Schroedinger Symmetries in Lifshitz Holography

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Elias Kiritsis, Niels Obers, to appear
Eric Bergshoeff, Jan Rosseel, to appear

Outline of the Talk

- The model and Lifshitz Asymptotics.
- The sources and their transformations.
- Boundary geometry: torsional Newton-Cartan geometry.
- Gauging the Schroedinger algebra.
- Variation of the on-shell action: the vevs.
- Ward identities.
- Conclusions.

Lifshitz Asymptotics

$$S = \int d^4x\sqrt{-g}\left(R - \frac{1}{2}(\partial\Phi)^2 - \frac{1}{4}Z(\Phi)F^2 - \frac{1}{2}W(\Phi)B^2 - V(\Phi)\right)$$

$$B_\mu = A_\mu - \partial_\mu \chi$$

- Asymptotics and gauge choice for general z obtained from generalization of z=2 results [Christensen, JH, Obers, Rollier, 2013]:

$$ds^2 = \frac{dr^2}{Rr^2} - e^{\underline{t}}e^{\underline{t}} + \delta_{\underline{ij}}e^{\underline{i}}e^{\underline{j}} \qquad R = R_{(0)} + \ldots$$

$$\begin{aligned} e^{\underline{t}} &= r^{-z}\alpha_{(0)}^{1/3}\tau_{(0)a}dx^a + \ldots \\ e^{\underline{i}} &= r^{-1}\alpha_{(0)}^{-1/3}e^{\underline{i}}_{(0)a}dx^a + \ldots \end{aligned}$$

$$\Phi = \Phi_\star + r^\Delta\Phi_{(0)} + \ldots$$

WLOG: $\Phi_\star = 0$

- $R_{(0)}$ and $\alpha_{(0)}$ determined by the equations of motion.

- Bulk local Lorentz transformations induce boundary local Galilean transformations.

Boundary Gauge Field

$$\alpha = \alpha_{(0)} + \dots$$

$$B_\mu = A_\mu - \partial_\mu \chi$$

$$\begin{aligned} A_a - \alpha e_a^t &= r^{z-2} A_{(0)a} + \dots \\ A_r &= (z-2) r^{z-3} \chi_{(0)} + \dots \\ \chi &= r^{z-2} \chi_{(0)} + \dots \end{aligned}$$

- The dilatation weight of $A_{(0)a}$ is fixed by Galilean boost invariance of the bulk metric.
- In order that the boundary gauge field transforms as a gauge connection we need that the residual gauge transformations have a parameter: $\Lambda = r^{z-2} \Lambda_{(0)} + \dots$
- In order that the Stueckelberg field is also a boundary Stueckelberg field ($\delta\chi_{(0)} = \Lambda_{(0)}$) we need: $\chi = r^{z-2} \chi_{(0)} + \dots$
- The gauge choice which leads to these residual gauge transformations is the one given for A_r .

Local Transformations Sources

$$\begin{aligned}
\delta\tau_{(0)a} &= -z\Lambda^D_{(0)}\tau_{(0)a} \\
\delta e^{\underline{i}}_{(0)a} &= \lambda^{\underline{i}}{}_{(0)}\tau_{(0)a} + \lambda^{\underline{i}}{}_{(0)\underline{j}}e^{\underline{j}}_{(0)a} - \Lambda^D_{(0)}e^{\underline{i}}_{(0)a} \\
\delta A_{(0)a} &= -\lambda^{\underline{i}}{}_{(0)}e_{(0)\underline{i}a} + \partial_a\Lambda_{(0)} + (z-2)\Lambda^D_{(0)}A_{(0)a} + (z-2)\chi_{(0)}\partial_a\Lambda^D_{(0)} \\
\delta\chi_{(0)} &= (z-2)\Lambda^D_{(0)}\chi_{(0)} + \Lambda_{(0)} \\
\delta\Phi_{(0)} &= \Delta\Lambda^D_{(0)}\Phi_{(0)}
\end{aligned}$$

- The local tangent space transformations are the same for all z>1. The dilatation weights follow from the leading order r dependence.

- We did not write the diffeomorphisms.

Invariants

- Inverse vielbeins $v^a_{(0)}\,, e^a_{(0)\underline{i}}$:

$$\begin{aligned} v^a_{(0)}\tau_{(0)a} &= -1 \qquad e^a_{(0)\underline{i}}\tau_{(0)a} = 0 \\ v^a_{(0)}e^{\underline{i}}_{(0)a} &= 0 \qquad e^a_{(0)\underline{i}}e^{\underline{j}}_{(0)a} = \delta^{\underline{j}}_{\underline{i}} \end{aligned}$$

- Invariants (boosts, rotations and gauge trafos) and with well-defined dilatation weight: $\tau_{(0)a}$ together with

$$\begin{aligned} \Pi^{ab}_{(0)} &= \delta^{\underline{ij}}e^a_{(0)\underline{i}}e^b_{(0)\underline{j}} \\ \hat{v}^a_{(0)} &= v^a_{(0)} + \Pi^{ab}_{(0)}B_{(0)b} \\ \bar{\Pi}_{(0)ab} &= \delta_{\underline{ij}}e^{\underline{i}}_{(0)a}e^{\underline{j}}_{(0)b} + \tau_{(0)a}B_{(0)b} + \tau_{(0)b}B_{(0)a} \\ \Phi^N_{(0)} &= -v^a_{(0)}B_{(0)a} - \frac{1}{2}\Pi^{ab}_{(0)}B_{(0)a}B_{(0)b} \end{aligned}$$

- where $B_{(0)a} = A_{(0)a} - \partial_a\chi_{(0)}$

- $\tau_{(0)a}$ and $\Pi^{ab}_{(0)}$ are degenerate temporal and spatial metrics. Invariant density: $e_{(0)} = \det\left(\tau_{(0)a}, e^{\underline{i}}_{(0)a}\right)$

T(orsional)NC Geometry

- $\Gamma^{Tc}_{(0)ab}$ is a gauge, boost and rotation invariant torsional connection given by

$$\Gamma^{Tc}_{(0)ab} = -\hat{v}^c_{(0)}\partial_a\tau_{(0)b} + \frac{1}{2}\Pi^{cd}_{(0)}\left(\partial_a\bar{\Pi}_{(0)bd} + \partial_b\bar{\Pi}_{(0)ad} - \partial_d\bar{\Pi}_{(0)ab}\right)$$

- This connection is metric compatible:

$$\begin{aligned}\nabla^T_{(0)a}\tau_{(0)b} &= 0\,,\\ \nabla^T_{(0)a}\Pi^{bc}_{(0)} &= 0\,,\end{aligned}$$

- Torsion: $\Gamma^{Tc}_{(0)[ab]} = -\frac{1}{2}\hat{v}^c_{(0)}\left(\partial_a\tau_{(0)b} - \partial_b\tau_{(0)a}\right)$

- NC: $\partial_a\tau_{(0)b} - \partial_b\tau_{(0)a} = 0$

- T(wistless)TNC: $\tau_{(0)[a}\partial_b\tau_{(0)c]} = 0 \leftrightarrow \Pi^{ac}_{(0)}\Pi^{bd}_{(0)}\left(\partial_c\tau_{(0)d} - \partial_d\tau_{(0)c}\right) = 0$

Gauging the Schroedinger Algebra I

- The transformations of the sources can be written as local Schroedinger transformations:

$$\mathcal{A}_{(0)a} = H\tau_{(0)a} + P_{\underline{i}}e^{\underline{i}}_{(0)a} + Db_{(0)a} + Mm_{(0)a} + G_{\underline{i}}\omega_{(0)a}{}^{\underline{i}} + \frac{1}{2}J_{\underline{ij}}\omega_{(0)a}{}^{\underline{ij}}$$

$$\begin{aligned}
\delta\mathcal{A}_{(0)a} &= \partial_a\tilde{\Sigma}_{(0)} + [\mathcal{A}_{(0)a}, \tilde{\Sigma}_{(0)}] \\
&= \mathcal{L}_{\xi_{(0)}}\mathcal{A}_{(0)a} + \xi^b_{(0)}\mathcal{F}_{(0)ab} + \partial_a\Sigma_{(0)} + [\mathcal{A}_{(0)a}, \Sigma_{(0)}] \\
\tilde{\Sigma}_{(0)} &= \xi^a_{(0)}\mathcal{A}_{(0)a} + \Sigma_{(0)} \\
\Sigma_{(0)} &= M\Lambda_{(0)} + G_{\underline{i}}\lambda^{\underline{i}}_{(0)} + \frac{1}{2}J_{\underline{ij}}\lambda^{\underline{ij}}_{(0)} + D\Lambda^D_{(0)}
\end{aligned}$$

- where $m_{(0)a} = A_{(0)a} - (z-2)\chi_{(0)}b_{(0)a}$ and the Schr algebra is:

$$\begin{aligned}
&[D, H] = zH\,, && [D, P_{\underline{i}}] = P_{\underline{i}}\,,\\
&[D, G_{\underline{i}}] = -(z-1)G_{\underline{i}}\,, && [D, M] = -(z-2)M\,,\\
&[H, G_{\underline{i}}] = P_{\underline{i}}\,, && [P_{\underline{i}}, G_{\underline{j}}] = -\delta_{\underline{ij}}M\,,\\
&[J_{\underline{ij}}, P_{\underline{k}}] = \delta_{\underline{ik}}P_{\underline{j}} - \delta_{\underline{jk}}P_{\underline{i}}\,, && [J_{\underline{ij}}, G_{\underline{k}}] = \delta_{\underline{ik}}G_{\underline{j}} - \delta_{\underline{jk}}G_{\underline{i}}\,,\\
&[J_{\underline{ij}}, J_{\underline{kl}}] = \delta_{\underline{ik}}J_{\underline{jl}} - \delta_{\underline{il}}J_{\underline{jk}} - \delta_{\underline{jk}}J_{\underline{il}} + \delta_{\underline{jl}}J_{\underline{ik}}\,.
\end{aligned}$$

Gauging the Schroedinger Algebra II

- Local translations are equivalent to diffeomorphisms up to local Sch symmetries by setting certain components of the Yang-Mills curvature $\mathcal{F}_{(0)ab}$ expanded in the Schroedinger Lie algebra equal to zero.

- Solving these curvature constraints turns some of the gauge connections such as the ones for boosts and rotations $\omega_{(0)a}{}^{\underline{ij}}$, $\omega_{(0)a}{}^{\underline{i}}$ into dependent gauge fields in agreement with the expressions that follow from the TNC vielbein postulates:

$$
\begin{aligned}
\mathcal{D}_{(0)a}\tau_{(0)b} &= \partial_a\tau_{(0)b} - \tilde{\Gamma}^{Tc}_{(0)ab}\tau_{(0)c} + zb_{(0)a}\tau_{(0)b} = 0 \\
\mathcal{D}_{(0)a}e^{\underline{i}}_{(0)b} &= \partial_a e^{\underline{i}}_{(0)b} - \tilde{\Gamma}^{Tc}_{(0)ab}e^{\underline{i}}_{(0)c} - \omega_{(0)a}{}^{\underline{i}}\tau_{(0)b} - \omega_{(0)a}{}^{\underline{i}}{}_{\underline{j}}e^{\underline{j}}_{(0)b} + b_{(0)a}e^{\underline{i}}_{(0)b} = 0
\end{aligned}
$$

- The connection is related to the TNC connection $\Gamma^{Tc}_{(0)ab}$ by replacing ordinary derivatives in $\Gamma^{Tc}_{(0)ab}$ by dilatation covariant ones.

Definition of the Vevs

- We assume a counterterm action exists that makes the variational problem well defined, that respects all the local symmetries of the bulk theory and that it is local up to an anomaly term which is assumed to be a local expression proportional to log r.

- This is sufficient to define the vevs, derive their transformations under the Schroedinger group and write down the Ward identities.

- Following [Ross, Saremi, 2009], [Ross, 2011] we use the HIM boundary stress tensor [Hollands, Ishibashi, Marolf, 2005].

$$\delta S_{\text{ren}} = -\int_{\partial\mathcal{M}} d^3x e \left(S^{\underline{t}}{}_a \delta e^a_{\underline{t}} + S^{\underline{i}}{}_a \delta e^a_{\underline{i}} + T_\varphi \delta\varphi + T^{\underline{i}} \delta A_{\underline{i}} + T_\chi \delta\chi + T_\Phi \delta\Phi - \mathcal{A}\frac{\delta r}{r} \right)$$

$$\varphi = e^a_{\underline{t}} \left(A_a - \alpha e^{\underline{t}}_a \right)$$

Variation of the On-Shell Action

$$\delta S_{\text{ren}} = -\int_{\partial\mathcal{M}} d^3xe \left(S^{\underline{t}}{}_a \delta e^a_{\underline{t}} + S^{\underline{i}}{}_a \delta e^a_{\underline{i}} + T_\varphi \delta\varphi + T^{\underline{i}} \delta A_{\underline{i}} + T_\chi \delta\chi + T_\Phi \delta\Phi - \mathcal{A}\frac{\delta r}{r} \right)$$

$$\begin{aligned}
e &= r^{-z-2}\alpha_{(0)}^{-1/3} e_{(0)} + \dots \\
e^a_{\underline{t}} &= -r^z \alpha_{(0)}^{-1/3} v^a_{(0)} + \dots \\
e^a_{\underline{i}} &= r\alpha_{(0)}^{1/3} e^a_{(0)\underline{i}} + \dots \\
\varphi &= r^{2z-2}\alpha_{(0)}^{-1/3} A_{(0)\underline{t}} + \dots \\
A_{\underline{i}} &= r^{z-1}\alpha_{(0)}^{1/3} A_{(0)\underline{i}} + \dots \\
\chi &= r^{z-2}\chi_{(0)} + \dots \\
\Phi &= r^{\Delta}\Phi_{(0)} + \dots
\end{aligned}$$

$$\begin{aligned}
S^{\underline{t}}{}_a &= r^2 \alpha_{(0)}^{2/3} S^{\underline{t}}_{(0)a} + \dots \\
S^{\underline{i}}{}_a &= r^{z+1} S^{\underline{i}}_{(0)a} + \dots \\
T_\varphi &= r^{4-z}\alpha_{(0)}^{2/3} T^{\underline{t}}_{(0)} + \dots \\
T^{\underline{i}} &= r^3 T^{\underline{i}}_{(0)} + \dots \\
T_\chi &= r^4 \alpha_{(0)}^{1/3} \langle O_\chi \rangle + \dots \\
T_\Phi &= r^{z+2-\Delta}\alpha_{(0)}^{1/3} \langle O_\Phi \rangle + \dots \\
\mathcal{A} &= r^{z+2}\alpha_{(0)}^{1/3} \mathcal{A}_{(0)} + \dots
\end{aligned}$$

$$\begin{aligned}
\delta S^{\text{os}}_{\text{ren}} = & -\int_{\partial\mathcal{M}} d^3xe_{(0)} \Big(-S^{\underline{t}}_{(0)a} \delta v^a_{(0)} + S^{\underline{i}}_{(0)a} \delta e^a_{(0)\underline{i}} + T^{\underline{t}}_{(0)} \delta A_{(0)\underline{t}} + T^{\underline{i}}_{(0)} \delta A_{(0)\underline{i}} \\
& + \langle O_\chi \rangle \delta\chi_{(0)} + \langle \tilde{O}_\Phi \rangle \delta\Phi_{(0)} - \mathcal{A}_{(0)} \frac{\delta r}{r} \Big)
\end{aligned}$$

Ward Identities

$$
\begin{aligned}
0 &= T^a_{(0)}\hat{e}^{\underline{i}}_{(0)a} + \mathcal{T}^b_{(0)a}\tau_{(0)b}e^{a\underline{i}}_{(0)} && \text{boost}\\
0 &= \mathcal{T}^b_{(0)a}\hat{e}^{\underline{i}}_{(0)b}e^{a\underline{j}}_{(0)} - (\underline{i}\leftrightarrow\underline{j}) && \text{rotation}\\
\mathcal{A}_{(0)} &= -z\mathcal{T}^b_{(0)a}\tau_{(0)b}\hat{v}^a_{(0)} + \mathcal{T}^b_{(0)a}\hat{e}^{\underline{i}}_{(0)b}e^{a\underline{i}}_{(0)} \\
&\quad +2(z-1)T^a_{(0)}\tau_{(0)a}\Phi^N_{(0)} + \Delta\langle\tilde{O}_\Phi\rangle\Phi_{(0)} && \text{scale}\\
\langle O_\chi\rangle &= \frac{1}{e_{(0)}}\partial_a\left(e_{(0)}T^a_{(0)}\right) && \text{gauge}\\
0 &= \nabla^T_{(0)b}\mathcal{T}^b_{(0)a} + 2\Gamma^{Tc}_{(0)[bc]}\mathcal{T}^b_{(0)a} - 2\Gamma^{Tb}_{(0)[ac]}\mathcal{T}^c_{(0)b} \\
&\quad +T^b_{(0)}\hat{e}^{\underline{i}}_{(0)b}\mathcal{D}_{(0)a}B_{(0)\underline{i}} + T^b_{(0)}\tau_{(0)b}\partial_a\Phi^N_{(0)} + \langle\tilde{O}_\Phi\rangle\partial_a\Phi_{(0)} && \text{diffeo}
\end{aligned}
$$

- Gauge, rotation and boost invariant vevs with well-defined scaling weight:

$$
\begin{aligned}
\mathcal{T}^a_{(0)b} &= -\left(S^{\underline{t}}_{(0)a} + T^{\underline{t}}_{(0)}\partial_a\chi_{(0)}\right)v^a_{(0)} + \left(S^{\underline{i}}_{(0)a} + T^{\underline{i}}_{(0)}\partial_a\chi_{(0)}\right)e^a_{(0)\underline{i}} && \text{bdry stress-energy}\\
T^a_{(0)} &= -T^{\underline{t}}_{(0)}v^a_{(0)} + T^{\underline{i}}_{(0)}e^a_{(0)\underline{i}} && \text{mass current}
\end{aligned}
$$

- The diffeo Ward identity involves torsion terms and force terms. $\Phi^N_{(0)}$ is the Newton potential.

Conclusions

- We have defined all the sources for asymptotically locally Lifshitz space-times. An important role is played by the boundary gauge field (containing the Newton potential) which sources the mass current.
- The sources transform under the Schroedinger algebra which is induced by bulk diffeomorphisms, gauge transformations and local Lorentz transformations.
- The boundary geometry is torsional Newton-Cartan and can be obtained by gauging the Schroedinger algebra.
- The on-shell action is Schroedinger invariant.
- Not in this talk: special conformal trafos, Killing symmetries, conserved currents and Lifshitz vacuum.