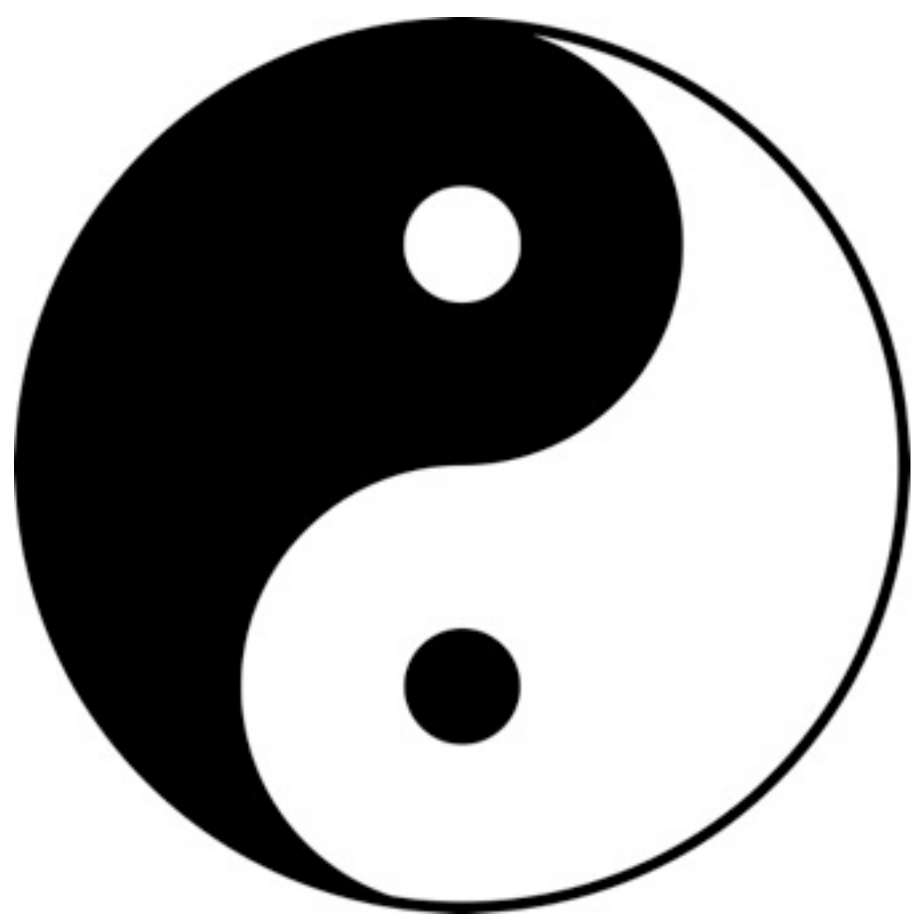
The Geometry of Duality



Double Field Theory

Hull & Zwiebach

- From sector of String Field Theory. Features some stringy physics, including T-duality, in simpler setting
- Strings see a doubled space-time
- Needed for non-geometric backgrounds
- Doubled space fully dynamic
- *Strong constraint* restricts to subsector in which extra coordinates auxiliary: get conventional field theory locally. Duality covariant sugra.

What is DFT geometry?

- Recent work: find finite gauge transformations & use to understand doubled geometry
- Hohm & Zwiebach: finite gauge transformations with non-associative composition. Non-associative geometry?
- Park; Berman, Cederwall & Perry: Manifold, but finite transformations only work up to certain local symmetries. Effectively works only for subgroup of gauge group. Gerbe structure?

Double trouble?

- CMH: doubled space from string theory is manifold, even for non-geometric backgrounds, giving different picture
- Recent proposals: try to relate finite DFT gauge transformations to diffeomorphisms of doubled space.
- Problems arise as these are different groups
- Constant 'metric' η in DFT. Is doubled geometry flat?

New Results:

arXiv:1406.7794

- Simple explicit form of finite gauge transformations
- Associative, works for full symmetry group
- Doubled space is a manifold, not flat
- Gives geometric understanding of 'generalised tensors' & relation to generalised geometry
- Transition functions give global picture

Strings on T^d

$$X = X_L(\sigma + \tau) + X_R(\sigma - \tau), \qquad \tilde{X} = X_L - X_R$$

X conjugate to momentum, $\tilde{X}$ to winding no.

$$dX = *d\tilde{X} \qquad\qquad \partial_a X = \epsilon_{ab}\partial^b \tilde{X}$$

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Need "auxiliary" $\tilde{X}$ for interacting theory

i) Vertex operators $e^{ik_L \cdot X_L}, \quad e^{ik_R \cdot X_R}$

ii) String field Kugo & Zwiebach $\Phi[x, \tilde{x}, a, \tilde{a}]$

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Strings on torus see **DOUBLED GEOMETRY**!

Doubled Torus 2d coordinates

Transform linearly under $O(d, d; \mathbb{Z})$

$$X \equiv \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix}$$

Sigma model on doubled torus Tseytlin; Hull

T-duality

- Takes S^1 of radius R to S^1 of radius 1/R
- Exchanges momentum p and winding w
- Exchanges S^1 coordinate X and dual S^1 coordinate $\tilde{X}$
- Acts on "doubled circle" with coordinates $(X, \tilde{X})$
- On d torus, T-duality group $O(d, d; \mathbb{Z})$
- Stringy symmetry, absent in field theory

Strings on a Torus

- States: momentum p, winding w
- String: Infinite set of fields $\psi(p,w)$
- Fourier transform to doubled space: $\psi(x,\tilde{x})$
- "Double Field Theory" from closed string field theory. Some non-locality in doubled space
- Subsector? e.g. $g_{ij}(x,\tilde{x}),\ b_{ij}(x,\tilde{x}),\ \phi(x,\tilde{x})$

Double Field Theory

(With weak section condition, not strong one)

- Double field theory on doubled torus
- General solution of string theory: involves doubled fields $\psi(x, \tilde{x})$
- DFT needed for non-geometric backgrounds
- *Real* dependence on *full* doubled geometry, dual dimensions not auxiliary or gauge artifact. Double geom. *physical* and *dynamical*

Hull & Zwiebach

DFT gives O(D,D) covariant formulation

O(D,D) Covariant Notation

$$X^M \equiv \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix} \qquad \partial_M \equiv \begin{pmatrix} \partial^i \\ \partial_i \end{pmatrix}$$

$$\eta_{MN} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad M = 1, ..., 2D$$

$$\Delta \equiv \frac{\partial^2}{\partial x^i \partial \tilde{x}_i} = \frac{1}{2}\partial^M \partial_M$$

Constraint $\partial^M \partial_M A = 0$

on all fields and parameters

Weak Constraint or weak section condition

Arises from SFT constraint

$$L_0^- \Psi = 0, \qquad L_0^- = L_0 - \bar{L}_0$$

Projectors and Cocycles

Naive product of constrained fields doesn't satisfy constraint

$$L_0^- \Psi_1 = 0, L_0^- \Psi_2 = 0 \quad \text{but} \quad L_0^- (\Psi_1 \Psi_2) \neq 0$$

$$\Delta A = 0, \Delta B = 0 \quad \text{but} \quad \Delta (AB) \neq 0$$

String product has explicit projection

Leads to a symmetry that is not a Lie algebra, but is a homotopy lie algebra.

Double field theory requires projections.

SFT has non-local cocycles in vertices, DFT should too

Cocycles and projectors not needed in cubic action

- Weakly constrained DFT non-local
- ALL doubled geometry dynamical, evolution in all doubled dimensions
- Restrict to simpler theory: STRONG CONSTRAINT
- Fields then depend on only half the doubled coordinates
- Locally, just conventional SUGRA written in duality symmetric form

Strong Constraint for DFT Hohm, H &Z

$$\partial^M \partial_M (AB) = 0 \qquad\qquad (\partial^M A)\,(\partial_M B) = 0$$

on all fields and parameters

If impose this, then it implies weak form, but product of constrained fields satisfies constraint.

This gives **Restricted DFT,** a subtheory of DFT

Locally, it implies fields only depend on at most half of the coordinates, fields are restricted to null subspace N.
Looks like conventional field theory on subspace N

- If fields supported only on submanifold N of doubled space M, recover Siegel's duality covariant form of (super)gravity on N

- In general get this only locally. In each 2D-dim patch of doubled space, fields supported on D-dim sub-patch, but sub-patches don't fit together to form a manifold with smooth fields.

- In string theory, T-duality acts on torus or fibres of torus fibration, relates local modes and winding
- Winding modes: doubling of torus or fibres
- Other topologies may not have windings, or have different numbers of momenta and windings. No T-duality. No doubling?
- DFT 'background independent' HHZ. Can write on doubling of any space. What is double if not derived from string theory?

Generalised Metric Formulation

Hohm, H &Z

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij} - b_{ik}g^{kl}b_{lj} \end{pmatrix} .$$

2 Metrics on double space $\mathcal{H}_{MN},\ \eta_{MN}$

$$\mathcal{H}^{MN} \equiv \eta^{MP}\mathcal{H}_{PQ}\eta^{QN}$$

Constrained metric $\mathcal{H}^{MP}\mathcal{H}_{PN} = \delta^{M}{}_{N}$

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Covariant O(D,D) Transformation

$$h^P{}_M h^Q{}_N \mathcal{H}'_{PQ}(X') = \mathcal{H}_{MN}(X)$$

$$X' = hX \qquad h \in O(D,D)$$

O(D,D) covariant action

$$S = \int dx d\tilde{x}\, e^{-2d}\, L$$

$$L = \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\,\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_N\mathcal{H}^{KL}\,\partial_L\mathcal{H}_{MK}$$
$$- 2\,\partial_M d\,\partial_N\mathcal{H}^{MN} + 4\mathcal{H}^{MN}\,\partial_M d\,\partial_N d$$

Gauge Transformation

$$\delta_\xi\mathcal{H}^{MN} = \xi^P\partial_P\mathcal{H}^{MN}$$
$$+ (\partial^M\xi_P - \partial_P\xi^M)\,\mathcal{H}^{PN} + (\partial^N\xi_P - \partial_P\xi^N)\,\mathcal{H}^{MP}$$

Rewrite as "Generalised Lie Derivative"

$$\delta_\xi\mathcal{H}^{MN} = \widehat{\mathcal{L}}_\xi\mathcal{H}^{MN}$$

Generalised Lie Derivative

$$A_{N_1 \dots}^{M_1 \dots}$$

$$\widehat{\mathcal{L}}_\xi A_M{}^N \equiv \xi^P \partial_P A_M{}^N$$
$$+(\partial_M \xi^P - \partial^P \xi_M) A_P{}^N + (\partial^N \xi_P - \partial_P \xi^N) A_M{}^P$$

$$\widehat{\mathcal{L}}_\xi A_M{}^N = \mathcal{L}_\xi A_M{}^N - \eta^{PQ} \eta_{MR}\, \partial_Q \xi^R\, A_P{}^N$$
$$+ \eta_{PQ} \eta^{NR}\, \partial_R \xi^Q\, A_M{}^P$$

Gauge Algebra

Parameters Σ^M

Gauge Algebra $\quad [\delta_{\Sigma_1}, \delta_{\Sigma_2}] = \delta_{[\Sigma_1,\Sigma_2]_C}$

$$\left[\, \widehat{\mathcal{L}}_{\xi_1} \,, \widehat{\mathcal{L}}_{\xi_2} \,\right] = -\widehat{\mathcal{L}}_{[\xi_1,\xi_2]_{\rm C}}$$

C-Bracket:

$$[\Sigma_1, \Sigma_2]_C \equiv \; [\Sigma_1, \Sigma_2] - \frac{1}{2}\,\eta^{MN}\eta_{PQ}\,\Sigma^P_{[1}\,\partial_N\,\Sigma^Q_{2]}$$

Lie bracket + metric term

Parameters $\Sigma^M(X)$ restricted to N
Decompose into vector + 1-form on N
C-bracket reduces to Courant bracket on N

Same covariant form of gauge algebra found in similar context by Siegel

Jacobi Identities not satisfied!

$$J(\Sigma_1, \Sigma_2, \Sigma_3) \equiv [\,[\Sigma_1, \Sigma_2]\,, \Sigma_3\,] + \text{cyclic} \neq 0$$

for both C-bracket and Courant-bracket

How can bracket be realised as a symmetry algebra?

$$[\,[\delta_{\Sigma_1}, \delta_{\Sigma_2}]\,, \delta_{\Sigma_3}\,] + \text{cyclic} = \delta_{J(\Sigma_1, \Sigma_2, \Sigma_3)}$$

Symmetry is Reducible

Parameters of the form $\Sigma^M = \eta^{MN}\partial_N \chi$
do not act

Gauge algebra determined up to such transformations

cf 2-form gauge field $\delta B = d\alpha$
Parameters of the form $\alpha = d\beta$
do not act

Symmetry is Reducible

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Gauge algebra determined up to such transformations

cf 2-form gauge field $\delta B = d\alpha$
Parameters of the form $\alpha = d\beta$
do not act

Resolution:

$$J(\Sigma_1, \Sigma_2, \Sigma_3)^M = \eta^{MN}\partial_N \chi$$

$\delta_{J(\Sigma_1,\Sigma_2,\Sigma_3)}$ does not act on fields

What is the Geometry of Generalised Tensors?

Doubled space coordinates $$X^M = \begin{pmatrix} x^m \\ \tilde{x}_m \end{pmatrix}$$

O(D,D) covariant vectors and tensors

$$V^M = \begin{pmatrix} v^m \\ \tilde{v}_m \end{pmatrix} \qquad \mathcal{H}_{MN}$$

Suggestive of tensors on doubled space, but transformations not those of diffeomorphisms on doubled space, as generated by generalised Lie derivative, not usual Lie derivative.

If not tensors on doubled space, what are they?

Finite transformations

Not diffeomorphisms of doubled space, as algebra given by C-bracket, not Lie bracket.

What do you get by exponentiating infinitesimal transformations?

Hohm, Zwiebach

cf exponentiating usual Lie derivative

$$A'_m(x) \; = \; e^{\mathcal{L}_\xi} \, A_m(x)$$

gives transformations induced by coordinate transformation

$$x'^m \; = \; e^{-\xi^k \partial_k} \, x^m$$

HZ write finite transformations for DFT in form with

$$X \to X' = f(X)$$

and generalised vectors transforming as

$$A'_M(X') \;=\; \mathcal{F}_M{}^N A_N(X)$$

$$\mathcal{F}_M{}^N \;\equiv\; \frac{1}{2}\Big(\frac{\partial X^P}{\partial X'^M}\,\frac{\partial X'_P}{\partial X_N} + \frac{\partial X'_M}{\partial X_P}\,\frac{\partial X^N}{\partial X'^P}\Big)$$

For conventional diffeos, would have

$$\mathcal{F}_M{}^N = \frac{\partial X^N}{\partial X'^M}$$

Important property: η_{MN} invariant

Looks a bit like a conventional geometry.
But there's a catch....
Exponentiating gen. Lie derivative

$$A'_M(X) \;=\; e^{\widehat{\mathcal{L}}_\xi}\, A_M(X)\,,$$

gives transformations of fields that form a group
(violation of Jacobi's doesn't act on fields)

These induce transformations of coordinates

$$X'^M \;=\; e^{-\Theta^K(\xi)\partial_K}\, X^M \qquad\qquad \Theta^K(\xi) \;\equiv\; \xi^K + \mathcal{O}(\xi^3)\,,$$

Not a group. Strange composition law.
Non-associative geometry? Hohm, Lust, Zwiebach

Algebraic Structure

Write parameters $\xi^M(X)$ as ξ^A

Composite index A=(M,X) combining discrete index M and continuous variables X

C-bracket defines constants

$$\left([\xi_1, \xi_2]_C\right)^A = -2 f_{BC}{}^A \xi_1^B \xi_2^C$$

Use as structure constants for closed algebra **k**

$$[T_A, T_B] = f_{AB}{}^C T_C$$

Not Lie:

$$[[T_A, T_B], T_C] + \text{cyclic permutations} = g_{ABC}{}^D T_D$$

Finite transformations $k(\xi)$

give algebra K with multiplication

$$k_1 \cdot k_2 = k_{12}$$

$$k(\xi_1) \cdot k(\xi_2) = k(\xi_{12})$$

For infinitesimal parameters $k(\xi) \sim 1 + \xi^A T_A + \ldots$

$$\xi_{12} = \xi_1 + \xi_2 - \frac{1}{2}[\xi_1, \xi_2]_C + \ldots$$

Failure of C-bracket Jacobi identities $\Longrightarrow$ Non-associativity

$$(k_1 \cdot k_2) \cdot k_3 \neq k_1 \cdot (k_2 \cdot k_3)$$

Representations on Generalised Tensors?

If represent by generalised Lie derivative acting on Generalised Tensors

$$k(\xi) \longrightarrow R(k) = \exp(\widehat{\mathcal{L}}_\xi)$$

$$T'(X) = \exp(\widehat{\mathcal{L}}_\xi) T(X)$$

Perfectly consistent

$$R(k_1)R(k_2) = R(k_1 \cdot k_2)$$

$$\Big(R(k_1)R(k_2)\Big)\, R(k_3) = R(k_1)\, \Big(R(k_2)R(k_3)\Big)$$

Key point is redundant gauge transformations z are represented trivially, R(z)=1. R(k) generate Lie group of DFT gauge symmetries, the quotient of K by z's

Hohm-Zwiebach proposal:

Represent K by new transformations S(k) acting on Generalised Tensors

$$T'(X') = \mathcal{F}(X, X')\, T(X)$$

Idea is to try to rewrite active transformation as an passive one taking X to X'(X).
HZ find transformation reproducing R(k) transformation.

But now apparent inconsistency as

$$S(k_1)S(k_2) \neq S(k_1 \cdot k_2)$$

To deal with this, they propose new composition of transformations

$$S(k_1) \star S(k_2) \equiv S(k_1 \cdot k_2)$$

Non-associativity of K

$$(k_1 \cdot k_2) \cdot k_3 \neq k_1 \cdot (k_2 \cdot k_3)$$

leads to non-associativity of star product:

$$\Big(S(k_1) \star S(k_2)\Big) \star S(k_3) \neq S(k_1) \star \Big(S(k_2) \star S(k_3)\Big)$$

In particular, each $k(\xi)$

gives a coordinate transformation

$$X \to X'(X)$$

$$X'^M \equiv e^{-\Theta^P(\xi)\partial_P} X^M$$

$$\Theta^M = \xi^M + \frac{1}{12}(\xi^N \partial_N \xi^L)\partial^M \xi_L + \mathcal{O}(\xi^4)$$

These coordinate transformations are composed not in the usual associative way $X''(X'(X))$

but are combined non-associatively using a star product.

Does this imply some kind of non-associative geometry?

Hohm, Lust, Zwiebach

Then each k in K is mapped to a diffeomorphism s(k) of the doubled spacetime

$$s(k(\xi)) = e^{-\Theta^P(\xi)\partial_P}$$

and these diffeomorphisms are not combined using the multiplication of the diffeomorphism group, but according to a non-associative star product.

This then attempts to impose a new algebraic structure on the set of diffeomorphisms, and this raises a number of issues.

TOY MODEL

Consider 2 different Lie groups G,G' of same dimension

e.g. $$G = SU(2) \times SU(2) \times SU(2), \quad G' = GL(3,\mathbb{R})$$

Consider a non-homomorphic map

$$S : G \to G' \qquad (G,\cdot),\ (G',\circ)$$

$$S(g_1) \circ S(g_2) \neq S(g_1 \cdot g_2)$$

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e.g. if G,G' resepectively have generators T_A, t_A

$$g = \exp(\xi^A T_A) \in G \to S(g) = \exp(f(\xi)^A t_A) \in G'$$

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e.g. if G,G' resepectively have generators T_A, t_A

$$g = \exp(\xi^A T_A) \in G \to S(g) = \exp(f(\xi)^A t_A) \in G'$$

Can then formally try to define star product on G'

$$S(g_1) \star S(g_2) \equiv S(g_1 \cdot g_2)$$

- Attempts to define a G multiplication on points of G’

- Attempts to ‘realise’ G transformations as G’ ones

- Algebraic structure of Lie group determines geometry. Can’t impose group on ‘wrong’ geometry

- Similar to ‘realising’ DFT gauge transformations as diffeomorphisms of doubled space?

- Park; Berman, Cederwall, Perry map DFT gauge transformations to diffeomorphisms of doubled space, essentially by restricting to subgroup for which this is possible

- Another way to understand finite transformations?

- What is finite transformation of generalised tensors?

- What is the geometry significance of generalised tensors?

Constraint $\partial^M \partial_M A = 0$

Strong Constraint for restricted DFT

$$\partial^M \partial_M (AB) = 0 \qquad\qquad (\partial^M A)\,(\partial_M B) = 0$$

Generic solution in patch $\hat{U}$: fields and parameters independent of half the coordinates:

$$\tilde{\partial}^i = 0$$

$$X^M = \begin{pmatrix} x^m \\ \tilde{x}_m \end{pmatrix} \qquad \partial_M = \begin{pmatrix} \partial_m \\ \tilde{\partial}^m \end{pmatrix} \qquad \eta_{MN} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

Fields live on null patch U, coordinates x: $\phi(x^m)$

U 'physical' spacetime

Vectors

$$V^M = \begin{pmatrix} v^m \\ \tilde{v}_m \end{pmatrix}$$

Generalised Lie derivative

$$\widehat{\mathcal{L}}_V W^M \;=\; V^P \partial_P W^M + W^P \left(\partial^M V_P - \partial_P V^M\right)$$

Vectors $$V^M = \begin{pmatrix} v^m \\ \tilde{v}_m \end{pmatrix}$$

Generalised Lie derivative

$$\widehat{\mathcal{L}}_V W^M \ = \ V^P \partial_P W^M + W^P \left(\partial^M V_P \ - \partial_P V^M\right)$$

has the components

$$(\widehat{\mathcal{L}}_V W)^m = \mathcal{L}_v w^m$$

$$(\widehat{\mathcal{L}}_V W)_m = \mathcal{L}_v \tilde{w}_m + w^p(\partial_m \tilde{v}_p \ - \partial_p \tilde{v}_m)$$

$\mathcal{L}_v$ is usual Lie derivative

$$\mathcal{L}_v w^m = v^p \partial_p w^m - w^p \partial_p v^m$$
$$\mathcal{L}_v \tilde{w}_m = v^p \partial_p \tilde{w}_m + \tilde{w}_p \partial_m v^p$$

Under infinitesimal transformation $\delta W^M = \widehat{\mathcal{L}}_V W^M$

$$\delta w^m = \mathcal{L}_v w^m$$

$$\delta \tilde{w}_m = \mathcal{L}_v \tilde{w}_m + w^p(\partial_m \tilde{v}_p - \partial_p \tilde{v}_m)$$

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Introduce a gerbe connection b with transformations

$$\delta_v b_{mn} = \mathcal{L}_v b_{mn} + \partial_m \tilde{v}_n - \partial_n \tilde{v}_m$$

Define $\hat{w}_m = \tilde{w}_m - b_{mn} w^n$

Under infinitesimal transformation $\delta W^M = \widehat{\mathcal{L}}_V W^M$

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Then $\delta \hat{w}_m = \mathcal{L}_v \hat{w}_m$

Under infinitesimal transformation $\delta W^M = \widehat{\mathcal{L}}_V W^M$

$$\delta w^m = \mathcal{L}_v w^m$$
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Introduce a gerbe connection b with transformations

$$\delta_v b_{mn} = \mathcal{L}_v b_{mn} + \partial_m \tilde{v}_n - \partial_n \tilde{v}_m$$

Define $\hat{w}_m = \tilde{w}_m - b_{mn} w^n$

Then $\delta \hat{w}_m = \mathcal{L}_v \hat{w}_m$

$\hat{w}$ transforms as 1-form under v-transformations and is invariant under $\tilde{v}$ transformations!

COVARIANT TRANSFORMATIONS

Then given $$W^M = \begin{pmatrix} w^m \\ \tilde{w}_m \end{pmatrix}$$

can define $$\hat{W}^M = \begin{pmatrix} w^m \\ \hat{w}_m \end{pmatrix} = \begin{pmatrix} w^m \\ \tilde{w}_m - b_{mn} w^n \end{pmatrix}$$

$$\delta \hat{W}^M = \mathcal{L}_v \hat{W}^M$$

It is invariant under $\tilde{v}$ transformations

COVARIANT TRANSFORMATIONS

Then given $W^M = \begin{pmatrix} w^m \\ \tilde{w}_m \end{pmatrix}$

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$$\delta \hat{W}^M = \mathcal{L}_v \hat{W}^M$$

It is invariant under $\tilde{v}$ transformations

Gives finite transformations!

$$x \to x'(x) = e^{-v^m \partial_m} x$$

$$w'^m(x') = w^n(x) \frac{\partial x'^m}{\partial x^n} \qquad \hat{w}'_m(x') = \hat{w}_n(x) \frac{\partial x^n}{\partial x'^m}$$

Can also find the transformation of $\tilde{w}$

Standard finite transformations of gerbe connection:

$$b'_{mn}(x') = [b_{pq}(x) + (\partial_p \tilde{v}_q - \partial_q \tilde{v}_p)(x)] \frac{\partial x^p}{\partial x'^m} \frac{\partial x^q}{\partial x'^n}$$

gives

$$\tilde{w}'_m(x') = \left[\tilde{w}_n(x) + (\partial_n \tilde{v}_q - \partial_q \tilde{v}_n) w^q(x)\right] \frac{\partial x^n}{\partial x'^m}$$

$$w'^m(x') = w^n(x) \frac{\partial x'^m}{\partial x^n}$$

DFT and GENERALISED GEOMETRY

Consider case fields restricted to submanifold N of M

w transforms as a tangent vector on N and $\hat{w}$ transforms as a cotangent vector under diff(N).
Both invariant under $\tilde{v}$ transformations.

$w \oplus \hat{w}$ is a section of $(T \oplus T^*)N$

This is Hitchin's generalised tangent bundle on N

$$w \oplus \tilde{w}$$

is section of E, which is $T \oplus T^*$ twisted by a gerbe

$$0 \to T^* \to E \to T \to 0$$

Then ‘generalized vectors’ $W^M = \begin{pmatrix} w^m \\ \tilde{w}_m \end{pmatrix}$

are not really vectors on doubled space, but are sections of generalised tangent bundle over ‘physical space’ N, twisted by a gerbe

$v^m(x)$ symmetries are diffeomorphisms of N
$\tilde{v}_m(x)$ symmetries are b-field gauge transformations on N

Gauge symmetry of DFT

$$\mathrm{Diff}(N) \ltimes \Lambda^2_{closed}(N)$$

Global O(D,D)

2D dimensional doubled space M, D dim. subspace N

3 kinds of vectors $V^M(X)$

Vector fields on M:

Sections of TM,

transform under diff(M)

Hatted generalised vector fields $\hat{W}$ on M:

Sections of $(T \oplus T^*)N$

transform under diff(N)

Generalised vector fields W on M

Sections of E(N)

transform under $\mathrm{Diff}(N) \ltimes \Lambda^2_{closed}(N)$

Extends to tensors, generalised tensors and untwisted generalised tensors

Generalised Metric

$$\mathcal{H}_{MN} = \begin{pmatrix} g_{mn} - b_{mk}g^{kl}b_{ln} & b_{mk}g^{kn} \\ -g^{mk}b_{kn} & g^{mn} \end{pmatrix}$$

Finite transformations give usual ones for g,b

Untwisted form of generalised metric

$$\hat{\mathcal{H}}_{MN} = \begin{pmatrix} g_{mn} & 0 \\ 0 & g^{mn} \end{pmatrix}$$

Natural metric on $T \oplus T^*$

Constant O(D,D) Metric

Matrix with constant components:

$$\eta_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

If this is tensor on M, then it is flat metric and this would greatly restrict possible M. Not invariant under Diff(M)

Constant O(D,D) Metric

Matrix with constant components:

$$\eta_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

If this is tensor on M, then it is flat metric and this would greatly restrict possible M. Not invariant under Diff(M)

If it is generalised tensor, section of $E^* \otimes E^*(N)$

$$\hat{\eta}_{MN} = \eta_{MN}$$

Invariant under DFT gauge transformations, natural object in DFT. Metric for E(N), not T(M)

No restriction on geometry

Conclusions

- Doubled space M is manifold, need not be flat
- If fields live on submanifold N, DFT gives conventional field theory on N
- Generalised tensors in $E \otimes E \cdots \otimes E(N)$ not $T \otimes T \cdots \otimes T(M)$
- E(N) is $(T \oplus T^*)N$ twisted by gerbe
- DFT gauge transformations just diffeos and b-field gauge transformations on N

- DFT: sugra in duality symmetric formulation, using generalised geometry on N

- Covariant formulation of generalised geometry, indep. of choice of duality frame

- More generally, this applies locally in patches. Use DFT gauge and O(D,D) symmetries in transition functions.

- DFT extends field theory to non-geometric spaces: T-folds, with T-duality transition functions.