ADE FIVEBRANES, MOCK MODULAR FORMS AND UMBRAL MOONSHINE

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OUTLINE

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Review of Mathieu and Umbral Moonshine

The elliptic genus of a K3 surface X has a decomposition into characters of the N=4 SCA:

$$Z^X_{ell} = 20\mathrm{ch}^{\tilde{R}}_{1,\frac{1}{4},0} + 2c(-1/8)\mathrm{ch}^{\tilde{R}}_{1,\frac{1}{4},\frac{1}{2}} + \sum_{n=1}^{\infty} c(n-1/8)\mathrm{ch}^{\tilde{R}}_{1,n+\frac{1}{4},\frac{1}{2}}$$

$$H^{(2)}(\tau) = -2q^{-1/8} + \sum_{n\geq 1} 2c(n-1/8)q^{n-1/8}$$

$$\left.\begin{array}{rcl} c(7/8) & = & 45 \\ c(15/8) & = & 231 \\ c(23/8) & = & 770 \\ c(31/8) & = & 2277 \end{array}\right\}$$

dimensions of M24 irreps

(Eguchi, Ooguri, Tachikawa)

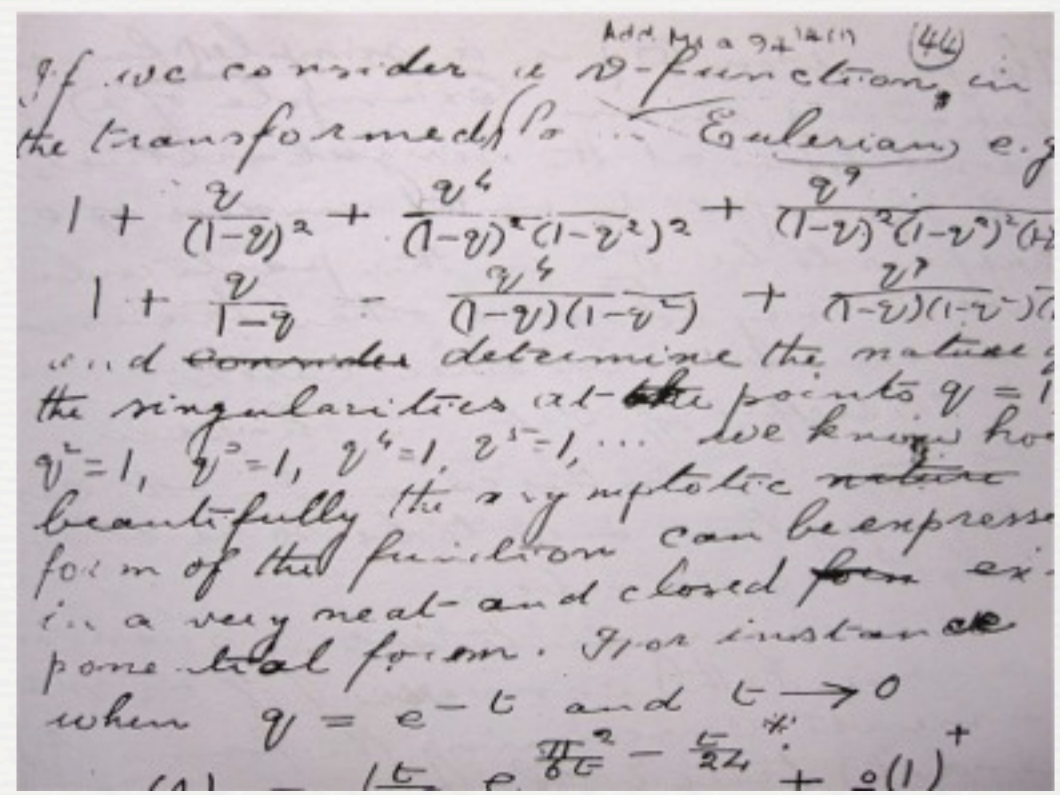
$$c(n-1/8) = \sum_i^{26} r^i_n \mathrm{dim} R_i$$

This gives us a q series with coefficients related to representations of M24 with positive multiplicity and is reminiscent of the monstrous moonshine connection between representations of the Monster group and the modular function

$$J(\tau) = q^{-1} + \text{const} + 196884q + \cdots$$

$$196883 + 1$$

However $H^{(2)}(\tau)$ is not a modular function, rather it is a mock modular function or mock theta function. Mock theta functions first appeared in 1920 in the last letter Ramanujan wrote to Hardy. He wrote them down as q expansions but did not explain how he found them or fully define their properties.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If we consider a ϑ-function in the transformed Eulerian e.g.

$$1 + \frac{q}{(1-q)^2} + \frac{q^4}{(1-q)^2(1-q^2)^2} + \frac{q^9}{(1-q)^2(1-q^2)^2 \ldots}$$

$$1 + \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^2)} + \frac{q^9}{(1-q)(1-q^2)\ldots}$$

and ~~consider~~ determine the nature of the singularities at the points $q = 1$, $q^2=1$, $q^3=1$, $q^4=1$, $q^5=1$, ... we know how beautifully the asymptotic form of the function can be expressed in a very neat and closed exponential form. For instance when $q = e^{-t}$ and $t \to 0$

$$\ldots = \sqrt{\ldots}\, e^{\frac{\pi^2}{6t} - \frac{t}{24}} + o(1)$$

Recall a modular form of weight k is a holomorphic function f

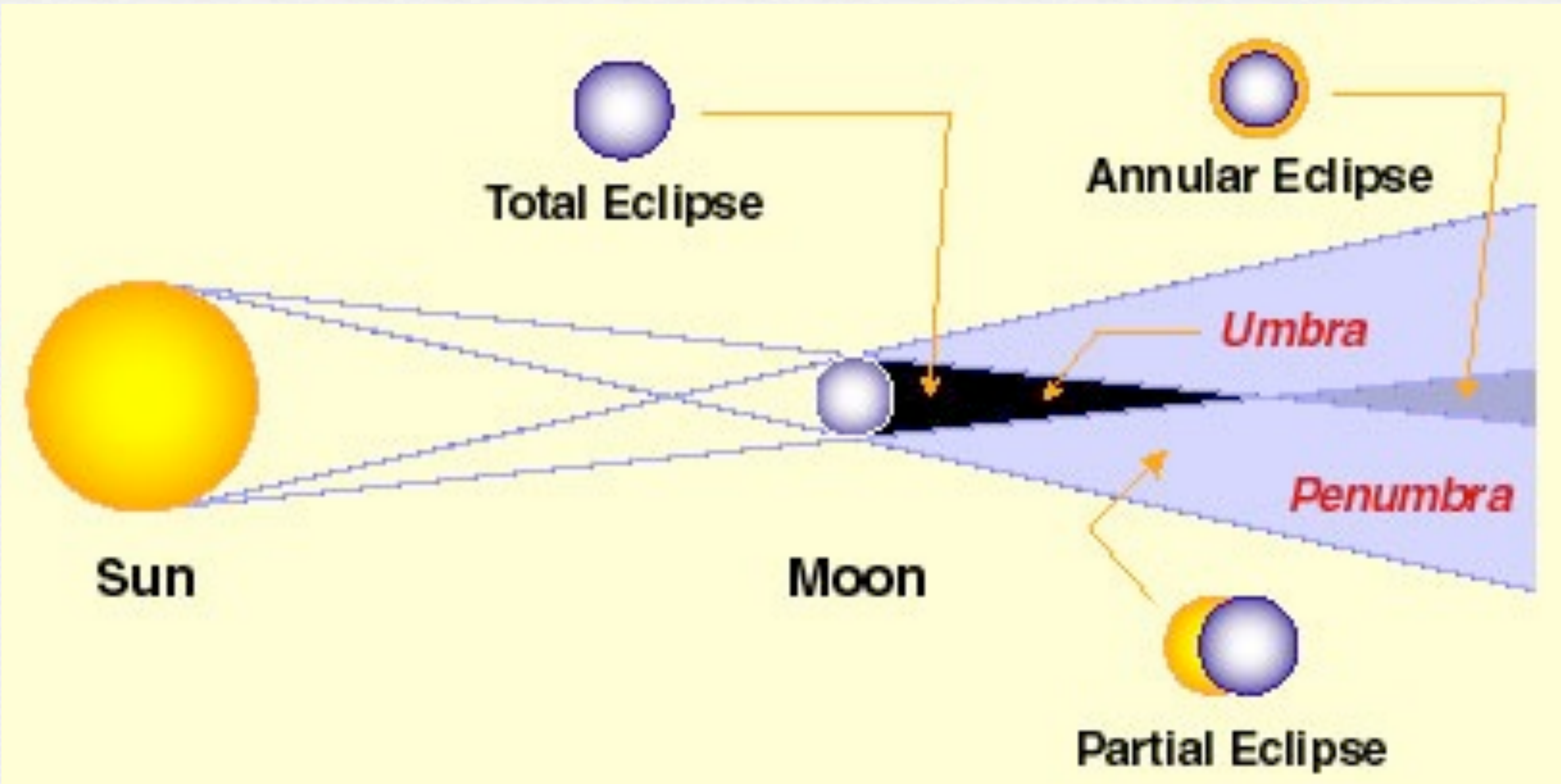
$$f\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^k f(\tau) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

$h(\tau)$ is a mock modular form of weight k if there is a pair $(h(\tau), g(\tau))$ where $g(\tau)$ is a holomorphic modular form of weight 2-k, known as the shadow of $h(\tau)$, such that the non-holomorphic function

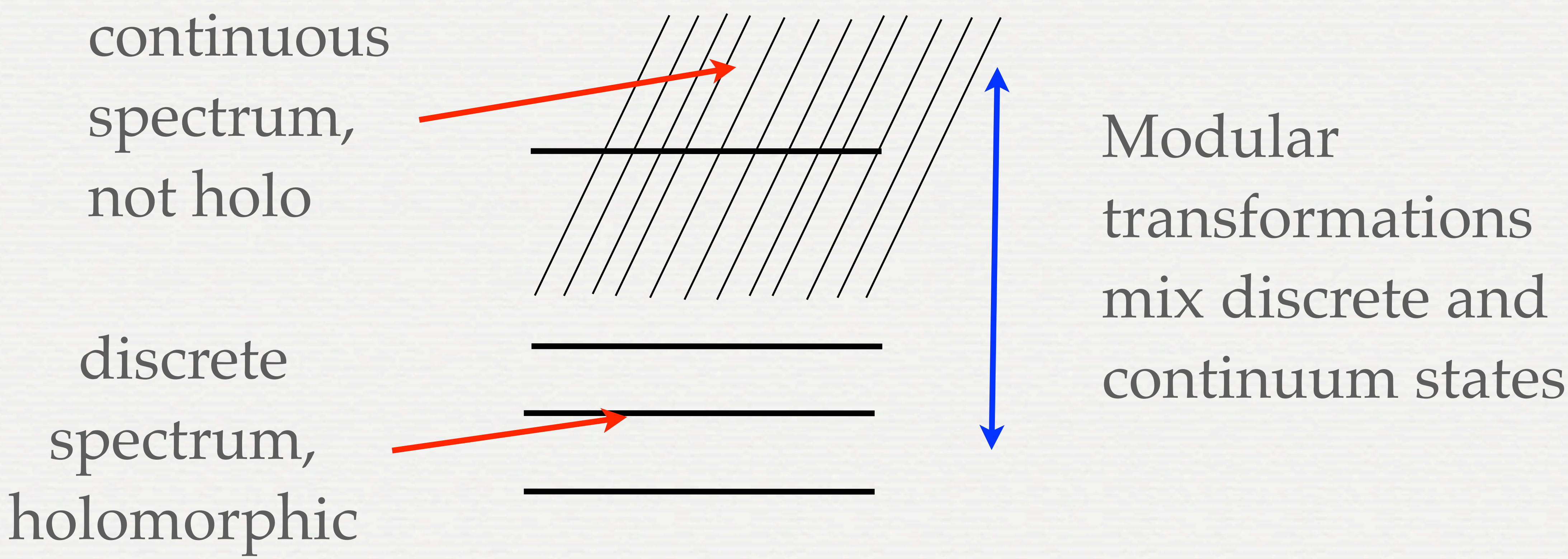
$$\hat{h}(\tau)=h(\tau)+\text{const}\int_{-\bar{\tau}}^{\infty}\overline{g(-\bar{z})}(z+\tau)^{-k}dz$$

transforms like a weight k modular form

The pair $(H^{(2)}(\tau), 24\eta(\tau)^3)$ is such a pair.


Total Eclipse
Annular Eclipse
Umbra
Penumbra
Sun
Moon
Partial Eclipse

Mock modularity appears in black hole state counting (DMZ) and in the computation of the elliptic genus of non-compact sigma models (Troost, Ashok, Eguchi, Sugawara). There is a simple physical explanation for the tension between holomorphy and modularity in these examples.



In Mathieu Moonshine we have a triple $(H^{(2)}(\tau), 24\eta(\tau)^3, M_{24})$ of a mock modular form, its shadow, and a finite group that acts on both (M24 has a 24-dimensional permutation representation). Can we generalize this structure? After much theoretical and "experimental" work the answer is yes, and the generalization reveals a great deal of additional structure.

Let X be a root system with A,D,E components, total rank 24 and with all components having equal Coxeter number. There are 23 such root systems:

$$A_1^{24},\ A_2^{12},\ A_3^{8},\ A_4^{6},\ A_5^{4}D_4,\ A_6^{4},\ A_7^{2}D_5^{2},\ A_8^{3},\ A_9^{2}D_6,\ A_{11}D_7E_6,\ A_{12}^{2},\ A_{15}D_9,\ A_{17}E_7,\ A_{24},$$

$$D_4^{6},\ D_6^{4},\ D_8^{3},\ D_{10}E_7^{2},\ D_{12}^{2},\ D_{16}E_8,\ D_{24},$$

$$E_6^{4},\ E_8^{3}.$$

From each X we (M. Cheng, J. Duncan and I) construct six mathematical objects

- L^X A Niemeier lattice with root system X (add weight vectors via glue code to construct an even, self-dual lattice)
- H^X A vector-valued mock modular form of weight 1/2
- S^X A vector-valued modular form of weight 3/2 which is the shadow of H^X.
- G^X A finite group exhibiting moonshine for H^X and equal to Aut(L^X)/Weyl(X) (global symmetry group)
- Γ^X A genus zero subgroup of SL(2,R) constructed from the ADE data of X
- T^X A hauptmodul for Γ^X (analog of J function)

Table 3: Umbral Groups

X	A_1^{24}	A_2^{12}	A_3^8	A_4^6	$A_5^4D_4$	A_6^4	$A_7^2D_5^2$
ℓ	2	3	4	5	6	7	8
G^X	M_{24}	$2.M_{12}$	$2.AGL_3(2)$	$GL_2(5)/2$	$GL_2(3)$	$SL_2(3)$	Dih_4
$\bar{G}^X$	M_{24}	M_{12}	$AGL_3(2)$	$PGL_2(5)$	$PGL_2(3)$	$PSL_2(3)$	2^2

X	A_8^3	$A_9^2D_6$	$A_{11}D_7E_6$	A_{12}^2	$A_{15}D_9$	$A_{17}E_7$	A_{24}
ℓ	9	10	12	13	16	18	25
G^X	Dih_6	4	2	4	2	2	2
$\bar{G}^X$	Sym_3	2	1	2	1	1	1

X	D_4^6	D_6^4	D_8^3	$D_{10}E_7^2$	D_{12}^2	$D_{16}E_8$	D_{24}
ℓ	6+3	10+5	14+7	18+9	22+11	30+15	46+23
G^X	$3.Sym_6$	Sym_4	Sym_3	2	2	1	1
$\bar{G}^X$	Sym_6	Sym_4	Sym_3	2	2	1	1

X	E_6^4	E_8^3
ℓ	12+4	30+6,10,15
G^X	$GL_2(3)$	Sym_3
$\bar{G}^X$	$PGL_2(3)$	Sym_3

These tables can be decoded to yield G^X, Γ^X, T^X. For example consider $X = A_6^4$.

G^X is the group of 2x2 matrices of determinant one over the field with 3 elements.

The genus zero group is $\Gamma_0(7)$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,Z), \quad c = 0 \bmod 7$$

with hauptmodul $\dfrac{\eta(\tau)^4}{\eta(7\tau)^4}$

Table 2: Niemeier Root Systems and Principal Moduli

X	A_1^{24}	A_2^{12}	A_3^8	A_4^6	$A_5^4D_4$	A_6^4	$A_7^2D_5^2$
ℓ	2	3	4	5	6	7	8
π^X	$\frac{2^{24}}{1^{24}}$	$\frac{3^{12}}{1^{12}}$	$\frac{4^8}{1^8}$	$\frac{5^6}{1^6}$	$\frac{2^16^5}{1^53^1}$	$\frac{7^4}{1^4}$	$\frac{2^28^4}{1^44^2}$
Γ^X	$2B$	$3B$	$4C$	$5B$	$6E$	$7B$	$8E$

X	A_8^3	$A_9^2D_6$	$A_{11}D_7E_6$	A_{12}^2	$A_{15}D_9$	$A_{17}E_7$	A_{24}
ℓ	9	10	12	13	16	18	25
π^X	$\frac{9^3}{1^3}$	$\frac{2^110^3}{1^35^1}$	$\frac{2^23^112^3}{1^34^16^2}$	$\frac{13^2}{1^2}$	$\frac{2^116^2}{1^28^1}$	$\frac{2^13^118^2}{1^26^19^1}$	$\frac{25^1}{1^1}$
Γ^X	$9B$	$10E$	$12I$	$13B$	$16B$	$18D$	$(25Z)$

X	D_4^6	D_6^4	D_8^3	$D_{10}E_7^2$	D_{12}^2	$D_{16}E_8$	D_{24}
ℓ	6+3	10+5	14+7	18+9	22+11	30+15	46+23
π^X	$\frac{2^66^6}{1^63^6}$	$\frac{2^410^4}{1^45^4}$	$\frac{2^314^3}{1^37^3}$	$\frac{2^33^218^3}{1^36^29^3}$	$\frac{2^222^2}{1^211^2}$	$\frac{2^23^15^130^2}{1^26^110^115^2}$	$\frac{2^146^1}{1^123^1}$
Γ^X	$6C$	$10B$	$14B$	$18C$	$22B$	$30G$	$46AB$

X	E_6^4	E_8^3
ℓ	12+4	30+6,10,15
π^X	$\frac{2^43^412^4}{1^44^46^4}$	$\frac{2^33^35^330^3}{1^36^310^315^3}$
Γ^X	$12B$	$30A$

Examples of H_r^X for $X = A_1^{24}, A_2^{12}, A_3^8$

$$H_1^{(2)}(\tau) = 2q^{-1/8}\left(-1 + 45q + 231q^2 + 770q^3 + 2277q^4 + 5796q^5 + \cdots\right)$$

$$H_1^{(3)}(\tau) = 2q^{-1/12}\left(-1 + 16q + 55q^2 + 144q^3 + 330q^4 + 704q^5 + \cdots\right)$$

$$H_2^{(3)}(\tau) = 2q^{2/3}\left(10 + 44q + 110q^2 + 280q^3 + 572q^4 + 1200q^5 + \cdots\right)$$

$$H_1^{(4)}(\tau) = 2q^{-1/16}\left(-1 + 7q + 21q^2 + 43q^3 + 94q^4 + 168q^5 + \cdots\right)$$

$$H_2^{(4)}(\tau) = 2q^{3/4}\left(8 + 24q + 56q^2 + 112q^3 + 216q^4 + 392q^5 + \cdots\right)$$

$$H_3^{(4)}(\tau) = 2q^{7/16}\left(3 + 14q + 28q^2 + 69q^3 + 119q^4 + 239q^5 + \cdots\right)$$

Dimensions of non-trivial irreps of M24

Dimensions of non-trivial irreps of 2.M12

Dimensions of non-trivial irreps of 2.AGL3(2)

These are special forms obeying an optimal growth condition first formulated by Dabholkar, Murthy and Zagier.

How can we explain all this structure?
Start first with K3/M24 connection

No classical K3 surface has M24 symmetry and no K3 SCFT has M24 symmetry (Mukai, Kondo, Gaberdiel, Hohenneger, Volpato)

However there is much structure reminiscent of CFT

Existence of M24 modules

$$K^{(2)} = \oplus_{n=0}^{\infty} K^{(2)}_{n-1/8}, \quad \mathrm{dim} K^{(2)}_{n-1/8} = c(n-1/8)$$

Existence of McKay-Thompson series for all elements of M24, even twining/twists for commuting pairs (g,h) in M24. Cheng, Gaberdiel, Hohenegger, Volpato, Eguchi, Hikami, Gannon, Persson, Ronellenfitsch.

It seems we need a CFT, or something CFT-like, but with some new ingredients

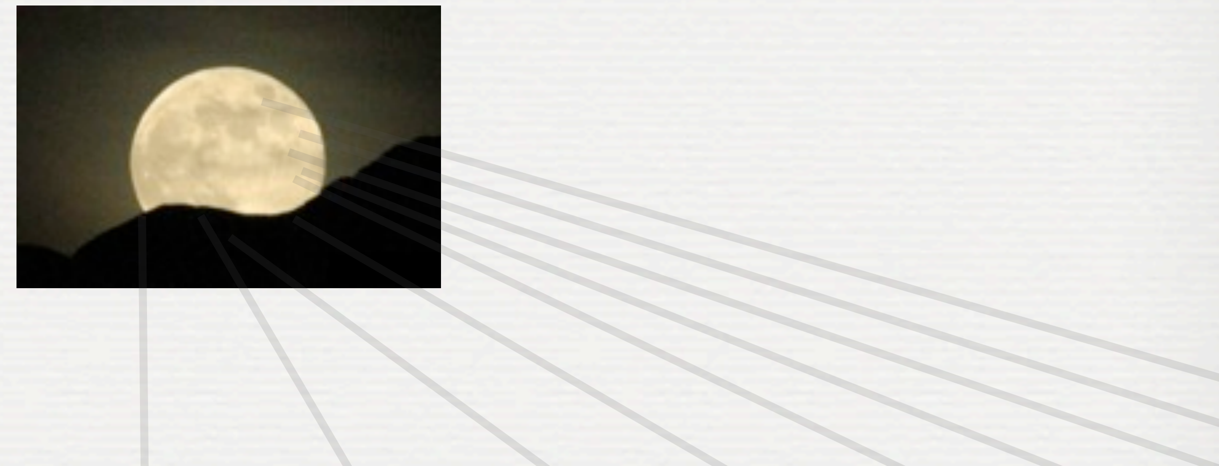
Tie together symmetries that exist at different points in K3 moduli space to obtain a larger symmetry group (Taormina, Wendland, Cheng, Harrison)

Try to identify M24 as a symmetry not of the SCFT, but only of the spectrum and algebra of BPS states (M. Gaberdiel, C. Keller, G. Moore, JH, in progress)

Construct a VOA structure associated to cones in lattices of indefinite signature whose theta functions are related to the mock modular forms of Umbral Moonshine (J. Duncan and JH, String-Math 2014 and to appear)

Look for a formalism in which mock modular forms and their symmetries appear in the counting of spacetime BPS states (S. Murthy, JH).

Approach of this talk



U m b r a l M o o n s h i n e

H^X

L^X S^X

X

T^X G^X

Γ^X

The c=12 SCFT

$$K3 \times \frac{SL(2,R)_k}{U(1)} \times \frac{SU(2)_k}{U(1)}$$

describing the near horizon limit of k fivebranes wrapped on K3 and the BPS spacetime states it leads to is the right framework for understanding Umbral Moonshine.

Why might this idea be correct?

The massless representations don't really transform correctly under M24 and in Monstrous Moonshine getting rid of massless states played an important role. This construction removes the massless state contribution to the holomorphic part of a BPS counting function.

Umbral Moonshine involves mock modular forms and these are known to appear in the computation of modular objects such as the elliptic genus for non-compact CFTs like the SL(2,R)/U(1) theory.

This CFT has an ADE classification and Umbral Moonshine has an ADE classification.

There are two sets of groups in Umbral Moonshine, the moonshine groups G^X which (in some cases) are related to finite subgroups of SU(2) and genus zero groups Γ^X that are subgroups of SL(2,R).

Physical Origin and Properties of the SCFT

Consider k coincident NS 5-branes in say type IIA string theory on R^{10}. The CFT was analyzed by CHS: super version of

Level k SU(2) WZW $\otimes$ Throat-dilaton $\otimes$ 6 free fields

Seiberg
Ooguri
Vafa
Giveon
Kutasov

and take the near horizon limit with asymptotic $g_s \to 0$

If we also take the fivebranes to wrap K3 we end up with the SCFT of interest

$$K3 \times \frac{SL(2,R)_k}{U(1)} \times \frac{SU(2)_k}{U(1)}$$

cigar or 2d BH

N=2 minimal model

To turn this into a consistent, spacetime supersymmetric background for string theory one must

Do a Z_k orbifold in order to project onto the integer $U(1)_R$ charges required for the GSO projection to a theory with spacetime susy.

Add in the remaining free superfields for the two remaining spacetime directions as well as the usual covariant ghost system.

The resulting theory then has an ADE classification of modular invariant partition functions reflecting the ADE classification of fivebranes.

BPS State Counting

We want to count spacetime BPS states in this theory. These are counted by the second BPS index

$$\chi_2^{(m,ADE)} = Tr(-1)^{F_S} J_{sp}^2 q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24}$$

where the trace is over R and NS sectors with GSO projection, and J is the generator of a spatial rotation (here around the asymptotic circle of the cigar).

Generalizing results of Troost, Ashok, Eguchi and Sugawara the full, non-holomorphic, modular invariant answer can be computed using the path integral description of WZW coset models. The holomorphic, but only mock modular answer can be computed by using the discrete characters of the component SCFTs or by projecting onto the holomorphic part of the first computation.

For m=2 fivebranes

holonomy of U(1) gauge field

$$\chi_2^{(2,A)} = \int_{E(\tau)} \frac{du_1 du_2}{\tau_2} P(\tau,u) T(\tau,u)$$

$$P(\tau,u) = \frac{\eta(\tau)^6 Z_{ell}^{K3}(\tau,u)}{2\theta_1(\tau,u)^2} \qquad T(\tau,u) = (2\tau_2)^{1/2} e^{-2\pi u_2^2/\tau_2} |\theta_1(\tau,u)|^2$$

Gaiotto, Zagier

$$\chi_2^{(2,A)}(\tau) = -\frac{1}{2}\eta(\tau)^3 \hat{H}^{(2)}(\tau)$$

$$\widehat{H}^{(2)}(\tau) = H^{(2)}(\tau) + 24 \sum_{k\in\mathbb{Z}} \mathrm{sgn}(4k+1) q^{-(4k+1)^2/8} \left(-1 + \mathrm{Erf}\left[\frac{4k+1}{2}\sqrt{2\pi\tau_2}\right]\right)$$

M24 mock modular form

non-holomorphic completion

It should be noted this formula computes BPS states and so depends only on the elliptic genus of K3 and is independent of moduli. Note also that there was no decomposition into N=4 characters or funny business with massless representations. The M24 mock modular form just emerges from counting spacetime BPS states.

This computation can be extended to any number m of fivebranes and to any allowed A,D,E modular invariant. If we compute it for choices of m that correspond to pure A,D or E examples of Umbral Moonshine, i.e. for

$$A_1^{24}, A_2^{12}, A_3^8, A_4^6, A_6^4, A_8^3, A_{12}^2, A_{24}, D_4^6, D_6^4, D_8^3, D_{24}, E_6^4, E_8^3$$

$$\chi_2^{(m,ADE)}(\tau) = -\frac{m-1}{2} \sum_{r=1}^{m-1} S_{m,r}(\tau) \hat{H}_r^{(X)}(\tau)$$

Completion of Umbral mock modular forms

Of course the computation can also be done for m fivebranes of type A,D,E that do not correspond to the Umbral cases. In these cases there is in general no common factor like m-1 that can be factored out.

For the Umbral cases dividing by m-1 still gives integer multiplicities in the counting of BPS states. Perhaps there is some kind of ``fractional" fivebrane? The Umbral cases occur whenever m-1 divides 24.

Some Fun Math

1. Riemann identities for Appell-Lerch sums

We are all familiar with the Riemann identity expressing equality between numbers of fermions and bosons in the RNS formalism: $\theta_{00}^4 - \theta_{01}^4 - \theta_{10}^4 = 0$
and you are probably also familiar with its generalization expressing equality of fermion and boson SO(8) quantum numbers in the RNS and GS formalisms:

$$\theta_{00}\theta_{00}\theta_{00}\theta_{00} - \theta_{01}\theta_{01}\theta_{01}\theta_{01} - \theta_{10}\theta_{10}\theta_{10}\theta_{10} + \theta_{11}\theta_{11}\theta_{11}\theta_{11} = 2\theta_{11}\theta_{11}\theta_{11}\theta_{11}$$

where on the left the arguments are x,y,u,v in that order and on the right the arguments are, in order

$$x_0 = \frac{1}{2}(x+y+u+v)$$
$$y_0 = \frac{1}{2}(x+y-u-v)$$
$$u_0 = \frac{1}{2}(x-y+u-v)$$
$$v_0 = \frac{1}{2}(x-y-u+v)$$

An analogous identity holds with some of the theta functions replaced by Appell-Lerch type sums. With

$$B_{1,ab} = B_1 \begin{bmatrix} a \\ b \end{bmatrix} (x, z; \tau) = e^{i\pi b(1-a/2)} \frac{\vartheta_{ab}(v;q)v^{1/2}}{\eta^3} \sum_{n\in\mathbb{Z}} \frac{(-1)^{bn} q^{(n+\frac{1}{2}-\frac{a}{2})(n+\frac{3}{2}-\frac{a}{2})/2} y^{n+\frac{1}{2}-\frac{a}{2}}}{1+(-1)^b v q^{n+\frac{1}{2}-\frac{a}{2}}}$$

we have the identity (with the same arguments as before)

$$\theta_{00}\theta_{00}B_{1,00} - \theta_{01}\theta_{01}B_{1,01} - \theta_{10}\theta_{10}B_{1,10} + \theta_{11}\theta_{11}B_{1,11} = 2\theta_{11}\theta_{11}B_{1,11}$$

strongly suggesting that there is a GS formalism for this background.

In Zwegers famous 2002 Ph.D thesis that gave a fundamental description of mock modular forms an important role was played the generalized two variable Appell-Lerch sum

$$\mu(\tau,u,v) = \frac{e^{\pi i u}}{\theta_1(v,\tau)} \sum_{n\in Z} \frac{(-1)^n e^{\pi i (n^2+n)\tau + 2\pi i n v}}{1 - e^{2\pi i n \tau + 2\pi i u}}$$

In our work we encountered a three variable generalization. Its existence is essentially a consequence of there being three U(1) currents in the SCFT

$$\mu^{k,j,j'}(\tau; v,u,w) = \frac{y_v^{j'/k}}{\theta_{11}(\tau;u)} \sum_{n\in\mathbb{Z}} \frac{q^{n^2/k} q^{j'n/k}}{1 - y_v q^n} \left(\frac{y_u\, y_w^{1-2/k}}{y_v^{1-2/k}} \right)^n C^{j-1,k}_{2n+j'} \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](\tau; w-v)$$

RNS characters of the minimal N=2 model

$$y_x = e^{2\pi i x}$$

Open Problems

1. Is there any sensible generalization to fivebranes with mixed ADE structure?

2. Is it possible to use this construction to understand the group actions and or to construct explicitly the modules implied by Umbal Moonshine?

3. Is there a physical explanation for the presence of Niemeier lattices, that is rank 24, even self-dual lattices?

4. Is there an algebraic structure underlying Umbral Moonshine analogous to the VOA structure underlying Monstrous Moonshine?

5. Is there some physical insight to be obtained from an understanding of why these large, often sporadic groups, appear in some supersymmetric string compactifications?

Conclusions

1. The fundamental mystery of moonshine remains, but some interesting new points of view are being developed in which spacetime supersymmetry plays an important role.

2. Non-compact SCFT such as those associated with fivebranes provide a natural way to generate mock modular forms through the elliptic genus and BPS state counting.

3. While I'm not sure what it is, I feel confident that these new moonshine phenomena have something interesting to teach us. They involve many fundamental mathematical and physical structures that appear to be tied together in novel ways.