S-duality
&
Fivebrane Instantons

Sergei Alexandrov

Laboratoire Charles Coulomb
Montpellier

S.A., S.Banerjee arXiv:1405.0291

arXiv:1403.1265

Calabi-Yau compactifications

The goal: to find the complete non-perturbative effective action in 4d for type II string theory compactified on arbitrary CY

Type II string theory

compactification on a Calabi-Yau

$\mathcal{N}$=2 supergravity in 4d coupled to matter

- supergravity multiplet (metric)
- vector multiplets (gauge fields & scalars)
- hypermultiplets (only scalars)

The low-energy effective action is determined by the metric on the *moduli space* parameterized by scalars of *vector* and *hypermultiplets*

Moduli space

$$\mathcal{M}_{VM} \times \mathcal{M}_{HM}$$

- $\mathcal{M}_{VM}$ – *special Kähler* (given by $F(z)$)
- $F(z)$ is classically exact (no corrections in string coupling g_s)

known

- $\mathcal{M}_{HM}$ – *quaternion-Kähler*
- receives all types of g_s-corrections

unknown

The (concrete) goal: to find the non-perturbative geometry of $\mathcal{M}_{HM}$

Quantum corrections

Tree level HM metric

includes α'-corrections (perturbative + worldsheet instantons)

In the image of the c-map (captured by the prepotential $F(z)$)

+

g_s - corrections

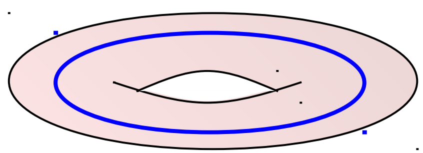
perturbative corrections

one-loop correction controlled by χ_X

Antoniadis,Minasian,Theisen,Vanhove '03, Robles-Llana,Saueressig,Vandoren '06,

S.A. '07

non-perturbative corrections



Instantons – (Euclidean) world volumes wrapping non-trivial cycles of CY

- D-brane instantons

$$e^{-2\pi|Z_\gamma|/g_s - 2\pi \mathrm{i}(q_\Lambda \zeta^\Lambda - p^\Lambda \tilde{\zeta}_\Lambda)}$$

S.A.,Pioline,Saueressig,Vandoren '08, S.A. '09 (in type IIA picture)

- NS5-brane instantons

$$e^{-2\pi|k|\mathcal{V}/g_s^2 - \mathrm{i}\pi k\sigma}$$

remains to find

Quantum corrections

Tree level HM metric

includes α'-corrections
(perturbative +
worldsheet instantons)

In the image of the c-map
(captured by the prepotential $F(z)$)

+

g_s - corrections

perturbative corrections

non-perturbative corrections

Instantons –
(Euclidean) world volumes
wrapping non-trivial cycles of CY

- D-brane instantons

$$e^{-2\pi|Z_\gamma|/g_s - 2\pi\mathrm{i}(q_\Lambda\zeta^\Lambda - p^\Lambda\tilde{\zeta}_\Lambda)}$$

S.A.,Pioline,Saueressig,Vandoren '08,
S.A. '09 (in type IIA picture)

- NS5-brane instantons

$$e^{-2\pi|k|\mathcal{V}/g_s^2 - \mathrm{i}\pi k\sigma}$$

The idea: Type IIA Type IIB

d_{cycle}	Type IIA		d_{cycle}	Type IIB
			0	D(-1)
1	×		2	D1
3	D2	mirror ⟷ symmetry	4	D3
5	×		6	$\boxed{\text{D5}}$

S-duality

One-instanton approximation — S.A.,Persson,Pioline '10

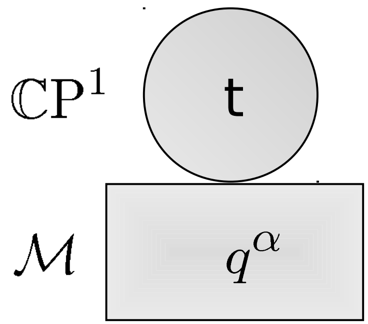
Twistor approach

The idea: one should work at the level of the twistor space

Quaternionic structure:

quaternion algebra of *almost* complex structures
$J^i J^j = \varepsilon^{ijk} J^k - \delta^{ij}$

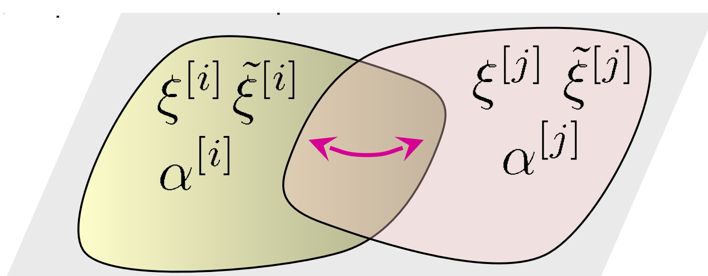
Twistor space



- $\mathcal{Z}_\mathcal{M}$ is a *Kähler manifold*
- carries *holomorphic contact structure* $\mathcal{X} \equiv \mathrm{d}\alpha + \xi^\Lambda \mathrm{d}\tilde{\xi}_\Lambda$
- symmetries of $\mathcal{M}$ can be lifted to *holomorphic* symmetries of $\mathcal{Z}_\mathcal{M}$

Holomorphicity

The geometry is determined by *contact transformations* between sets of Darboux coordinates



$$\mathcal{X}^{[i]} \mapsto \mathcal{X}^{[j]} = \lambda \mathcal{X}^{[i]}$$

generated by
holomorphic functions
$h^{[ij]}(\xi_{[i]}, \tilde{\xi}^{[i]}, \alpha^{[i]})$

Contact Hamiltonians and instanton corrections

gluing conditions

$$\begin{pmatrix} \xi^\Lambda_{[j]} \\ \tilde\xi^{[j]}_\Lambda \\ \alpha^{[j]} \end{pmatrix} = e^{\{h^{[ij]},\,\cdot\,\}} \begin{pmatrix} \xi^\Lambda_{[i]} \\ \tilde\xi^{[i]}_\Lambda \\ \alpha^{[i]} \end{pmatrix}$$

contact bracket

$$\{h,\xi^\Lambda\} = -\partial_{\tilde\xi_\Lambda} h + \xi^\Lambda \partial_\alpha h$$
$$\{h,\tilde\xi_\Lambda\} = \partial_{\xi^\Lambda} h$$
$$\{h,\alpha\} = h - \xi^\Lambda \partial_{\xi^\Lambda} h$$

integral equations for “twistor lines”

$$\xi^\Lambda_{[i]}(q^\alpha,t) = A^\Lambda + t^{-1}Y^\Lambda - t\bar Y^\Lambda + \frac{1}{2}\sum_j \oint_{C_j} \frac{\mathrm{d}t'}{2\pi\mathrm{i}t'}\,\frac{t'+t}{t'-t}\left[\left(e^{\{h^{[ij]},\,\cdot\,\}}-1\right)\xi^\Lambda_{[i]}(t')\right]$$
$$\tilde\xi^{[i]}_\Lambda(q^\alpha,t) = \cdots$$

metric on $\mathcal{M}$

It is sufficient to find holomorphic functions corresponding to quantum corrections

D2-instantons in type IIA

$$\boxed{h^{[\gamma]}(\xi,\tilde\xi) = \frac{\Omega(\gamma)}{(2\pi)^2}\,\mathrm{Li}_2\left(e^{2\pi\mathrm{i}(q_\Lambda\xi^\Lambda - p^\Lambda\tilde\xi_\Lambda)}\right)}$$

$\gamma = (q_\Lambda, p^\Lambda)$ — D-brane charge

$\Omega(\gamma)$ — generalized DT invariants

$$\mathrm{Li}_2(x) = \sum_{n=1}^\infty \frac{x^n}{n^2}$$

S.A.,Pioline,Saueressig, Vandoren ’08, S.A. ’09

S.A.,Banerjee '14

S-duality in twistor space

The idea: to add all images of the D-instanton contributions under S-duality with a non-vanishing 5-brane charge

One must understand how S-duality is realized on twistor space and which constraints it imposes on the twistorial construction

Non-linear holomorphic representation of SL(2,Z) –duality group on $\mathcal{Z}_{\mathcal{M}}$

$$\xi^0 \mapsto \frac{a\xi^0+b}{c\xi^0+d} \qquad \xi^a \mapsto \frac{\xi^a}{c\xi^0+d}$$

$$\tilde\xi_a \mapsto \tilde\xi_a + \frac{c}{2(c\xi^0+d)}\kappa_{abc}\xi^b\xi^c$$

$$\begin{pmatrix}\tilde\xi_0\\ \alpha\end{pmatrix} \mapsto \begin{pmatrix} d & -c\\ -b & a\end{pmatrix}\begin{pmatrix}\tilde\xi_0\\ \alpha\end{pmatrix} + \begin{matrix}\text{non-linear}\\ \text{terms}\end{matrix}$$

$$\mathcal{X} \mapsto \frac{\mathcal{X}}{c\xi^0+d}$$

contact transformation

Transformation property of the contact bracket

$$\varrho\cdot\{h,f\} = \{\lambda^{-1}\varrho\cdot h, \varrho\cdot f\}$$

$$\varrho\cdot\mathcal{X} = \lambda\mathcal{X}$$

Condition for $\mathcal{Z}_{\mathcal{M}}$ to carry an isometric action of SL(2,Z)

$$h^{[i]}_{m,n} \mapsto \frac{h^{[i]}_{m',n'}}{c\xi^0+d} + \text{reg.}$$

$$\begin{pmatrix} m'\\ n'\end{pmatrix} = \begin{pmatrix} a & c\\ b & d\end{pmatrix}\begin{pmatrix} m\\ n\end{pmatrix}$$

Fivebrane instantons

The input: $h^{[\gamma]}_{0,p^0} = h^{[\gamma]}$ $\quad p^0 \neq 0$ — D5-brane charge

Apply SL(2,Z) transformation $\quad g = \begin{pmatrix} a & b \\ k/p^0 & p/p^0 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \qquad p^0 = \gcd(k,p)$

NS5-brane charge

Compute fivebrane contact Hamiltonians

$$h^{[\gamma]}_{k,p} = (p^0)^{-1}(k\xi^0 + p)\; g \cdot h^{[\gamma]}$$

$$n^a = \frac{p^a}{k},\quad n^0 = \frac{p}{k}$$

$\hat{q}_\Lambda$ — invariant charges

$$\boxed{h^{[\gamma]}_{k,p} = \frac{k\bar{\Omega}(\gamma)}{4\pi^2 p^0}(\xi^0+n^0)\exp\left[-k\alpha + kn^\Lambda\tilde{\xi}_\Lambda - kF^{\mathrm{cl}}(\xi+n) + \frac{p^0(p^0\hat{q}_0 - k\hat{q}_a(\xi^a+n^a))}{k^2(\xi^0+n^0)}\right]}$$

Encodes fivebrane instanton corrections to *all* orders of the instanton expansion

- One can evaluate the action on Darboux coordinates and write down the integral equations on twistor lines
- The contact structure is invariant under full U-duality group

Open issues

- Manifestly S-duality invariant description of D3-instantons

Quantum corrections in type IIB:

closely related to *(0,4) elliptic genus* and *mock modular forms*

S.A.,Manschot,Pioline ‘12

• α'-corrections:	pert.	w.s.inst.		
• pert. g_s-corrections:	1ℓ			
• instanton corrections:	D(-1)	D1	D3	D5 NS5

- NS5-brane instantons in the Type IIA picture

Can the integrable structure of D-instantons in Type IIA be extended to include NS5-brane corrections?

Inclusion of NS5-branes $\overset{?}{=}$ **quantum deformation** ⟹ **quantum dilog?**

- Resolution of one-loop singularity
- Resummation of divergent series over brane charges

(expected to be regularized by NS5-branes)

A-model topological wave function in the real polarization

Relation to the topological string wave function

$$\sum_{p,\gamma} h_{1,p}^{[\gamma]} \sim \sum_{n^\Lambda} e^{-\alpha+n^\Lambda\tilde\xi_\Lambda}\,\Psi_{\mathbb{R}}^{\rm top}(\xi^\Lambda+n^\Lambda)$$