2}(-e_3 + e_5 - e_6 + e_7)\, , \;\; \varepsilon_2 = e_4$$

with Cartan matrix $A_{ij} = \langle \varepsilon_i | \varepsilon_j \rangle := \varepsilon_i \bar{\varepsilon}_j + \varepsilon_j \bar{\varepsilon}_i$

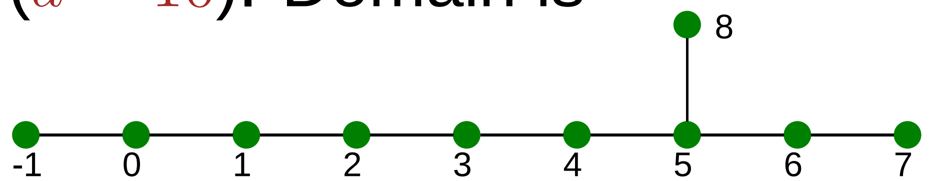
Root lattice $Q(E_8)$ = *non-commutative and non-associative ring* of integer octonions `O` ('octavians') `[Bruck, Coxeter, 1946]`

240 *roots* of $E_8 \;\equiv\;$ 240 *units* of $Q(E_8)$; highest root $\theta = 1$

Arithmetic structure (I)

Beyond general inequality details of spectrum depend on shape of domain. (‘Shape of the drum’ problem)

Focus on maximal supergravity ($d = 10$). Domain is determined by E_{10} Weyl group.



9-dimensional upper octonionic upper half plane: $u \equiv \vec{u} \in \mathbb{O}$

On $z = u + iv$ ($v > 0$) fundamental Weyl reflections act by

$$w_{-1}(z) = \frac{1}{\bar{z}}\,, \;\; w_0(z) = -\bar{z} + 1\,, \;\; w_j(z) = -\varepsilon_j \bar{z} \varepsilon_j$$

with ε_j = simple E_8 roots ($j = 1, \ldots, 8$).

Arithmetic structure (II)

Iteration of ‘modular’ action

$$w_{-1}(z) = \frac{1}{\bar{z}}\,,\; w_0(z) = -\bar{z} + 1\,,\; w_j(z) = -\varepsilon_j \bar{z} \varepsilon_j$$

generates whole Weyl group $W(E_{10})$. No (very) simple octonionic representation of an arbitrary element known.

Restricting to the even Weyl group $W^+(E_{10})$ gives ‘holomorphic’ transformations and one obtains

$$W^+(E_{10}) = PSL_2(\mathtt{O})$$

which should be interpreted as a modular group over the integer octonions $\equiv$ ‘octavians’ $\mathtt{O}$. [Feingold,Kleinschmidt,HN ’08]

Modular wavefunctions (I)

Fundamental Weyl reflections on wavefunction $\Psi(\rho, z)$

$$\Psi(w_I(\beta)) \equiv \Psi(\rho, w_I(z)) = \begin{cases} +\Psi(\rho, z) & \text{Neumann b.c.} \\ -\Psi(\rho, z) & \text{Dirichlet b.c.} \end{cases}$$

Use Weyl symmetry to *define* $\Psi(\rho, z)$ on the whole upper half plane, with Dirichlet boundary conditions $\Rightarrow \Psi(\rho, z)$ is

- Linear combination of eigenfunctions of Δ_{LB} on UHP
- Invariant under action of $W^+(E_{10}) = PSL_2(\mathtt{O})$.
  Anti-invariant under extension to $W(E_{10})$.

$\Rightarrow$ Wavefunction is an odd Maass wave form of $PSL_2(\mathtt{O})$

[cf. [Forte 2008] for work on $D = 4$ gravity with $PSL_2(\mathbb{Z})$]

Modular wavefunctions (II)

The spectrum of odd Maass wave forms is discrete but not known. For $PSL_2(\mathtt{O})$ the theory is not even developed.

For lower dimensional cases like pure $(3+1)$-dimensional Einstein gravity with $PSL_2(\mathbb{Z})$ many numerical results.

[Graham, Szépfalusy 1990; Steil 1994; Then 2003]

Summary of analysis so far:

> Quantum billiard wavefunction $\Psi(\rho, z)$ is an *odd Maass wave form* (Dirichlet b.c.) for $PSL_2(\mathtt{O})$.

NB: Neumann b.c.: discrete spectrum embedded in continuous spectrum (Eisenstein series), *etc.*

Interpretation (I)

In BKL limit 'Wavefunction of the Universe' is formally

$$|\Psi_{\text{full}}\rangle \sim \prod_{\mathbf{x}} |\Psi_{\mathbf{x}}\rangle$$

Product of quantum cosmological billiard wavefunctions, one for each spatial point (ultra-locality).

Each factor contains a Maass wave form of the type $\Psi_{\mathbf{x}}(\rho_{\mathbf{x}}, z_{\mathbf{x}}) = \sum R_{\mathbf{x}}(\rho_{\mathbf{x}}) F_{\mathbf{x}}(z_{\mathbf{x}})$ with

$$-\Delta_{\text{LB}} F(z) = E F(z)\,, \quad R_{\pm}(\rho) = \rho^{-\frac{d-2}{2} \pm i\sqrt{E - \left(\frac{d-2}{2}\right)^2}}$$

Beyond ultralocality: replace $|\Psi_{\text{full}}\rangle$ by wavefunction depending on full tower of E_{10} degrees of freedom!

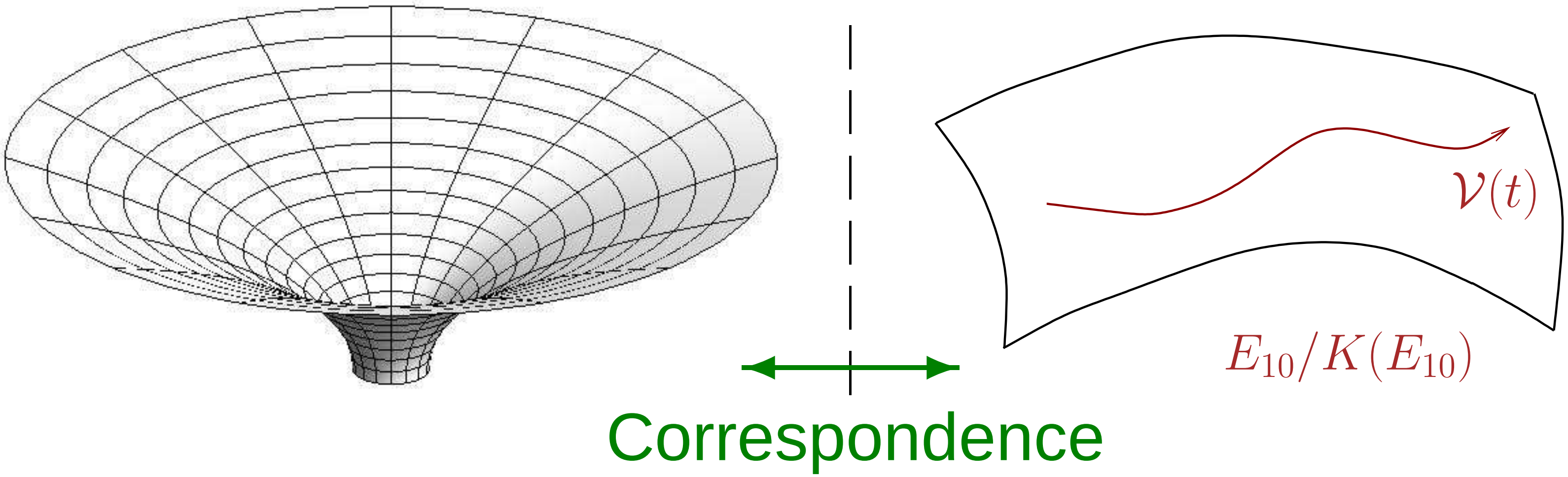
Interpretation (II)

- Absence of potential: can restrict to a well-defined Hilbert space with positive definite metric.
- Complexity and notion of positive frequency
  $\Rightarrow$ Arrow of time? [Isham 1991; Barbour 1993]
- All wavefunctions vanish at the singularity!
- Remain oscillating and complex and cannot be continued analytically past the singularity.
- Vanishing wavefunctions on singular geometries are one possible boundary condition. [DeWitt 1967]
- No way of going through the singularity. No bounce.
- ‘Semi-classical’ states are expected to spread (quantum ergodicity). [Non-relativistic intuition]

Generalization (I)

Classical cosmological billiards led to the E_{10} conjecture.

$D = 11$ supergravity can be mapped to a constrained null geodesic motion on infinite-dimensional $E_{10}/K(E_{10})$ coset space. [Damour, Henneaux, HN 2002]



Symmetric space $E_{10}/K(E_{10})$ has $10 + \infty$ many directions.

Cartan subalgebra — positive step operators

Generalization (II)

Features of the conjectured E_{10} correspondence:

- Billiard corresponds to 10 Cartan subalgebra generators
- ∞ many step operators to remaining fields and spatial dependence. [Verified only at low 'levels' but for many different models] [see e.g. Kleinschmidt,HN, IJMPA(2006)1619]
- Extension to E_{10} expected to overcome ultra-locality: space dependence via *dual fields* $\partial_x \varphi \sim \partial_t \tilde{\varphi}$ (but non-linear, cf. Geroch group) — only *kinetic terms*.
- Thus: replace $|\Psi_{\text{full}}\rangle$ by wavefunction depending on full tower of E_{10} degrees of freedom $\Rightarrow$
- Space (de-)emergent via an algebraic mechanism?

Generalization (III)

$$\mathcal{H}_0 \to \mathcal{H} \equiv \mathcal{H}_0 + \sum_{\alpha \in \Delta_+(E_{10})} e^{-2\alpha(\beta)} \sum_{s=1}^{\text{mult}(\alpha)} \Pi^2_{\alpha,s}$$

is the unique (quadratic) E_{10} Casimir. Formally like free Klein–Gordon; positive norm could remain consistent?

For the full theory there are more constraints than the Hamiltonian constraint $\mathcal{H}\Psi = 0$: diffeo, Gauss, etc.

- Global E_{10} symmetry provides ∞ conserved charges $\mathcal{J}$
- Evidence that constraints can be written as bilinears *à la Sugawara* $\mathfrak{L} \sim \mathcal{J}\mathcal{J}$. [Damour, Kleinschmidt, HN 2007; 2009]
- $\mathcal{H} \geq \mathcal{H}_0 \;\Rightarrow$ wave function still vanishes and is complex oscillatory for $\rho \to \infty$.

Aim: Quantize full geodesic model. $E_{10}(\mathbb{Z})$ [Ganor 1999]?

Supersymmetric extension (I)

$D = 11$ supergravity gravitino ψ_μ can be added to billiard analysis via $K(E_{10})$ representation. Work in supersymmetry gauge [Damour, Kleinschmidt,HN; de Buyl,Henneaux,Paulot 2005]

$$\psi_t = \Gamma_t \sum_{a=1}^{10} \Gamma^a \psi_a$$

Classically, separate billiard motion [Damour, Hillmann 2009]. Best in variable ($\Gamma_* = \Gamma^1 \cdots \Gamma^{10}$)

$$\varphi^a = g^{1/4} \Gamma_* \Gamma^a \psi^a \qquad \text{(no sum on } a\text{)}$$

Canonical Dirac bracket: $\left\{ \varphi^a_\alpha, \varphi^b_\beta \right\} = -iG^{ab}\delta_{\alpha\beta}$

Supersymmetric extension (II)

Quantize Clifford algebra using canonical anticommutators over a 2^{160}-dimensional Fock space vacuum $|\Omega\rangle$.

Have to implement supersymmetry constraint in quantum theory

$$\mathcal{S}_\alpha = i \sum_{a=1}^{10} \pi_a \varphi_\alpha^a \qquad (\alpha = 1, \dots, 32)$$

It obeys: $\{\mathcal{S}_\alpha, \mathcal{S}_\beta\} = \delta_{\alpha\beta}\mathcal{H}$ [Teitelboim 1977]

For quantum constraint choose 16 annihilation operators $\mathcal{S}_A$.

The state $|\Psi\rangle = \prod_{A=1}^{16} \mathcal{S}_A^\dagger \left(\Phi(\rho, z)|\Omega\rangle\right)$

solves the constraint iff $\Phi(\rho, z)$ solves the WDW equation.

Summary and outlook

Done:

- Quantum cosmological billiards wavefunctions involve automorphic forms of $PSL_2(\mathtt{O})$ for M theory.
- Extendable to supersymmetric case.

To do:

- Construct wavefunctions? Behaviour of wavepackets?
- Vanishing Wavefunctions $\Rightarrow$ Singularity resolution?
- More variables $\Rightarrow$ Constraints/Observables?
- An element of non-computabitility for $T \to 0$?

Thank you for your attention!