Open-string tree amplitudes and the Drinfeld associator

Johannes Brödel

ETH Zürich

based on joint work with Oliver Schlotterer, Stephan Stieberger and Tomohide Terasoma

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Introduction I

- Calculation of scattering amplitudes via Feynman graphs: cumbersome. Instead: symmetries (hidden) triggered revival of ***S-matrix approach.***
- Closed or recursive forms for scattering amplitudes are available for highly symmetric theories (and/or subsectors thereof):
  - Parke-Taylor form for tree-level MHV gluon scattering amplitudes [Parke Taylor]
  - recursive construction of all tree-level scattering amplitudes in $\mathcal{N}=4$ super-Yang–Mills (sYM) theory [Drummond, Henn]
  - closed form for gravity tree-level scattering amplitudes [Berends, Giele Kuijf]
- *Loop amplitudes in $\mathcal{N}=4$ sYM*: no general recursive results available *However, S-matrix approach successful:*
  - multiple polylogarithms, "symbol" [Bern, Dixon, Kosower Roiban, Spradlin, Vergu, Volovich]
  - kinematical limits (soft & collinear, multi-Regge kinematics) [Dixon, Drummond Duhr, Pennington]

Introduction II

- Integrability-based methods: employ OPE to determine hexagonal Wilson-loops from the GKP string/flux-tube excitations. All-loop results for specific subsectors/kinematical limits. [Basso, Sever Vieira][Gubser, Klebanov Polyakov]

This talk:

- construct all tree-level amplitudes in open string theory recursively.
- relate the α'-expansion of the $(N-1)$-point and N-point amplitude employing the *Drinfeld associator*.

Why string-theory tree amplitudes?

$\mathcal{N} = 4$ sYM
tree amplitudes
recursive form available

→

$\mathcal{N} = 4$ sYM
loop amplitudes
many results available, no closed form

↓ ↓

open string theory
tree amplitudes
Recursive form? Indeed!

→

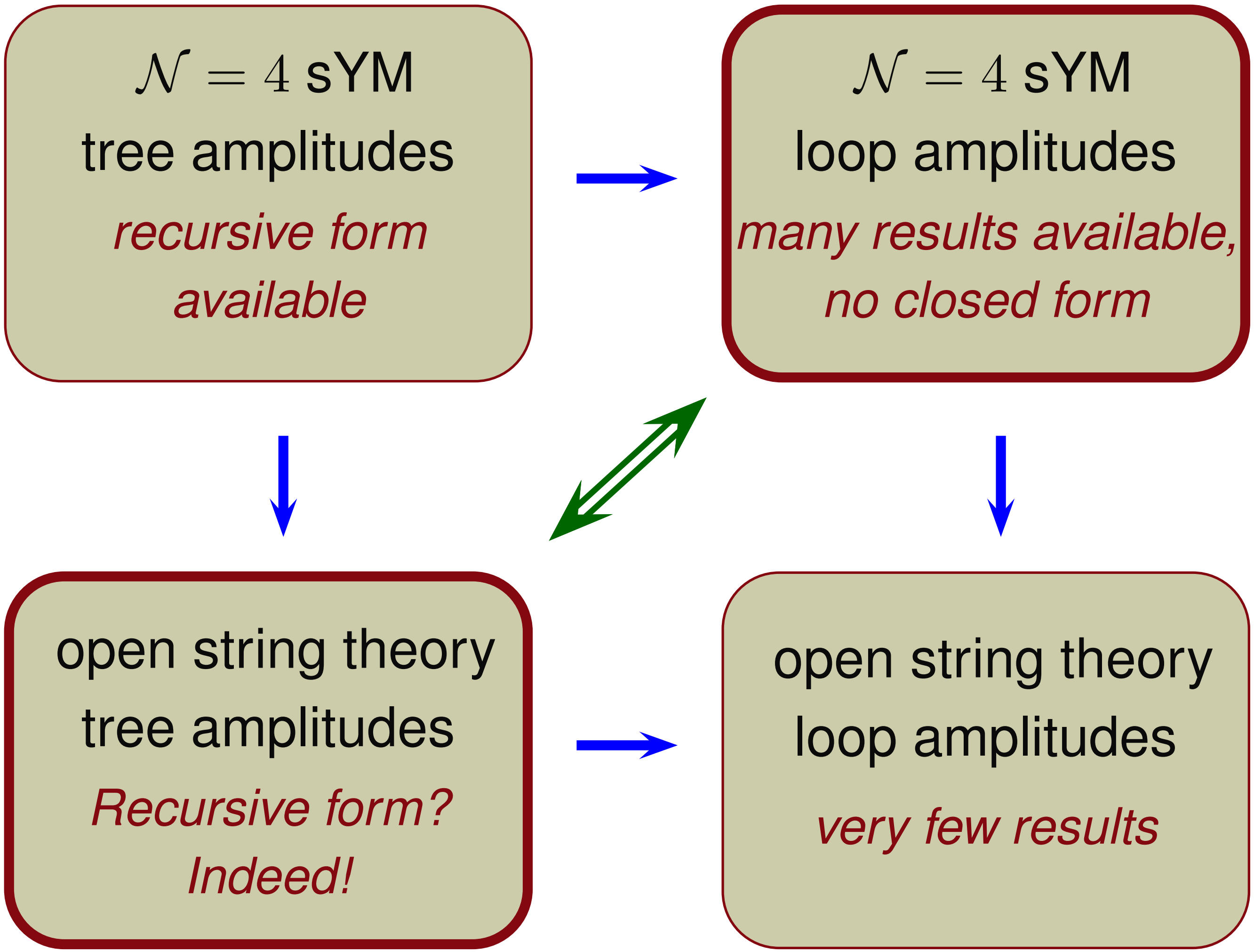
open string theory
loop amplitudes
very few results

soft/collinear limits, MRK
Grassmanian,
BDS / remainder function,
Flux-tube excitations,
Polylogarithms

Worldsheet integrals
soft/collinear limits
Drinfeld associator

Higher-genus integrals

Why string-theory tree amplitudes?



soft/collinear limits, MRK
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Worldsheet integrals
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Higher-genus integrals

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low-multiplicity loop amplitudes in $\mathcal{N}=4$ sYM & open-string trees:

multiple polylogarithms

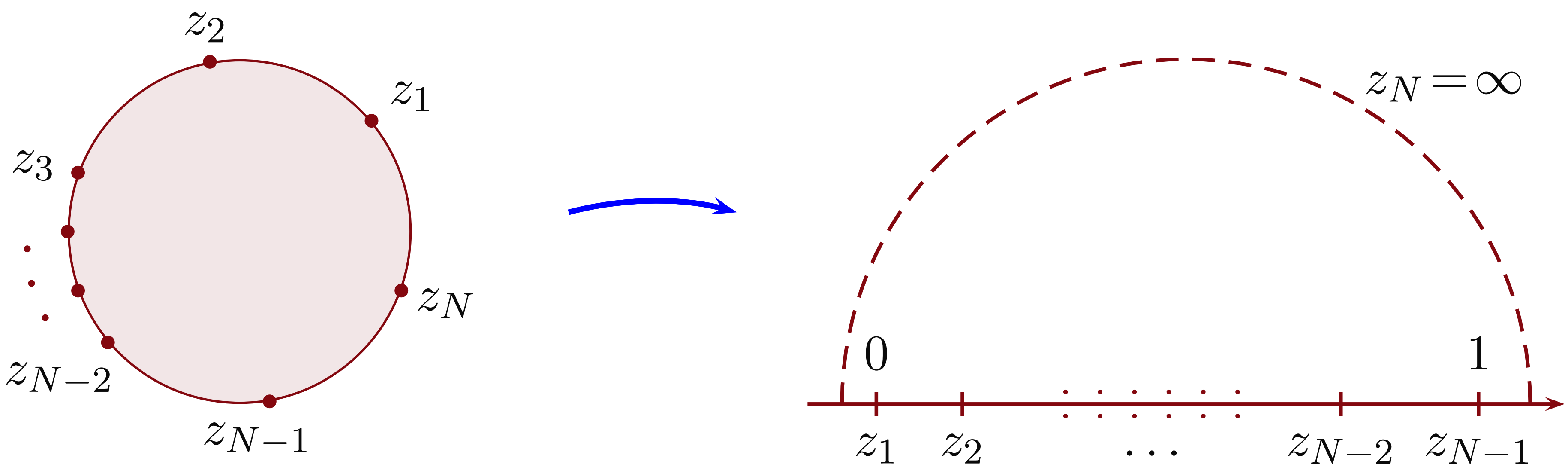
Open string-theory tree amplitudes...

- are a perfect testing ground for loop calculations in $\mathcal{N} = 4$ sYM theory:
  - no divergences
  - same building blocks: iterated integrals.
  - final result simple: multiple ζ-values.
- string loops and higher-point $\mathcal{N} = 4$ amplitudes: ***elliptic polylogarithms***
- leading orders in α' can be used to explore UV-properties of supergravities using the Kawai-Lewellen-Tye relations. [Kawai, Lewellen Tye]
- field-theory properties, such as the *Kleiss-Kuijf* and *Bern-Carrasco-Johansson* relations, can be easily derived from algebraic properties of worldsheet integrals. [Kleiss Kuijf][Bern, Carrasco Johansson][Bjerrum-Bohr, Damgaard Vanhove][Stieberger]

after all, string theory is a heavily constrained theory

$\Rightarrow$ ***should produce simple answers***

N-point tree-level open-string amplitude:



General structure:

[Mafra, Schlotterer, Stieberger]

$$\mathbf{A}^{\text{open}}_{\text{string}} = \boldsymbol{F}.\mathbf{A}_{\text{YM}}$$

String corrections:

well known, state-dependent

- functions of dimensionless Mandelstam variables: $s_{ij} = \alpha'(k_i + k_j)^2$
- no dependence on external states - just kinematical correction

4-point amplitude

[Veneziano]

$$A^{\text{open}}_{\text{string}}(1,2,3,4) = F^{(2)} A_{\text{YM}}(1,2,3,4)$$

String correction can be expanded in α' (uniform transcendentality):

$$F^{(2)} = \frac{\Gamma(1+s_{12})\Gamma(1+s_{23})}{\Gamma(1+s_{12}+s_{23})}$$

$$= 1 - \zeta_2\, s_{12}s_{23} + \zeta_3\, s_{12}s_{23}(s_{12}+s_{23}) - \zeta_4\, s_{12}s_{23}\left(s_{12}^2 + \frac{1}{4}s_{12}s_{23} + s_{23}^2\right)$$

$$+ \zeta_5\, s_{12}s_{23}\,\left(s_{12}^3 + 2s_{12}^2 s_{23} + 2s_{12}s_{23}^2 + s_{23}^3\right) - \zeta_2\zeta_3\, s_{12}^2 s_{23}^2\,(s_{12}+s_{23}) + \cdots$$

Multiple Zeta values:

$$\zeta_{n_1,\ldots,n_r} = \sum_{0<k_1<\cdots<k_r} \frac{1}{k_1^{n_1}\cdots k_r^{n_r}}, \quad n_l \geq 1, \quad n_r \geq 2\,, \qquad \text{weight: } w = \sum_{i=1}^{r} n_i$$

$\Rightarrow$ *only **single** ζ's in the four-point amplitude.*

5-point amplitude

[Mafra, Schlotterer Stieberger]

$$A^{\text{open}}_{\text{string}}(1,2,3,4,5) = F^{(23)} A_{\text{YM}}(1,2,3,4,5) + F^{(32)} A_{\text{YM}}(1,3,2,4,5)$$

Expansion in terms of multiple zeta values:

[Stieberger Taylor][Barreiro Medina]

$$\begin{aligned} F^{(23)} &= 1 - \zeta_2(s_{12}s_{23} + s_{12}s_{24} + s_{12}s_{34} + s_{13}s_{34} + s_{23}s_{34}) \\ &\quad + \zeta_3(s_{12}^2 s_{23} + s_{12}s_{23}^2 + s_{12}^2 s_{24} + 2s_{12}s_{23}s_{24} + s_{12}s_{24}^2 + \cdots) + \cdots \\ &\quad + \zeta_{3,5}(\ldots) + \cdots \end{aligned}$$

$$F^{(32)} = \zeta_2\, s_{13}s_{24} - \zeta_3(s_{13}^2 s_{24} + 2s_{12}s_{13}s_{24} + \cdots) + \cdots + \zeta_{3,5}(\ldots) + \cdots$$

6-point amplitude

$$\begin{aligned} A^{\text{open}}_{\text{string}}(1,2,3,4,5,6) &= F^{(234)} A_{\text{YM}}(1,2,3,4,5,6) + F^{(243)} A_{\text{YM}}(1,2,4,3,5,6) \\ &\quad + F^{(324)} A_{\text{YM}}(1,3,2,4,5,6) + F^{(342)} A_{\text{YM}}(1,3,4,2,5,6) \\ &\quad + F^{(423)} A_{\text{YM}}(1,4,2,3,5,6) + F^{(432)} A_{\text{YM}}(1,4,3,2,5,6) \end{aligned}$$

Structure of the string tree-level amplitude

Tree-level open string amplitude:

$$\mathbf{A}_{\text{open}}^{\text{string}} = \boldsymbol{F}.\mathbf{A}_{\text{YM}}$$

[Mafra, Schlotterer Stieberger]

Explicitely:

$$\begin{pmatrix} A_{\text{open}}(1,\Pi_1,N-1,N) \\ \vdots \\ A_{\text{open}}(1,\Pi_{(N-3)!},N-1,N) \end{pmatrix} = \begin{pmatrix} F_{\Pi_1}^{\sigma_1} & \cdots & F_{\Pi_1}^{\sigma_{(N-3)!}} \\ \vdots & \ddots & \vdots \\ F_{\Pi_{(N-3)!}}^{\sigma_1} & \cdots & F_{\Pi_{(N-3)!}}^{\sigma_{(N-3)!}} \end{pmatrix} \begin{pmatrix} A_{\text{YM}}(1,\sigma_1,N-1,N) \\ \vdots \\ A_{\text{YM}}(1,\sigma_{(N-3)!},N-1,N) \end{pmatrix}$$

where Π_i and $\sigma_i \in \mathcal{P}(\{2,3,\ldots,N-2\})$.

- $\mathbf{A}_{\text{open}}^{\text{string}}$, $\mathbf{A}_{\text{YM}}$: vectors of $(N-3)!$ basis elements of *color-ordered amplitudes* in open string theory and Yang-Mills theory [Bjerrum-Bohr Damgaard, Vanhove][Stieberger] [Bern, Carrasco Johansson]
- ***string corrections*** $\boldsymbol{F}$: $(N-3)!\times(N-3)!$-matrix

Structure of the string tree-level amplitude

Tree-level open string amplitude:

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- ***string corrections*** $\boldsymbol{F}$: $(N-3)!\times(N-3)!$-matrix
- **redundant information in $\boldsymbol{F}$:**

  the first line is sufficient to obtain all others by a suitable relabelling

  $\Rightarrow$ focus on permutation $\Pi_1 = 2,3,\ldots,N-2$ below

  $\Rightarrow$ consider only $(N-3)!$ objects F^σ in the first line of $\boldsymbol{F}$.

String corrections

Why stick with the matrix form? Expand $\boldsymbol{F}$ in α': [Schlotterer Stieberger]

$$\boldsymbol{F} = \mathbb{1}_{(N-3)!\times(N-3)!} + \zeta_2 P_2 + \zeta_3 M_3 + \zeta_2^2 P_4 + \zeta_2\zeta_3 P_2 M_3 + \zeta_5 M_5$$
$$+ \zeta_2^3 P_6 + \frac{1}{2}\zeta_3^2 M_3 M_3 + \zeta_7 M_7 + \zeta_2\zeta_5 P_2 M_5 + \zeta_2^2\zeta_3 P_4 M_3$$
$$+ \zeta_2^4 P_8 + \zeta_3\zeta_5 M_5 M_3 + \frac{1}{2}\zeta_2\zeta_3^2 P_2 M_3 M_3 + \frac{1}{5}\zeta_{3,5}[M_5, M_3]$$
$$+ \ldots + \left(9\zeta_2\zeta_9 + \frac{6}{25}\zeta_2^2\zeta_7 - \frac{4}{35}\zeta_2^3\zeta_5 + \frac{1}{5}\zeta_{3,3,5}\right)[M_3,[M_5,M_3]]$$
$$+ \ldots + \zeta_{3,5}\zeta_{3,7}\frac{208926}{894845}[M_3[M_3[M_7,M_5]]] + \ldots$$

- each matrix M_w and P_w contains entries of weight w exclusively
  $\Rightarrow$ degree-w polynomials in Mandelstam variables ($s_{ij\ldots} = \alpha'(k_i + k_j + \cdots)^2$)
- ***5-point amplitude:***

$$\boldsymbol{F}|_{w=0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} -(s_{13}+s_{23})s_{34} - s_{12}(s_{234}) & s_{13}s_{24} \\ s_{12}s_{34} & -(s_{12}+s_{23})s_{24} - s_{13}(s_{234}) \end{pmatrix}$$

String corrections

- rewriting in terms of non-commutative words available [Brown][Schlotterer Stieberger]
  $\Rightarrow$ *removes the unwieldy coefficients*
  $\Rightarrow$ *structure completely determined + known*

$$
\begin{aligned}
\boldsymbol{F} \to & \left(1 + f_2 P_2 + f_2^2 P_4 + f_2^3 P_6 + f_2^4 P_8 + f_2^5 P_{10} + f_2^6 P_{12} + \ldots\right) \\
& \times \left(1 + f_3\, M_3 + f_5\, M_5 + f_3^2\, M_3^2 + f_7\, M_7 + f_3 f_5\, M_5 M_3 + f_5 f_3\, M_3 M_5 \right. \\
& \quad + f_9\, M_9 + f_3^3\, M_3^3 + f_5^2\, M_5^2 + f_3 f_7\, M_7 M_3 + f_7 f_3\, M_3 M_7 + f_{11}\, M_{11} \\
& \quad + f_3^2 f_5\, M_5 M_3^2 + f_3 f_5 f_3\, M_3 M_5 M_3 + f_5 f_3^2\, M_3^2 M_5 + f_3^4\, M_3^4 \\
& \quad \left. + f_3 f_9\, M_9 M_3 + f_9 f_3\, M_3 M_9 + f_5 f_7\, M_7 M_5 + f_7 f_5\, M_5 M_7 + \ldots \right)
\end{aligned}
$$

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\begin{aligned}
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& \times (1 + f_3\ M_3 + f_5\ M_5 + f_3^2\ M_3^2 + f_7\ M_7 + f_3 f_5\ M_5 M_3 + f_5 f_3\ M_3 M_5 \\
& \quad + f_9\ M_9 + f_3^3\ M_3^3 + f_5^2\ M_5^2 + f_3 f_7\ M_7 M_3 + f_7 f_3\ M_3 M_7 + f_{11}\ M_{11} \\
& \quad + f_3^2 f_5\ M_5 M_3^2 + f_3 f_5 f_3\ M_3 M_5 M_3 + f_5 f_3^2\ M_3^2 M_5 + f_3^4\ M_3^4 \\
& \quad + f_3 f_9\ M_9 M_3 + f_9 f_3\ M_3 M_9 + f_5 f_7\ M_7 M_5 + f_7 f_5\ M_5 M_7 + \dots)
\end{aligned}
$$

Beautiful structure.

Missing:

Expansion of matrices $\boldsymbol{F}$ into M_w and P_w for all weights.

closed / recursive form?

Two very different methods

How to obtain the matrices M_w and P_w efficiently?

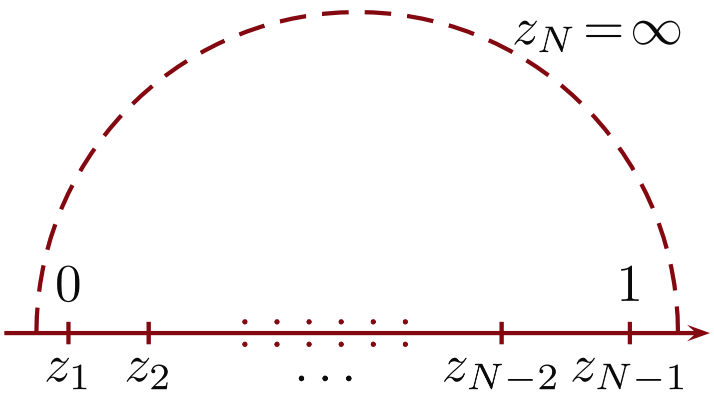
- ***Pedestrian***: formalize the calculation of F^{σ} from worldsheet integrals:
  - explore pole structure
  - employ polylogarithms to solve regular integrals

- ***Parachute***: use the *Drinfeld associator*

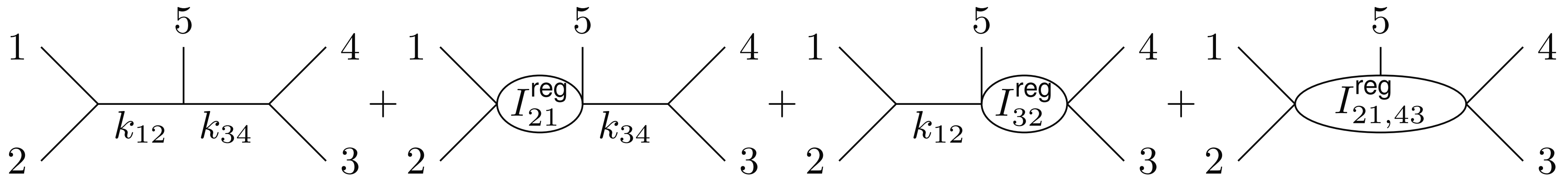
General form of the F's

$$F^{\sigma} = \prod_{i=2}^{N-2} \int_{\Pi} \mathrm{d}z_i \prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}} \, \sigma \left\{ \frac{s_{12}}{z_{12}} \left(\frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} \right) \cdots \left(\frac{s_{1,N-2}}{z_{1,N-2}} + \ldots + \frac{s_{N-3,N-2}}{z_{N-3,N-2}} \right) \right\}$$

$$= \prod_{i=2}^{N-2} \int_{\Pi} \mathrm{d}z_i \prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}} \, \sigma \left\{ \prod_{k=2}^{N-2} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \right\} .$$



Numerous poles $\Rightarrow$ can be expressed in terms of ***regular*** lower-point integrals:



$$F^{(23)} = 1 + s_{12}\, I^{\text{reg}}_{21}[s_{12}, s_{23}+s_{24}] + s_{34}\, I^{\text{reg}}_{21}[s_{34}, s_{13}+s_{23}] + s_{12}s_{34}\, I^{\text{reg}}_{21,43}$$

Functions F^{σ} can be expressed in terms of regular integrals I^{reg}.

Regular parts

Regular integrals I^{reg} $(z_1 = 0,\ z_{N-1} = 1,\ z_N = \infty)$

[Brödel, Schlotterer, Stieberger]

$$I^{\text{reg}}_{\{a_i\}} = \prod_{i=2}^{N-2} \int_0^{z_{i+1}} \frac{\mathrm{d}z_i}{z_i - a_i} \prod_{i<j}^{N-1} \underbrace{|z_{ij}|^{s_{ij}}}_{\text{expand...}} , \qquad a_i \in \{\, 0, z_{i+1}, z_{i+2}, \ldots, z_{N-2}, 1 \,\}$$

$$= \prod_{i=2}^{N-2} \int_0^{z_{i+1}} \frac{\mathrm{d}z_i}{z_i - a_i} \prod_{i<j}^{N-1} \sum_{n_{ij}=0}^{\infty} (s_{ij})^{n_{ij}} \underbrace{\frac{(\ln |z_{ij}|)^{n_{ij}}}{n_{ij}!}}_{\text{multiple polylogs}}$$

$$= \underbrace{\prod_{i=2}^{N-2} \int_0^{z_{i+1}} \frac{\mathrm{d}z_i}{z_i - a_i} \prod_{i<j}^{N-1} \sum_{n_{ij}=0}^{\infty} (s_{ij})^{n_{ij}} G(\{0, 1, z_l\}, z_k)}_{\text{integrate step by step to remove } z_l\text{'s from the argument of } G}$$

$$= \underbrace{\prod_{i<j}^{N-1} \sum_{n_{ij}=0}^{\infty} (s_{ij})^{n_{ij}} G(\{0, 1\}, 1)}_{\text{rewrite polylogs as multiple } \zeta\text{'s}}$$

Multiple polylogarithms and multiple zeta functions

$$\zeta_{n_1,\ldots,n_r} = \sum_{0<k_1<\cdots<k_r} \frac{1}{k_1^{n_1}\cdots k_r^{n_r}}$$

$$= (-1)^r G(\underbrace{0,0,\ldots,0,1}_{n_r},\ldots,\underbrace{0,0,\ldots,0,1}_{n_1};1) = \zeta_{(w)}$$

- multiple polylogs / multiple zeta values are divergent in general can be dealt with by *shuffle regularization* [Goncharov][Duhr]
- similar methods have been studied and used in many situations for example [Henn Huber][Dixon, Duhr Pennington][Brown Bogner][Duhr][Anastasiou, Duhr Dulat, Mistlberger]

Thus,

- rewriting the pole structure and using polylogs, any open-string tree-amplitude can - *in principle* - calculated at any order in α'
- bottleneck: extensive algebra, but efficient implementation buys several orders ofthe expansion in α'

Drinfeld associator

Knizhnik-Zamolodchikov (KZ) equation [Knizhnik Zamolodchikov]

$$\frac{\mathrm{d}\hat{\mathbf{F}}(z_0)}{\mathrm{d}z_0} = \left(\frac{e_0}{z_0} - \frac{e_1}{z_0 - 1}\right) \hat{\mathbf{F}}(z_0) \, .$$

- $z_0 \in \mathbb{C}\backslash\{0,1\}$, Lie-algebra generators e_0, e_1

Regularized boundary values

$$C_0 \equiv \lim_{z_0 \to 0} z_0^{-e_0} \hat{\mathbf{F}}(z_0) \, , \quad C_1 \equiv \lim_{z_0 \to 1} (1 - z_0)^{e_1} \hat{\mathbf{F}}(z_0)$$

are related by the *Drinfeld associator* Φ: [Drinfeld]

$$C_1 = \Phi(e_0, e_1) \, C_0 \, .$$

- C_0, C_1 and Φ are (real and single-valued) elements of the universal enveloping algebra of the Lie algebra generated by e_0 and e_1

Example:

Representation of the Drinfeld associator: [Le Murakami][Furusho][Drummond Ragoucy]

$$\Phi(e_0,e_1) = \sum_{w\in\{0,1\}} \tilde{w}[e_0,e_1]\zeta_{(w)}.$$

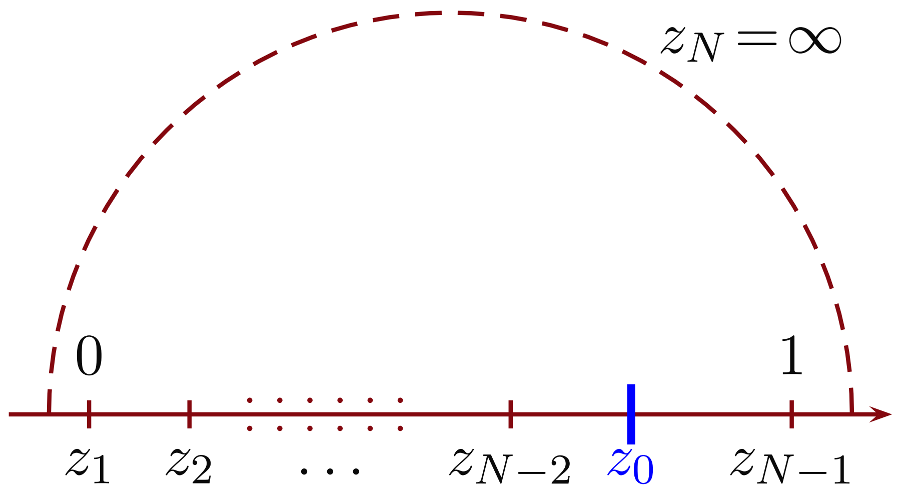
The Drinfeld associator generates the four-point amplitude. [Drummond Ragoucy]

$$\begin{aligned}\Phi(e_0,e_1) = \; & 1 + \zeta_{(0,0)}e_0.e_0 + \zeta_{(1,0)}e_0.e_1 + \zeta_{(0,1)}e_1.e_0 + \zeta_{(1,1)}e_1.e_1 + \\ & + \zeta_{(0,0,0)}e_0.e_0.e_0 + \zeta_{(1,0,0)}e_0.e_0.e_1 + \zeta_{(0,1,0)}e_0.e_1.e_0 + \zeta_{(1,1,0)}e_0.e_1.e_1 + \\ = \; & 1 + \zeta_2[e_0,e_1] + \zeta_3[e_0 - e_1,[e_0,e_1]] \\ & + \zeta_4([e_0,[e_0,[e_0,e_1]]] + \tfrac{1}{4}[e_1,[e_0,[e_1,e_0]]] \\ & - [e_1,[e_1,[e_1,e_0]]] + \tfrac{5}{4}[e_0,e_1]^2) + \ldots \quad .\end{aligned}$$

How is this construction related to open superstring tree amplitudes?

What is the role of z_0?

$$\frac{\mathrm{d}\hat{\mathbf{F}}(z_0)}{\mathrm{d}z_0} = \left(\frac{e_0}{z_0} - \frac{e_1}{z_0 - 1}\right)\hat{\mathbf{F}}(z_0)\,.$$



- z_0 is an auxiliary insertion point interpolating between the N-point and $(N-1)$-point amplitude

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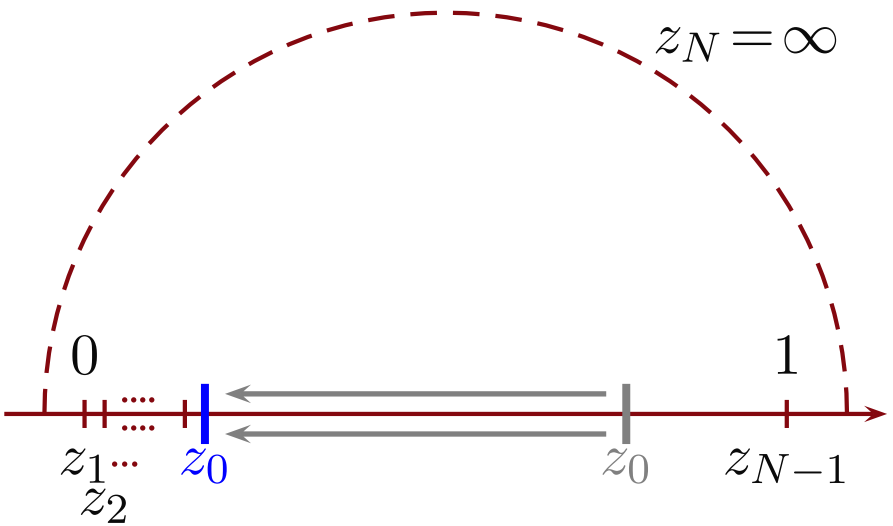
$$\frac{\mathrm{d}\hat{\mathbf{F}}(z_0)}{\mathrm{d}z_0} = \left(\frac{e_0}{z_0} - \frac{e_1}{z_0 - 1}\right)\hat{\mathbf{F}}(z_0), \qquad C_0 \equiv \lim_{z_0 \to 0} z_0^{-e_0}\hat{\mathbf{F}}(z_0)$$

$\frac{e_0}{z_0} \to C_0$



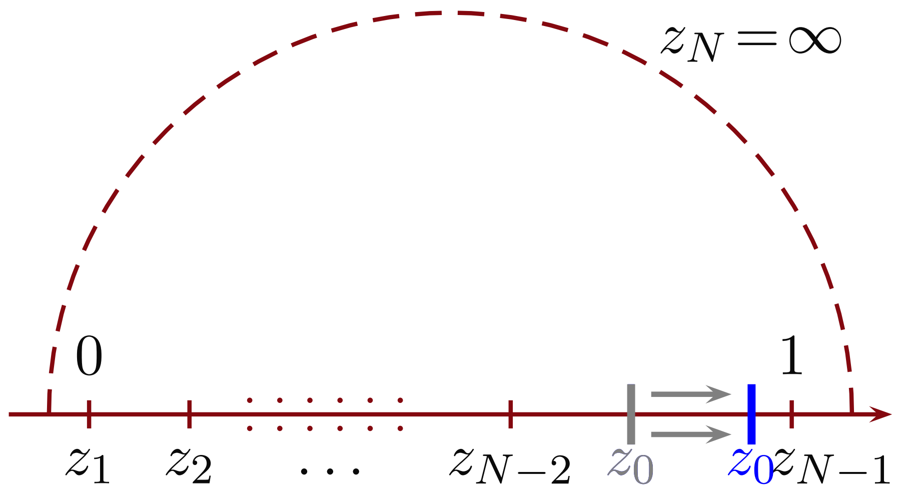
- in the limit $z_0 \to 0$, all insertion points are squeezed close to 0.
- from the integration region, the point 1 appears to be close to infinity.

$\Rightarrow$ situation is reminiscent of the $(N-1)$-point amplitude.

What is the role of z_0?

$$\frac{\mathrm{d}\hat{\mathbf{F}}(z_0)}{\mathrm{d}z_0} = \left(\frac{e_0}{z_0} - \frac{e_1}{z_0 - 1}\right)\hat{\mathbf{F}}(z_0), \qquad C_1 \equiv \lim_{z_0 \to 1} (1 - z_0)^{e_1}\hat{\mathbf{F}}(z_0)$$

$\to C_1$



- in the limit $z_0 \to 1$ one recovers the N-point situation

Open-string tree amplitude for any multiplicity

$$\frac{\mathrm{d}\hat{\mathbf{F}}(z_0)}{\mathrm{d}z_0} = \left(\frac{e_0}{z_0 - z_1} - \frac{e_1}{z_0 - z_{N-1}}\right) \hat{\mathbf{F}}(z_0)\,.$$

How to construct an auxiliary function $\hat{\mathrm{F}}(\mathbf{z_0})$ such that

- it depends on an auxiliary insertion point z_0 in the right way
- it contains the correct information for the N-point and $(N-1)$-point amplitude in C_1 and C_0 respectively?
- one can derive suitable matrices e_0 and e_1 that the KZ-equation is satisfied?

What about dimensions?

- *previously:* all string-theory information contained in the first line of an $(N-3)! \times (N-3)!$-matrix.
- *now:* an additional auxiliary position z_0 more. Objects of dim. $(N-2)!\,$.

auxiliary vector $\hat{\mathbf{F}}$

$$F^{\sigma} = \prod_{i=2}^{N-2} \int_0^{z_{i+1}} \mathrm{d}z_i \prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}} \; \sigma\left\{ \prod_{k=2}^{N-2} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \right\}$$

$$\hat{F}^{\sigma}_{\nu}(z_0) = \int_0^{z_0} \mathrm{d}z_{N-2} \prod_{i=2}^{N-3} \int_0^{z_{i+1}} \mathrm{d}z_i \prod_{i<j}^{N-1} |z_{ij}|^{s_{ij}} \prod_{k=2}^{N-2} (z_{0k})^{s_{0k}} \; \sigma\left\{ \prod_{k=2}^{\nu} \sum_{j=1}^{k-1} \frac{s_{jk}}{z_{jk}} \prod_{m=\nu+1}^{N-2} \sum_{n=m+1}^{N-1} \frac{s_{mn}}{z_{mn}} \right\}$$

$$\hat{\mathbf{F}} = \begin{pmatrix} \begin{pmatrix} \hat{F}^{\sigma_1}_{N-2} \\ \hat{F}^{\sigma_2}_{N-2} \\ \vdots \\ \hat{F}^{\sigma_{(N-3)!}}_{N-2} \end{pmatrix} \\ \vdots \\ \vdots \\ \begin{pmatrix} \hat{F}^{\sigma_1}_{1} \\ \hat{F}^{\sigma_2}_{1} \\ \vdots \\ \hat{F}^{\sigma_{(N-3)!}}_{1} \end{pmatrix} \end{pmatrix}$$

$\nu \in \{N-2, \ldots, 2\}$

$\hat{\mathbf{F}}$: vector of length $(N-2)!$

Plug $\hat{\mathbf{F}}$ into the KZ equation, solve it and obtain the $(N-2)! \times (N-2)!$-matrices e_0 and e_1:

$$\frac{\mathrm{d}\hat{\mathbf{F}}(z_0)}{\mathrm{d}z_0} = \left(\frac{e_0}{z_0 - z_1} - \frac{e_1}{z_0 - z_{N-1}} \right) \hat{\mathbf{F}}(z_0) \,.$$

- after applying the derivative, use partial fraction and integration by parts in order to obtain the right-hand-side
- matrices are *linear* in Mandelstam variables s_{ij} and thus in α'

What remains?

Need to show that regularized boundary values C_0 and C_1 derived from $\hat{\mathbf{F}}(\mathbf{z_0})$ indeed contain the desired information.

Boundary value C_0

$$C_0 \equiv \lim_{z_0 \to 0} z_0^{-e_0} \hat{\mathbf{F}}$$

- Consider the first subvector ($(N-3)!$ components):

$$\hat{F}^{\sigma}_{N-2}(z_0 \to 0) = z_0^{s_{\max}} F^{\sigma}\Big|_{s_{i,N-1}=s_{0i}} + \mathcal{O}(s_{0i}) \,,$$

with eigenvalue of e_0: [Terasoma]

$$s_{\max} = s_{12\ldots N-2} + \sum_{j=2}^{N-2} s_{0j} \,.$$

- Other components are at least $\mathcal{O}(z_0)$, thus suppressed. Resulting vector:

$$\left(z_0^{s_{\max}} F^{\sigma}, \mathbf{0}_{(N-3)(N-3)!}\right) .$$

- Soft limit $k_{N-1} \to 0$ is equivalent to setting $s_{0i} = s_{i,N-1} = 0$ (remove the kinematical contribution from the "*second point at infinity*")

$$C_0 = \left(F^{\sigma}\Big|_{k_{N-1}=0}, \mathbf{0}_{(N-3)(N-3)!}\right) .$$

Boundary value C_1

$$C_1 \equiv \lim_{z_0 \to 1} (1 - z_0)^{e_1} \hat{\mathbf{F}}$$

- Extract again the first $(N-3)!$ components:
- schematic form of the first $(N-3)!$ rows

$$(1 - z_0)^{e_1} = \begin{pmatrix} \mathbf{1}_{(N-3)! \times (N-3)!} & \mathbf{0}_{(N-3)! \times (N-3)(N-3)!} \\ \vdots & \vdots \end{pmatrix}$$

we can neglect all components of $\hat{\mathbf{F}}(z_0 \to 1)$ except

$$\hat{F}^{\sigma}_{N-2}(z_0 \to 1) = F^{\sigma} + \mathcal{O}(s_{0i}) \ .$$

- Setting again $s_{0i} = 0$ leads to

$$C_1 = (F^{\sigma}, \ldots) \ .$$

Example 1

Four-point amplitude

$$F^{(2)} = \int_0^1 \mathrm{d}z_2 \, |z_{12}|^{s_{12}} |z_{23}|^{s_{23}} \frac{s_{12}}{z_{12}} = \frac{\Gamma(1+s_{12})\Gamma(1+s_{23})}{\Gamma(1+s_{12}+s_{23})} \, .$$

Auxiliary vector contains two subvectors of length one:

$$\hat{\mathbf{F}} = \begin{pmatrix} \hat{F}_2^{(2)} \\ \hat{F}_1^{(2)} \end{pmatrix} = \int_0^{z_0} \mathrm{d}z_2 \, |z_{12}|^{s_{12}} |z_{23}|^{s_{23}} z_{02}^{s_{02}} \begin{pmatrix} s_{12}/z_{12} \\ s_{23}/z_{23} \end{pmatrix} \, .$$

KZ equation

$$\frac{\mathrm{d}}{\mathrm{d}z_0} \begin{pmatrix} \hat{F}_2^{(2)} \\ \hat{F}_1^{(2)} \end{pmatrix} = \left(\frac{e_0}{z_{01}} - \frac{e_1}{z_{03}} \right) \begin{pmatrix} \hat{F}_2^{(2)} \\ \hat{F}_1^{(2)} \end{pmatrix}$$

leads to matrices and boundary values (after setting $s_{02} = 0$):

$$e_0 = \begin{pmatrix} s_{12} & -s_{12} \\ 0 & 0 \end{pmatrix}, \; e_1 = \begin{pmatrix} 0 & 0 \\ s_{23} & -s_{23} \end{pmatrix}, \; C_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; C_1 = \begin{pmatrix} F^{(2)} \\ F^{(2)} - 1 \end{pmatrix} \, .$$

Thus,

$$\begin{pmatrix} F^{(2)} \\ F^{(2)} - 1 \end{pmatrix} = \left[\Phi(e_0, e_1)\right]_{2\times 2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \, .$$

Example 2

Five-point amplitude

$$\hat{\mathbf{F}} = \begin{pmatrix} \hat{F}_3^{(23)} \\ \hat{F}_3^{(32)} \\ \hat{F}_2^{(23)} \\ \hat{F}_2^{(32)} \\ \hat{F}_1^{(23)} \\ \hat{F}_1^{(32)} \end{pmatrix} = \int_0^{z_0} \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_2 \prod_{i<j}^{4} |z_{ij}|^{s_{ij}} \, z_{02}^{s_{02}} \, z_{03}^{s_{03}} \begin{pmatrix} X_{12}(X_{13}+X_{23}) \\ X_{13}(X_{12}+X_{32}) \\ X_{12}X_{34} \\ X_{13}X_{24} \\ (X_{23}+X_{24})X_{34} \\ (X_{32}+X_{34})X_{24} \end{pmatrix}$$

where $X_{ij} \equiv \frac{s_{ij}}{z_{ij}}$. Corresponding matrices and boundary values read

$$e_0 = \begin{pmatrix} s_{123} & 0 & -s_{13}-s_{23} & -s_{12} & -s_{12} & s_{12} \\ 0 & s_{123} & -s_{13} & -s_{12}-s_{23} & s_{13} & -s_{13} \\ 0 & 0 & s_{12} & 0 & -s_{12} & 0 \\ 0 & 0 & 0 & s_{13} & 0 & -s_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad C_0 = \begin{pmatrix} F^{(2)} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ s_{34} & 0 & -s_{34} & 0 & 0 & 0 \\ 0 & s_{24} & 0 & -s_{24} & 0 & 0 \\ s_{34} & -s_{34} & s_{23}+s_{24} & s_{34} & -s_{234} & 0 \\ -s_{24} & s_{24} & s_{24} & s_{23}+s_{34} & 0 & -s_{234} \end{pmatrix} \qquad C_1 = \begin{pmatrix} F^{(23)} \\ F^{(32)} \\ \vdots \\ \vdots \end{pmatrix}.$$

Finally, the 5-point result reads

$$\begin{pmatrix} F^{(23)} \\ F^{(32)} \\ \vdots \end{pmatrix} = \left[\Phi(e_0, e_1)\right]_{6\times 6} \begin{pmatrix} F^{(2)} \\ 0 \\ \mathbf{0}_4 \end{pmatrix} .$$

- there are no single ζ's in the four-point $F^{(2)}$
  $\Rightarrow$ all multiple ζ's originate in the Drinfeld associator.

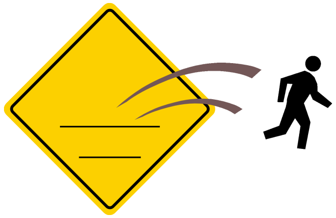
String corrections $\boldsymbol{F}$ can be calculated completely (in principle).

Matrices e_0 and e_1 through $N = 9$ and
results up to and including $N = 7$ are available at

http://mzv.mpp.mpg.de

Conclusions

- old-fashioned method formalized:
  $\Rightarrow$ applicable to any multiplicity N
  and to any order in α'



[Broedel, Schlotterer Stieberger]

- Drinfeld method calculationally favourable:

[Broedel, Schlotterer Stieberger, Terasoma]

  - S-matrix description, no integrals
  - results up to 9 points
  - matrices e_0 and e_1 can be obtained without KZ-equation

- similar methods have been investigated in the context of loop amplitudes in $\mathcal{N}=4$ super Yang-Mills theory [He Caron-Huot][Henn]

- KLT-relations: techniques carry over to closed-string amplitudes
  investigate ζ-structures [Stieberger][Kawai, Lewellen Tye]

- Similar formalism for closed-string tree amplitudes?
  $\Rightarrow$ Single-valued harmonic polylogarithms. [Schnetz][Stieberger]

- What about higher-genus surfaces? Other theories (e.g. $\mathcal{N}=4$ sYM)?

THANKS !

Extra slide I: Multiple polylogarithms

Definition:

$$G(a_1, a_2, \ldots, a_n; z) = \int_0^z \frac{\mathrm{d}t}{t - a_1} G(a_2, \ldots, a_n; t) \qquad w = n$$

$G(z) = G(; z) = 1$, except for $G(\vec{a}; 0) = G(; 0) = 0$.

Shuffle product:

$$\begin{aligned} G(a_1, \ldots, a_r; z) G(a_{r+1}, \ldots, a_{r+s}; z) &= G(a_1, \ldots, a_r \sqcup\!\sqcup \, a_{r+1}, \ldots, a_{r+s}; z) \\ &= \sum_{\sigma \in \Sigma(r,s)} G(a_{\sigma(1)}, \ldots, a_{\sigma(r+s)}; z) \end{aligned}$$

Why polylogarithms?

$$G(\underbrace{0, 0, \ldots, 0}_{w}; z) = \frac{1}{w!} (\ln\ z)^w \qquad G(\underbrace{1, 1 \ldots, 1}_{w}; z) = \frac{1}{w!} \ln^w (1 - z)$$

$$G(\underbrace{a, a, \ldots, a}_{w}; z) = \frac{1}{w!} \ln \left(1 - \frac{z}{a}\right)^w$$

Scaling property:

$$G(k\vec{a}; kz) = G(\vec{a}; z)\ , \qquad k \neq 0$$

Extra slide II: explicit integration

$$
\begin{aligned}
I^{\text{reg}}_{\text{Ex}_1} &= \frac{1}{2} s_{23}^2 \int_0^1 \frac{dz_3}{z_3-1} \int_0^{z_3} \frac{dz_2}{z_2} \left(\ln(z_3-z_2)\right)^2 \\
&= \frac{1}{2} s_{23}^2 \int_0^1 \frac{dz_3}{z_3-1} \int_0^{z_3} \frac{dz_2}{z_2} \left[(\ln(z_3))^2 + 2\ln z_3 \ln\left(1-\frac{z_2}{z_3}\right) + \left(\ln\left(1-\frac{z_2}{z_3}\right)\right)^2 \right] \\
&= s_{23}^2 \int_0^1 \frac{dz_3}{z_3-1} \int_0^{z_3} \frac{dz_2}{z_2} \big(G(0,0;z_3) + G(0;z_3)\, G(z_3;z_2) + G(z_3,z_3;z_2) \big)\,. \\
&= s_{23}^2 \int_0^1 \frac{dz_3}{z_3-1} \big(G(0,0;z_3) G(0;z_3) + G(0;z_3)\, G(0,z_3;z_3) + G(0,z_3,z_3;z_3) \big)\,. \\
&= s_{23}^2 \int_0^1 \frac{dz_3}{z_3-1} \big(3\,G(0,0,0;z_3) + G(0;z_3)\, G(0,1;1) + G(0,1,1;1) \big) \\
&= s_{23}^2 \big(3\,G(1,0,0,0;1) + G(1,0;1)\, G(0,1;1) + G(1;1)\, G(0,1,1;1) \big) \\
&= s_{23}^2 \big(3\,\zeta_4 - \zeta_2^2 \big) \\
&= \frac{1}{5}\, s_{23}^2 \zeta_2^2\,,
\end{aligned}
$$

. . . there is an obstruction. Consider

$$I^{\text{reg}}_{\{0,1\}} = \int_0^1 \frac{\mathrm{d}z_3}{z_3-1} \int_0^{z_3} \frac{dz_2}{z_2}\, G(z_3;z_2)\, G(1;z_2)$$
$$= \int_0^1 \frac{\mathrm{d}z_3}{z_3-1} \big(G(0,z_3,1;z_3) + G(0,1,z_3;z_3)\big)\,.$$

How to rewrite an integral of the form

$$G(\{0,a_1,a_2,\ldots,z,\ldots,a_n\}_w;z)$$

in terms of objects without z in the label?

Way to go:

- use the Hopf-algebra structure
- decompose polylog step by step using the coproduct
- express the result in the appropriate basis [Duhr]

Extra slide IIIb: Polylogarithmic identity

$$
\begin{aligned}
G(a_1,\ldots,a_{i-1},z,a_{i+1},\ldots,a_n;z) = {} & G(a_{i-1},a_1,\ldots,a_{i-1},\hat{z},a_{i+1},\ldots,a_n;z) \\
& - G(a_{i+1},a_1,\ldots,a_{i-1},\hat{z},a_{i+1},\ldots,a_n;z) \\
& - \int_0^z \frac{\mathrm{d}t}{t-a_{i-1}} G(a_1,\ldots,\hat{a}_{i-1},t,a_{i+1},\ldots,a_n;t) \\
& + \int_0^z \frac{\mathrm{d}t}{t-a_{i+1}} G(a_1,\ldots,a_{i-1},t,\hat{a}_{i+1},\ldots,a_n;t) \\
& + \int_0^z \frac{\mathrm{d}t}{t-a_1} G(a_2,\ldots,a_{i-1},t,a_{i+1},\ldots,a_n;t)\,.
\end{aligned}
$$

⇒ identity preserves *shuffle regularization*

⇒ several occurrences of z: change formula appropriately

Example:

$$
\begin{aligned}
G(0,z,1;z) = {} & G(0,0,1;z) - G(1,0,1;z) \\
& - \int_0^z \frac{\mathrm{d}t}{t-0} G(t,1;t) + \int_0^z \frac{\mathrm{d}t}{t-1} \underbrace{G(0,t;t)}_{-\zeta_2} + \int_0^z \frac{\mathrm{d}t}{t-0} G(t,1;t) \\
= {} & G(0,0,1;z) - G(1,0,1;z) - \zeta_2 G(1;z)
\end{aligned}
$$

Recursive application of the identity leads to formulæ for any weight