RECENT DEVELOPMENTS IN STRING THEORY

International Conference

Ascona, 21 – 25 July 2014

Quantum spectral curve of N=4 SYM and its BFKL limit

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ENS

UPMC PARIS UNIVERSITAS

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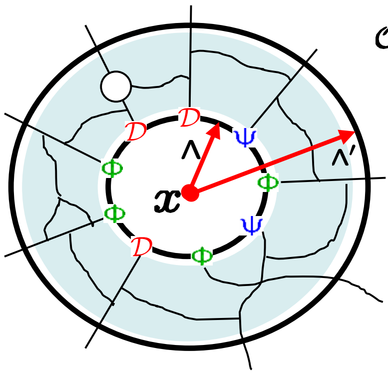
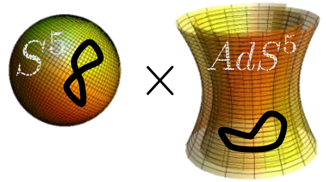
Integrability of AdS$_5$/CFT$_4$ spectral problem

$$\mathcal{S}_{SYM} = \frac{N}{g^2}\int d^4x\, \mathrm{Tr}\left(F^2 + (\mathcal{D}\Phi)^2 + \bar{\Psi}\mathcal{D}\Psi + \bar{\Psi}\Phi\Psi + [\Phi,\Phi]^2\right)$$



$$\mathcal{O}_j^{\Lambda'}(x) = \left[\left(\frac{\Lambda'}{\Lambda}\right)^{\Delta}\right]_{jk} \mathcal{O}_k^{\Lambda}(x)$$



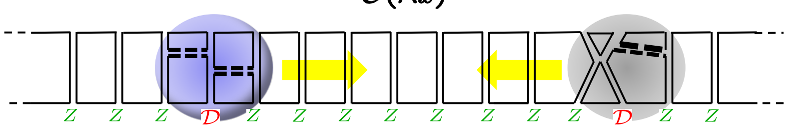
Minahan, Zarembo
Beisert,Kristjansen,Staudacher

Weak coupling expansion for SYM anomalous dimensions. Perturbative integrability: Spin chain

Metsaev-Tseytlin
Bena, Roiban, Polchinski
V.K.,Marshakov, Minahan, Zarembo
Beisert, V.K.,Sakai, Zarembo

Strong coupling from AdS-dual – classical superstring sigma model Classical integrability, algebraic curve

Beisert,Eden, Staudacher
Janik,
Beisert

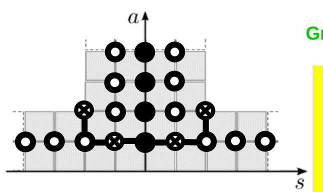


S-matrix Asymptotic Bethe ansatz

$$e^{iLp_j}\prod_{k\neq j=1}^{M}\hat{S}(p_j,p_k) = I$$

Gromov, Kazakov, Vieira

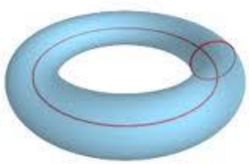
$$Y_{a,s}(u+\tfrac{i}{2})\,Y_{a,s}(u-\tfrac{i}{2}) = \frac{[1+Y_{a,s+1}]\,[1+Y_{a,s-1}]}{\left[1+\frac{1}{Y_{a+1,s}}\right]\left[1+\frac{1}{Y_{a-1,s}}\right]}$$



PSU(2,2|4)

Y-system + analyticity Thermodynamic Bethe ansatz (exact!)

Bombardelli,Fioravanti,Tateo
Gromov,V.K.,Kozak,Vieira
Arutyunov,Frolov



$$\log Y_{a,s} = L\,\delta_{a,0}\frac{\partial}{\partial u}\tilde{\epsilon}_a + \sum_{a',s'} K_{a,s;a',s'} * \log\left(1+Y_{a',s'}(u)\right)$$

Gromov, V.K., Tsuboi,
Gromov, V.K., Leurent, Tsuboi
V.K., Leurent, Volin

Cavaglia,Fioravanti,Tateo
Hegedus,Balog

Wronskian solution of Y/T-system via Baxter's Q-functions Finite system of integral non-linear equations (FiNLIE)

Gromov, V.K., Leurent, Volin

Gromov, V.K., Leurent, Volin

Q-system and Riemann-Hilbert eqs. for quantum spectral curve (P-μ)

BFKL dimensions for twist-2 operator

Maldacena
Gubser, Klebanov, Polyakov
Witten

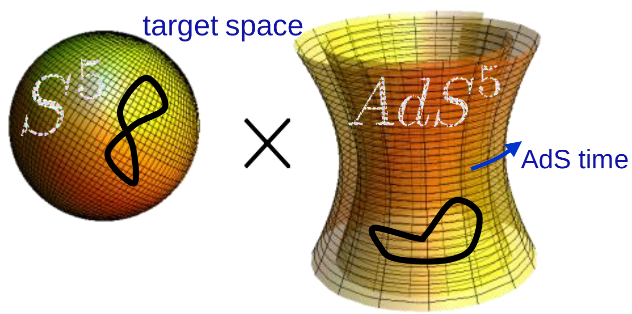
SYM is dual to supersting σ-model on $AdS_5 \times S^5$

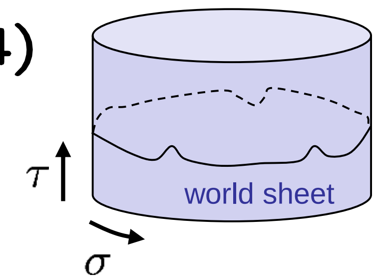
- Super-conformal N=4 SYM symmetry PSU(2,2|4) → isometry of string target space

- 2D σ-model on a coset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ $\quad G(\sigma,\tau) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in sl(4|4)$

fermions

$$J = -G^{-1}dG = J^{(0)} + J^{(1)} + J^{(2)} + J^{(3)} \in su(2,2|4)$$





- Metsaev-Tseytlin action

$$S_{MT} = g \ \mathrm{str} \int_{\mathcal{M}_2} \left[J^{(2)} \wedge *J^{(2)} - J^{(1)} \wedge J^{(3)} \right]$$

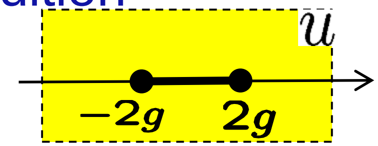
Dimension of YM operator $\Delta_A(g)$ **=** Energy of a string state

Classical integrability of superstring on $AdS_5 \times S^5$

- String eqs. of motion and constraints recast into flatness condition

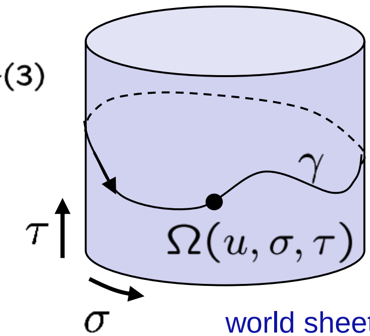
Mikhailov,Zakharov
Bena,Roiban,Polchinski

$$[(\partial_0 + \mathcal{A}_0(u)), (\partial_1 + \mathcal{A}_1(u))] = 0$$



for Lax connection - ***double valued*** w.r.t. spectral parameter u

$$\mathcal{A}(u) = J^{(0)} + \frac{u}{\sqrt{u^2-4g^2}} J^{(2)} + \frac{g}{\sqrt{u^2-4g^2}} *J^{(2)} + \left(\frac{u+2g}{u-2g}\right)^{1/4} J^{(1)} + \left(\frac{u-2g}{u+2g}\right)^{1/4} J^{(3)}$$



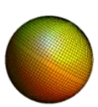
- Monodromy matrix $\quad \Omega(u) = P \exp \oint_\gamma \mathcal{A}(u) \quad \in PSU(2,2|4)$
  encodes infinitely many conservation lows

- Eigenvalues define quasi-momenta:

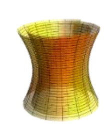
V.K.,Marshakov,Minahan,Zarembo
Beisert,V.K.,Sakai,Zarembo

$$\Omega(u) = U^{-1} \{ e^{i\hat{p}_1(u)}, e^{i\hat{p}_2(u)}, e^{i\hat{p}_3(u)}, e^{i\hat{p}_4(u)} || e^{i\check{p}_1(u)}, e^{i\check{p}_2(u)}, e^{i\check{p}_3(u)}, e^{i\check{p}_4(u)} \} U$$

- Asymptotics fixed by Cartan charges of PSU(2,2|4): $\{J_1, J_2, J_3 \,|\, \Delta, S_1, S_2\}$



$$\begin{pmatrix} \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \\ \hat{p}_4 \end{pmatrix} \simeq \frac{1}{2u} \begin{pmatrix} +J_1 + J_2 - J_3 \\ +J_1 - J_2 + J_3 \\ -J_1 + J_2 + J_3 \\ -J_1 - J_2 - J_3 \end{pmatrix}$$



$$\begin{pmatrix} \check{p}_1 \\ \check{p}_2 \\ \check{p}_3 \\ \check{p}_4 \end{pmatrix} \simeq \frac{1}{2u} \begin{pmatrix} +\Delta_1 - S_1 + S_2 \\ +\Delta_1 + S_1 - S_2 \\ -\Delta_1 - S_1 - S_2 \\ -\Delta_1 + S_1 + S_2 \end{pmatrix}$$

- Each quasi-momentum inherits the double-valuedness of Lax connection.

From classical to quantum Hirota in U(2,2|4) T-hook

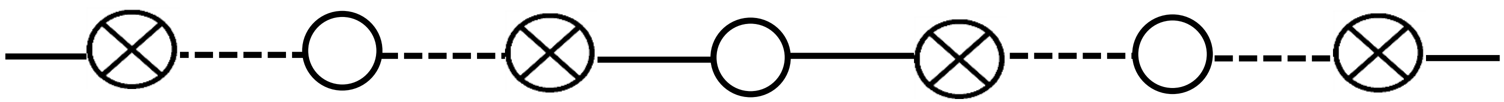
Gromov, V.K., Tsuboi
Gromov, V.K., Leurent,Tsuboi

- Quantization: replace classical spectral determinant by quantum spectral functional

$$w(p,u) = \text{sdet}\left(\mathbf{I} - e^{-ip}\cdot\Omega(u)\right) \qquad \Rightarrow \qquad \widehat{\mathcal{W}}(p,u) = \text{"Sdet"}\left(\mathbf{I} - e^{-i\,\partial_u}\cdot\hat{\Omega}(u)\right)$$

$$\boldsymbol{\Omega(u)} \Rightarrow \{\mathcal{Y}_1(u)|\mathcal{X}_1(u),\mathcal{X}_2(u)|\mathcal{Y}_2(u),\mathcal{Y}_3(u)|\mathcal{X}_3(u),\mathcal{X}_4(u)|\mathcal{Y}_4(u)\}$$

- We have to precise the order of operatorial factors along Dynkin diagram:



$$W = \left[(1-D\mathcal{Y}_1D)\frac{1}{(1-D\mathcal{X}_1D)}\frac{1}{(1-D\mathcal{X}_2D)}(1-D\mathcal{Y}_2D)\right]_+ \times \left[(1-D\mathcal{Y}_3D)\frac{1}{(1-D\mathcal{X}_3D)}\frac{1}{(1-D\mathcal{X}_4D)}(1-DY_4D)\right]_-$$
$$= \sum_{s=-\infty}^{\infty} D^s T_{1,s} D^s$$

$[\cdots]_\pm$ - expansion in $D^{\pm 1} = e^{\mp\frac{i}{2}\partial_u}$

- T-functions in general (a×s) irrep $T_{a,s}(u) = \text{Det}_{1\le k,j\le a}\, T_{1,s+k-j}\left(u + i\frac{k+j}{2}\right)$

solve the Hirota equation (and thus the Y-system in T-hook)

For spin chains :
Bazhanov,Reshetikhin
Cherednik
V.K.,Vieira (for the proof)

$$T_{a,s}\left(u+\frac{i}{2}\right)T_{a,s}\left(u-\frac{i}{2}\right) = T_{a,s-1}(u)\,T_{a,s+1}(u) + T_{a+1,s}(u)T_{a-1,s}(u)$$

- The best parameterization is in terms of Baxter-like Q-functions: Q-system

Q-system as a Grassmanian

Krichever,Lipan, Wiegmann,Zabrodin
Gromov, Vieira
V.K., Leurent, Volin.

- One-form on N single indexed Q-functions:

$$Q_{(1)} \equiv \sum_{j=1}^{N} Q_j(u)\xi^j, \qquad \{\xi^i, \xi^j\} = 0$$

- l-form encodes all Q-functions with l indices:

$$Q_{(l)} \equiv Q_{(1)}^{[-l+1]} \wedge Q_{(1)}^{[-l+3]} \wedge \ldots \wedge Q_{(1)}^{[l-1]}$$

Notations:

$$Q^{[n]} \equiv Q(u + \frac{in}{2})$$

$$Q^{\pm} \equiv Q(u \pm \frac{i}{2})$$

- Multi-index Q-function: coefficient of $\xi_{i_1} \wedge \xi_{i_2} \wedge \ldots \wedge \xi_{i_l}$

$$Q_{j_1,\ldots,j_k} = \det_{1 \le m,n \le k} Q_{j_m}^{[-1-k+2n]}$$

- Example for N=2: $Q_{(2)} = 2Q_{12}\,\xi_1 \wedge \xi_2, \qquad Q_{12} = Q_1^+ Q_2^- - Q_1^- Q_2^+$

- Notations in terms of subsets of indices:

$$\mathrm{Q}_{j_1,\ldots,j_k} \equiv \mathrm{Q}_I\,, \qquad I = \{j_1, \ldots, j_k\} \subset \{1, 2, \ldots, N\}$$

- Plücker's QQ-relations: $Q_I Q_{I,i,j} = Q_{I,i}^+ Q_{I,j}^- - Q_{I,i}^- Q_{I,j}^+$

Tsuboi
Gromov,V.K., Leurent, Tsuboi
V.K.,Leurent,Volin

(K|M)-graded Q-system

- Split the full set of K+M indices as $\{B\} \cup \{F\}$

$$B = \{1,2,\ldots,K\}, \quad F = \{K+1, K+2, \ldots, K+M\}$$

- Grading = re-labeling of F-indices (subset → complimentary subset of F)

$$Q_{I|J} \equiv \mathrm{Q}_{I, F\backslash J}\,, \quad I \in B, \;\; J \in F$$

- Examples for (4|4): $Q_{j|\emptyset} = \mathrm{Q}_{j5678}\,, \quad j = 1,2,3,4, \qquad Q_{12|57} = \mathrm{Q}_{1268}$

- Graded forms:

$$Q_{(n|p)} = \sum_{\{b\}\in B} \sum_{\{f\}\in F} Q_{b_1,b_2,\ldots,b_n|f_1,f_2,\ldots,f_p} \cdot \xi^{b_1} \wedge \xi^{b_2} \wedge \cdots \wedge \xi^{b_n} \wedge \xi^{f_1} \wedge \xi^{f_2} \wedge \cdots \wedge \xi^{f_p}$$

- New type of QQ-relations involving 2 indices of opposite grading:

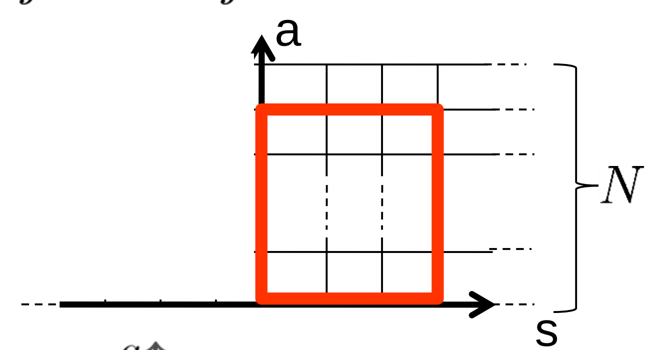
$$Q_{I|J,j} Q_{I,i|J} = Q^{+}_{I,i|J,j} Q^{-}_{I|J} - Q^{-}_{I,i|J,j} Q^{+}_{I|J}$$

Now we can label: $F = \{1,2,\ldots,M\}$

Wronskian solution of Hirota eq.

- Example: solution of Hirota equation in a band of width N in terms of exterior full-forms via 2N arbitrary functions

$$Q_j(u),\ \tilde{Q}_j(u)$$



Krichever,Lipan, Wiegmann,Zabrodin

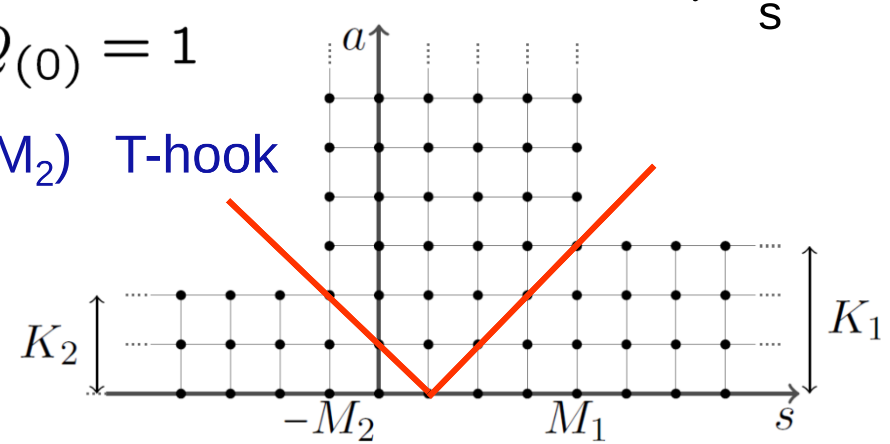
$$T_{a,s} = Q^{[s]}_{(a)} \wedge \tilde{Q}^{[-s]}_{(N-a)}$$

- For su(N) spin chain (half-strip) we impose:

$$\tilde{Q}(u) = Q^{[N]}, \qquad \tilde{Q}_{(0)} = Q_{(0)} = 1$$

- Solution of Hirota eq. for (K_1,K_2 | M_1+M_2) T-hook

Tsuboi
V.K.,Leurent,Volin



$$Q_{I_1,I_2|J} \qquad \{I_1, I_2|J\} \subset \{B_1, B_2|F\}$$

$$T_{a,s} = \begin{cases} Q^{[\tilde{s}]}_{(a,0|0)} \wedge Q^{[-\tilde{s}]}_{(K_1-a,K_2|M)} & \tilde{s} \geq \tilde{a} \\ Q^{[\tilde{a}]}_{(K_1,0|M_1-s)} \wedge Q^{[-\tilde{a}]}_{(0,K_2|M_2+s)} & \tilde{a} \geq |\tilde{s}| \\ Q^{[-\tilde{s}]}_{(K_1,K_2-a|M)} \wedge Q^{[+\tilde{s}]}_{(0,a|0)} & \tilde{s} \leq -\tilde{a} \end{cases}$$

$$\tilde{s} = s - \frac{-K_1 + K_2 + M_1 - M_2}{2}$$

$$\tilde{a} = a - \frac{K - M}{2}$$

Algebraic symmetries of Q-system

- **Hodge duality** is a simple relabeling:

$$Q^{A|J} \equiv Q_{\{1234\}\backslash A \,|\, \{1234\}\backslash J}$$

Example for (4|4): $Q^{1|134} = Q_{234|2}$

- Satisfy the same QQ-relations if we impose: $Q^{\emptyset|\emptyset} = Q_{1234|1234} = 1$
(related to unimodularity of PSU(2,2|4))

- **H-symmetry**: sl(4) × sl(4) rotation preserving QQ-relations
with i-periodic H-functions: $H^{++} = H$

$$Q_{A|J} \to \left(H_b^{[|A|+|J|]}\right)_A^{A'} \left(H_f^{[|A|+|J|]}\right)_J^{J'} Q_{I'|J'}, \qquad |X| \equiv \mathrm{Span}(X)$$

where $H_I{}^{I'} \equiv H_{i_1}{}^{i'_1} H_{i_2}{}^{i'_2} \ldots H_{i_{|I|}}{}^{i_{|I'|}}$

Examples: $Q_{i|\emptyset} \to \left(H_b^{+}\right)_i^j Q_{j|\emptyset}$ $\qquad Q_{i|j} \to (H_b)_i^{i'} \left(H_f\right)_j^{j'} Q_{i'|j'}$

*

Gromov, V.K., Leurent, Volin 2013

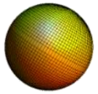
Analyticity of AdS/CFT Q-system

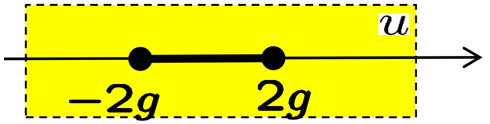
- AdS/CFT T-system is defined on (2,2|4)-hook and is solved via wronskians of Q-functions with specific analytic properties. Their simplest basic Q's:

$$\mathbf{P}_j \equiv \mathcal{Q}_{j|\emptyset}, \qquad \mathbf{Q}_j \equiv \mathcal{Q}_{\emptyset|j}, \qquad \mathbf{P}^j \equiv \mathcal{Q}^{j|\emptyset}, \qquad \mathbf{Q}^j \equiv \mathcal{Q}^{\emptyset|j} \qquad (j = 1,2,3,4)$$

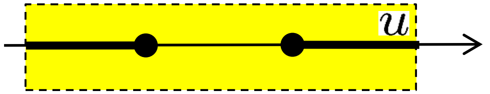
- Comparing characters of classical monodromy matrix and their quantum analogues -- T-functions, we relate these functions to classical quasimomenta

$$\mathbf{P}_j \sim (\mathbf{P}^j)^{-1} \sim \exp\left(-\int^u \widehat{p}_j(u')du'\right)$$





$$\mathbf{Q}^j \sim (\mathbf{Q}^j)^{-1} \sim \exp\left(-\int^u \breve{p}_j(u')du'\right)$$



- They inherit one-cut structure on their defining Riemann sheets (checked from TBA!)
- From asymptotics of (quasi)classical quasi-momenta:

$$\mathbf{P}_i \simeq A_i\, u^{-\tilde{M}_i}, \quad \mathbf{Q}_{\hat{i}} \simeq B_i\, u^{\hat{M}_i - 1}, \qquad \mathbf{P}^i \simeq A^i\, u^{\tilde{M}_i - 1}, \quad \mathbf{Q}^{\hat{i}} \simeq B^i\, u^{-\hat{M}_i}$$

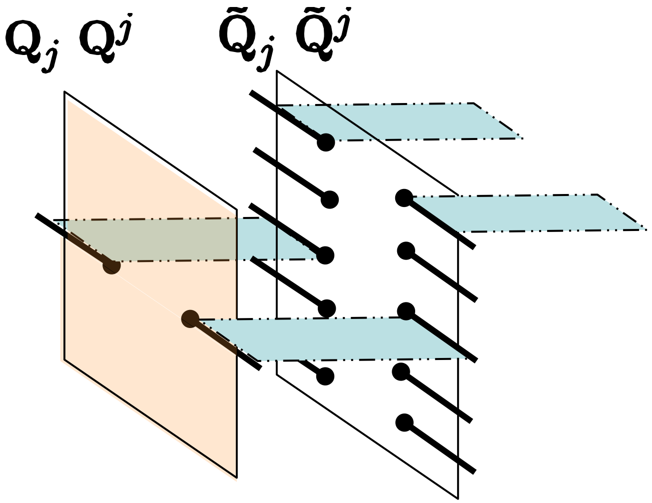
$$\tilde{M}_a = \left\{\frac{1}{2}(J_1 + J_2 - J_3 + 2), \frac{1}{2}(J_1 - J_2 + J_3), \frac{1}{2}(-J_1 + J_2 + J_3 + 2), \frac{1}{2}(-J_1 - J_2 - J_3)\right\}$$

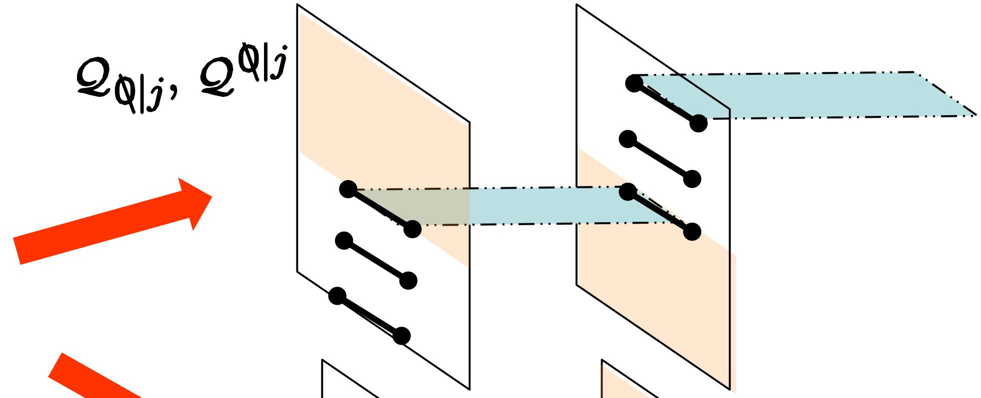
$$\hat{M}_a = \left\{\frac{1}{2}(\Delta - S_1 - S_2 + 2), \frac{1}{2}(\Delta + S_1 + S_2), \frac{1}{2}(-\Delta - S_1 + S_2 + 2), \frac{1}{2}(-\Delta + S_1 - S_2)\right\}$$

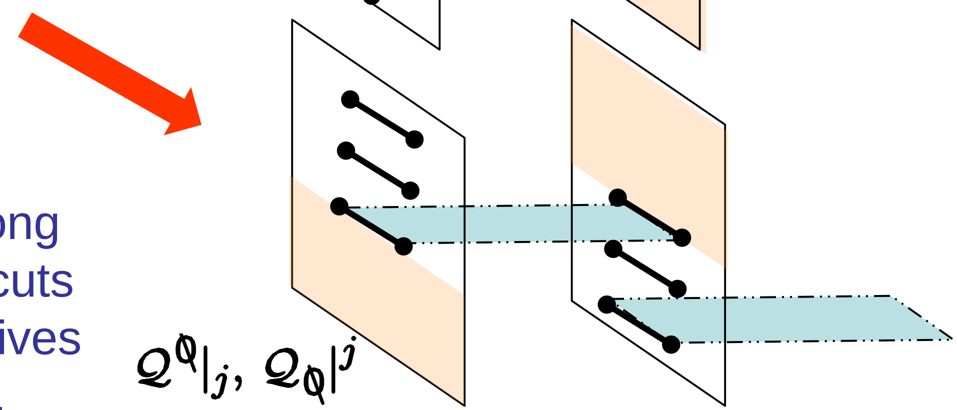
- To fix all Q-functions (and Riemann-Hilbert equations for AdS/CFT spectrum) we have to know the monodromy around the branch points.
- The very existence of Q-system imposes strong restrictions on analyticity!

H(ω)-transformation from upper- to lower-analytic Q's

- Structure of cuts of Q-functions:







- We can "flip" all short cuts to long ones going through the short cuts from above or from below. It gives the upper or lower-analytic P's.

- Q-system allows to choose all Q-functions upper-analytic or all lower-analytic. Both representations are physically equivalent → related by H-rotations with periodic coefficients rising and lowering indices $\hat{\omega}_{ab}(u+i) = \hat{\omega}_{ab}(u)$

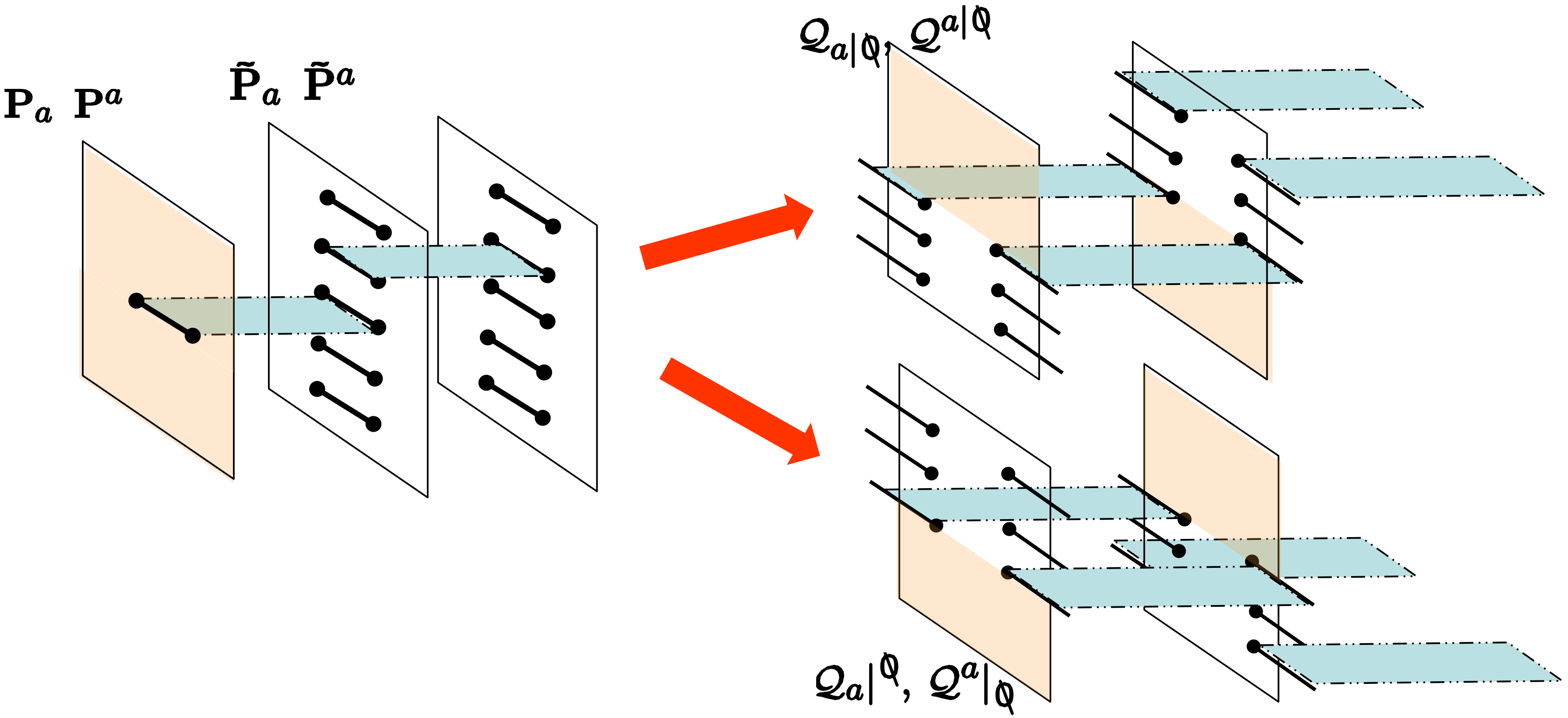
$$\mathcal{Q}_{\emptyset|j} = \omega_{jk}\mathcal{Q}_{\emptyset}|^k \qquad \mathcal{Q}_{\emptyset}|^j = \omega^{jk}\mathcal{Q}_{\emptyset|k}$$

$$\omega^{ij} = (\omega)^{-1}_{kl} = -\frac{1}{2}\epsilon^{ijkl}\omega_{kl}, \quad \mathrm{Pf}(\omega) = 1$$

- True only for 4×4 antisym. matrices: Exceptional role of PSU(2,2|4) !

H(μ)-transformation from upper- to lower-analytic Q's

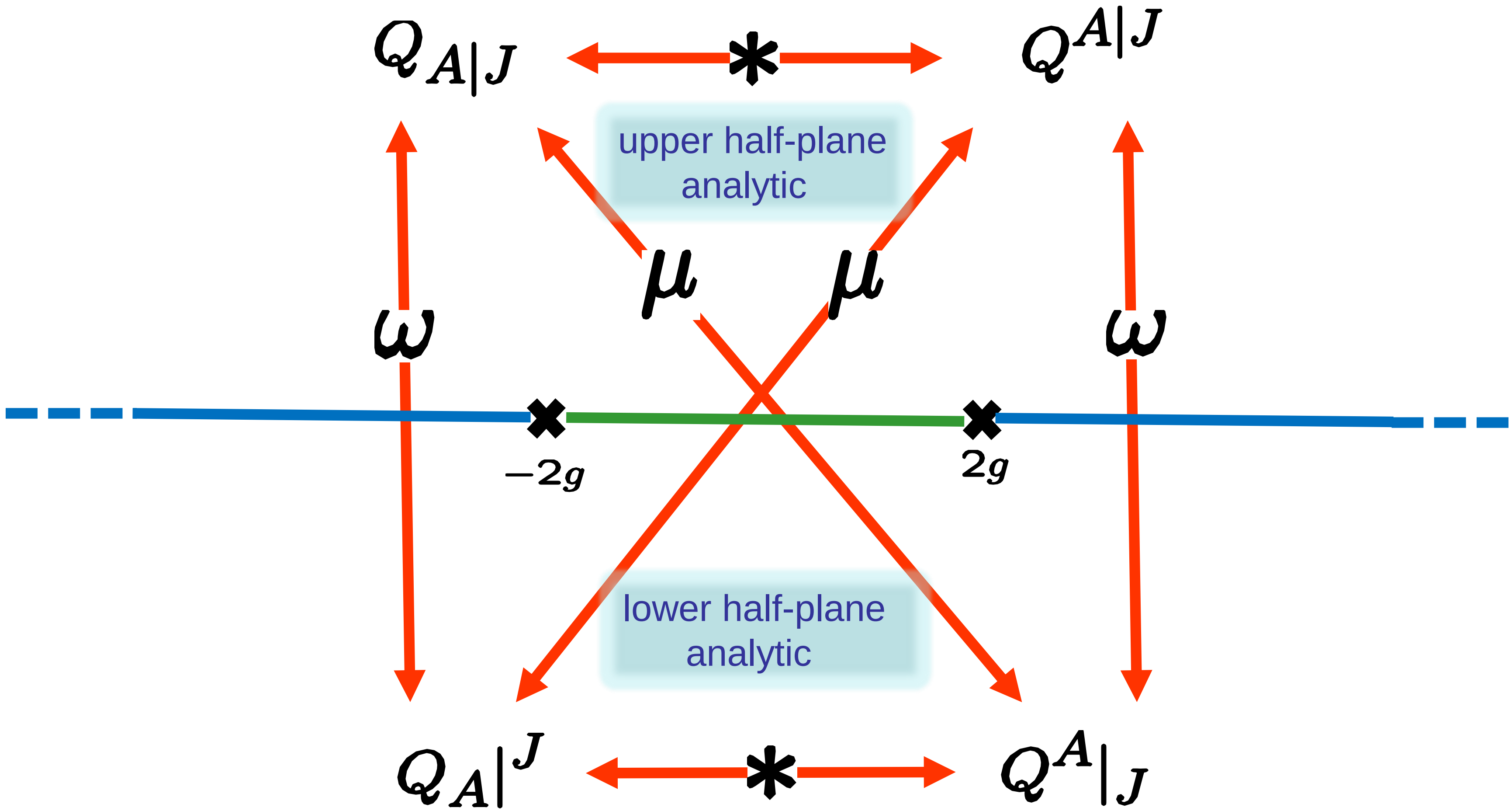
- Structure of cuts of P-functions: the same picture, but with the exchange of roles of long and short cuts



- Upper-analytic or all lower-analytic functions with long cuts related by H-rotation with periodic coefficients rising and lowering indices: $\check{\mu}_{ab}(u+i)=\check{\mu}_{ab}(u)$

$$\mathcal{Q}_{a|\emptyset}=\mu_{ab}\mathcal{Q}^b|_{\emptyset} \qquad \mathcal{Q}^a|_{\emptyset}=\mu^{ab}\mathcal{Q}_{b|\emptyset}$$

Analytic Q-system



μ - lowers or rises “bosonic” indices and flips UHP and LHP analyticity

ω - lowers or rises “fermionic” indices and flips UHP and LHP analyticity

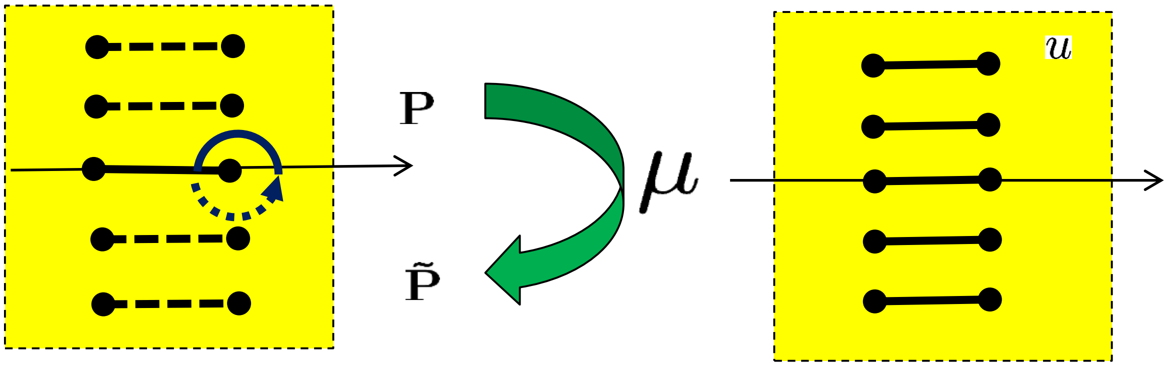
$*$ - flips all upper and lower indices by Hodge transformation

Pμ-system and reduction to SL(2) sector $\mathrm{Tr}(\nabla^S Z^L)$

- H(μ) transformation defines monodromy through short cuts:

$$\tilde{\mathbf{P}}_a = \mu_{ab}\mathbf{P}^b \qquad \tilde{\mu}_{ab}(u) = \mu_{ab}(u+i)$$

$\tilde{\mathbf{P}}$ is the analytic continuation of $\mathbf{P}$ through the cut:



- Pμ -system containts the equation for μ (follows from a QQ-relation):

$$\mu_{ab}^{++} - \mu_{ab} = \mathbf{P}_a\tilde{\mathbf{P}}_b - \mathbf{P}_b\tilde{\mathbf{P}}_a$$

- SL(2)-reduction: $\mathbf{P}^i = -\chi^{ij}\mathbf{P}_j\,, \quad \mathbf{Q}^i = -\chi^{ij}\mathbf{Q}_j\,, \quad \chi^{ij} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

- Cut structure on defining sheet and asymptotics at $u \to \infty$

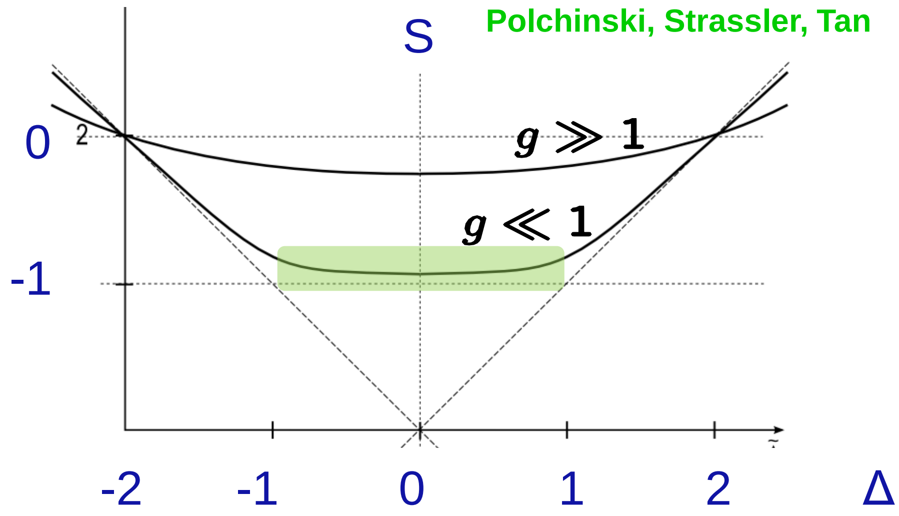
$$\begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \mathbf{P}_4 \end{pmatrix} \sim \begin{pmatrix} A_1 u^{-\frac{L}{2}} \\ A_2 u^{-\frac{L+2}{2}} \\ A_3 u^{\frac{L}{2}} \\ A_4 u^{\frac{L-2}{2}} \end{pmatrix} \qquad \begin{pmatrix} \mu_{12} \\ \mu_{13} \\ \mu_{14} \\ \mu_{24} \\ \mu_{34} \end{pmatrix} \sim \begin{pmatrix} u^{\Lambda-L} \\ u^{\Lambda+1} \\ u^{\Lambda} \\ u^{\Lambda-1} \\ u^{\Lambda+L} \end{pmatrix}, \qquad \Lambda = 0,\ \pm\Delta,\ \pm(S-1)$$

Alfimov Gromov V.K. 2014

BFKL Dimension from Quantum Spectral Curve

- QSC allows for analytic continuation of exact dimension $\Delta(S,g)$ to continuous spins $-1 < S < \infty$. We need to find the appropriate analytic continuation of Q-functions.

Janik Gromov, V.K.
Gromov, Levkovich-Maslyuk, Sizov, Valatka



- BFKL is a double scaling limit:

$$w = S+1 \to 0, \quad g \to 0, \quad \Lambda = \frac{g^2}{S+1} \quad -\text{fixed}$$

- We will restore from QSC the leading order (LO) BFKL approximation for $\Delta(S,g)$ already known up to NLO from direct summation of Feynman graphs

Kotikov, Lipatov

$$\frac{S+1}{4g^2} = \Psi(\Delta) + g^2\delta(\Delta) + \mathcal{O}(g^4)$$

where

$$\Psi(\Delta) = -\psi\left(\frac{1+\Delta}{2}\right) - \psi\left(\frac{1-\Delta}{2}\right) + 2\psi(1)$$

$$\delta(\Delta) = 4\Psi''(\Delta) + 6\zeta_3 + 2\zeta_2\Psi(\Delta) - \frac{\pi^3}{\cos\frac{\pi\Delta}{2}} - 4\Phi(\tfrac{1}{2} - \tfrac{\Delta}{2}) - 4\Phi(\tfrac{1}{2} + \tfrac{\Delta}{2}),$$

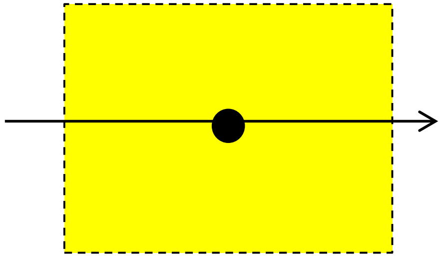
$$\Phi(x) = \sum_{k=0}^{\infty} \frac{(-)^k}{(x+k)^2}[\psi(k+1+x) - \psi(1)]$$

- In particular, near the Regge pole $\Delta - 1 \simeq \frac{-8g^2}{w} + w\zeta_3\left(\frac{-4g^2}{w}\right)^3 + \mathcal{O}\left(\left(\frac{g^2}{w}\right)^4\right)$

- BFKL is an excellent test for the whole AdS/CFT integrability: it sums up "wrapped" graphs omitted in asymptotic Bethe ansatz

Kotikov, Lipatov, Rej, Staudacher
Bajnok, Janik, Lukowsky
Lukowski, Rej, Velizhanin,Orlova

P-functions at LO BFKL



- We can split **P** into regular and singular parts

$$\mathbf{P} = \frac{\tilde{\mathbf{P}} + \mathbf{P}}{2} + \sqrt{u^2 - 4g^2}\left(\frac{\tilde{\mathbf{P}} - \mathbf{P}}{2\sqrt{u^2 - 4g^2}}\right)$$

- In the regime $g \ll |u| \ll 1$ singular part gives poles at $u = 0$

$$\sqrt{u^2 - 4g^2} \equiv \sqrt{u^2 - 4\Lambda w} = u - \frac{2\Lambda}{u} w - \frac{2\Lambda^2}{u^3} w^2 + O(w^3)$$

reminder: $w = S + 1$, $\Lambda = \frac{g^2}{S+1}$

- Only poles at $u \to \infty$

$$\begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \mathbf{P}_4 \end{pmatrix} \simeq \begin{pmatrix} u^{-1} \\ u^{-2} \\ A_3\, u \\ A_4\, u^0 \end{pmatrix}$$

$$\begin{aligned} A_4 &= \frac{1}{96i}((5-w)^2 - \Delta^2)((1+w)^2 - \Delta^2) \\ A_3 &= \frac{1}{32i}((1-w)^2 - \Delta^2)((3-w)^2 - \Delta^2) \end{aligned}$$

- Due to asymptotics and parity **P**'s are fixed at LO up to a single constant

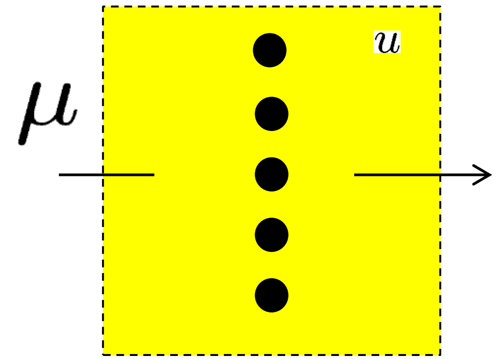
$$\mathbf{P}_1 = \frac{1}{u},, \qquad \mathbf{P}_2 = \frac{1}{u^2}, \qquad \mathbf{P}_3 = A_3^{(0)} u + \frac{c_{3,1}^{(1)}}{\Lambda u}, \qquad \mathbf{P}_4 = A_4^{(0)}$$

- To fix it we go through the cut. Uniformized by Zhukovsky map $u = \sqrt{\Lambda w}(x + 1/x)$

$$\mathbf{P}_a = \sum_{n=-1}^{\infty} \frac{c_{a,n}}{[x(u)]^n} \qquad c_{a,n}(\Lambda, w) = (\sqrt{\Lambda w})^{n-4} \sum_{k=0}^{+\infty} c_{a,n}^{(k)} w^k$$

μ-functions at LO BFKL

- A “ladder” of cuts generating poles at $u = i\mathbb{Z}$



- Asymptotics $u \to \infty$
  suggests that
  μ are polynomials at LO

$$\begin{pmatrix} \mu_{12} \\ \mu_{13} \\ \mu_{14} \\ \mu_{24} \\ \mu_{34} \end{pmatrix} \simeq \begin{pmatrix} u^0 \\ u^3 \\ u^2 \\ u^1 \\ u^4 \end{pmatrix}$$

- They also can be multiplied by a regular periodic function $\cosh(\pi u) + \text{const}$

- Now we apply the Pμ-equation $\tilde{\mathbf{P}}_a = \mu_{ab}\mathbf{P}^b$ using the rule $\tilde{x}(u) = \frac{1}{x(u)}$

  and fix μ-functions by parity and regularity conditions:

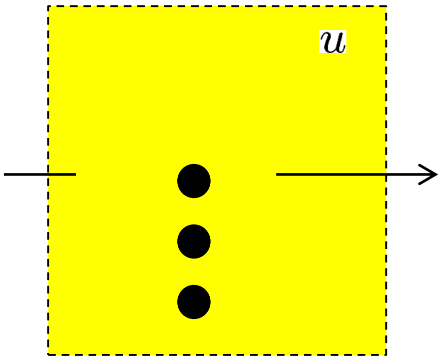
$$\mu^+_{12} = \frac{\cosh^2 \pi u}{\pi^2 \Lambda^2 w^2} \frac{-4i}{(\Delta^2 - 1)^2},$$
$$\mu^+_{13} = \frac{\cosh^2 \pi u}{\pi^2 \Lambda^2 w^2} \frac{u(4u^2 + 1)}{48},$$
$$\mu^+_{14} = \frac{\cosh^2 \pi u}{\pi^2 \Lambda^2 w^2} \frac{(4u^2 + 1)}{32},$$
$$\mu^+_{24} = \frac{\cosh^2 \pi u}{\pi^2 \Lambda^2 w^2} \frac{u}{4},$$
$$\mu^+_{34} = \frac{\cosh^2 \pi u}{\pi^2 \Lambda^2 w^2} \frac{i(\Delta^2 - 1)^2}{12288}(4u^2 - 3)(4u^2 + 1).$$

- At the same time we fix the missing coefficient in **P** $c^{(1)}_{3,1} = -\frac{i(\Delta^2 - 1)^2}{96}$

Analytic properties of Q-functions

- Natural objects for approaching BFKL are **Q**-functions: their asymptotics contain conformal charges, including Δ

$$\begin{pmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \\ \mathbf{Q}_3 \\ \mathbf{Q}_4 \end{pmatrix} \sim \begin{pmatrix} u^{\frac{\Delta+1-w}{2}} \\ u^{\frac{\Delta-3+w}{2}} \\ u^{\frac{-\Delta+1-w}{2}} \\ u^{\frac{-\Delta-3+w}{2}} \end{pmatrix}$$



- A "ladder" of cuts generating poles at $u = i\mathbb{Z}_-$

- From purely algebraic relations of Q-system we get a 4-th order finite difference equation with 4 solutions giving all 4 **Q**-functions:

$$0 = \mathbf{Q}^{[+4]}D_0 - \mathbf{Q}^{[+2]}\left[D_1 - \mathbf{P}_a^{[+2]}\mathbf{P}^{a[+4]}D_0\right] + \frac{1}{2}\mathbf{Q}\left[D_3 + \mathbf{P}_a\mathbf{P}^{a[+4]}D_0 + \mathbf{P}_a\mathbf{P}^{a[+2]}D_1\right] + \text{c.c.}$$

- The coefficients depend only on **P**-functions: $D_m = \det_{1\le a,k\le 4} (\mathbf{P}^a)^{[4-2k+2\delta_{k,m}]}$

- Plugging here the LO **P**-functions we get an equation factorized as follows

$$\left[(u+2i)^2 D + (u-2i)^2 D^{-1} - 2u^2 - \frac{17-\Delta^2}{4}\right] \times \left[D + D^{-1} - 2 - \frac{1-\Delta^2}{4u^2}\right]\mathbf{Q} = 0$$

- 2-nd order equation is the Faddeev-Korchemsky-Baxter eq. for BFKL pomeron !

Finding the BFKL dimension

- We need to find the NLO for $\mu, \mathbf{P}, \mathbf{Q}$ and the LO for ω
  For that we also have to solve the Q-ω system at the leading order

- Using explicit LO solution for $\mathbf{Q}$ and for $\tilde{\mathbf{Q}} = \omega\mathbf{Q}$ we find at the pole in u=0

$$\mathbf{Q}_3(u) \simeq 2iw\Lambda\mathbf{Q}_3(0)\frac{\Psi(\Delta)}{u} + \text{regular}(u) + \mathcal{O}(w^2)$$

$$\Psi(\Delta) = -\psi\left(\frac{1+\Delta}{2}\right) - \psi\left(\frac{1-\Delta}{2}\right) + 2\psi(1)$$

Finding the BFKL Dimension

- On the other hand, from the explicit knowledge of NLO $\mathbf{P}$ we find the 4'th order NLO equation for $\mathbf{Q}$ which factorizes again, to give for j=1,3

$$\mathbf{Q}_j\left(\frac{\Delta^2-1-8u^2}{4u^2}+w\frac{\left(\Delta^2-1\right)\Lambda-u^2}{2u^4}\right)+\mathbf{Q}_j^{--}\left(1-\frac{iw/2}{u-i}\right)+\mathbf{Q}_j^{++}\left(1+\frac{iw/2}{u+i}\right)=0$$

with explicit solution for $\mathbf{Q}=\alpha\mathbf{Q}_1+\beta\mathbf{Q}_3$

$$\mathbf{Q}=\frac{\sqrt{\omega}(u^2-2\Lambda\omega)}{iu-\frac{\omega}{4}-i\sqrt{2\Lambda w}}\frac{\Gamma\left(iu-\frac{\omega}{4}+i\sqrt{2\Lambda w}\right)}{\Gamma\left(-iu-\frac{w}{4}-i\sqrt{2\Lambda w}\right)}\;{}_3F_2\left(\frac{1-\Delta}{2},\frac{1+\Delta}{2},-iu-\frac{\omega-i\sqrt{32\Lambda w}}{4};-\frac{w}{2},2i\sqrt{2\Lambda w}+1;1\right)$$

- Comparing its value at the pole $\mathbf{Q}=\frac{4\sqrt{-2\Lambda}\cos\left(\frac{\pi\Delta}{2}\right)}{\pi}\left(1-\frac{iw}{2u}\right)+\mathcal{O}(w^2)$

  and using that $\mathbf{Q}_1(0)=0$ we fix $\beta\mathbf{Q}_3(0)=\frac{\sqrt{-8\Lambda}\cos\left(\frac{\pi\Delta}{2}\right)}{\pi}$

- This allows to fix the dimension and restore the LO Kotikov-Lipatov formula

$$\frac{S+1}{4g^2}=-\psi\left(\frac{1}{2}-\frac{\Delta}{2}\right)-\psi\left(\frac{1}{2}+\frac{\Delta}{2}\right)+2\psi(1)+\mathcal{O}(g^2)$$

Conclusions, comments, future directions

- We proposed a concise system of matrix Riemann-Hilbert equations – Quantum Spectral Curve - for exact spectrum of anomalous dimensions of planar N=4 SYM theory in 4D.

- BFKL dimension in LO is recovered; regular BFKL expansion (NLO,NNLO,…) is possible.

  Consequences for scattering theory in Regge limit and a link to QCD pomeron.

- Hopefully efficient for numeric. In particular, the full curve $\Delta(S,g)$ could be restored numerically.

- Applicable for Wilson loops and quark-antiquark potential in N=4 SYM

  Gromov, Kazakov, Leurent, Volin

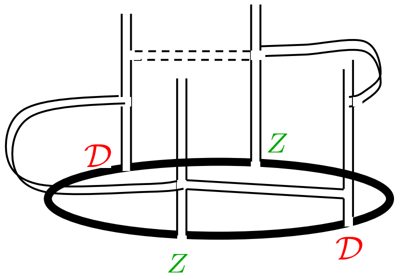
- Very efficient for various approximations: weak coupling (9 loops!) and strong coupling (3 loops) expansions exact slope and curvature functions: **Volin**

$$\Delta(S,g) - \Delta_0 = \Delta'(g)\, S + \Delta''(g)\, S^2 + O(S^3)$$

Basso **Gromov, Levkovich-Maslyuk, Sizov, Valatka**

Perturbative Konishi: integrability versus Feynman graphs

$\mathcal{O}_{\text{Konishi}} = \text{Tr}\ [\mathcal{D}, Z]^2$



- Integrability allows to sum exactly enormous numbers of Feynman diagrams of N=4 SYM

$$\begin{aligned}
\Delta = \ & 4 + 12\,g^2 - 48\,g^4 + 336\,g^6 + 96\,g^8\,(-26 + 6\,\zeta_3 - 15\,\zeta_5) \\
& -96\,g^{10}\,(-158 - 72\,\zeta_3 + 54\,\zeta_3^2 + 90\,\zeta_5 - 315\,\zeta_7) \\
& -48\,g^{12}\,(160 + 5472\,\zeta_3 - 3240\,\zeta_3\,\zeta_5 + 432\,\zeta_3^2 - 2340\,\zeta_5 - 1575\,\zeta_7 + 10206\,\zeta_9) \\
& +48\,g^{14}\,(-44480 + 108960\,\zeta_3 + 8568\,\zeta_3\,\zeta_5 - 40320\,\zeta_3\,\zeta_7 - 8784\,\zeta_3^2 + 2592\,\zeta_3^3 \\
& \quad - 4776\,\zeta_5 - 20700\,\zeta_5^2 - 26145\,\zeta_7 - 17406\,\zeta_9 + 152460\,\zeta_{11}) \\
& +96\,g^{16}\,(566752 - 869760\,\zeta_3 - 45360\,\zeta_3\,\zeta_5 - 64890\,\zeta_3\,\zeta_7 + 241920\,\zeta_3\,\zeta_9 + 82656\,\zeta_3^2 - 33912\,\zeta_3^2\,\zeta_5 + 20736\,\zeta_3^3 \\
& \quad - 204984\,\zeta_5 + 231840\,\zeta_5\,\zeta_7 + 24840\,\zeta_5^2 + 227799\,\zeta_7 + 97164\,\zeta_9 + 135927\,\zeta_{11} - 1104246\,\zeta_{13} \\
& \quad + 7128\,\frac{\zeta_{11} - \zeta_3\,\zeta_{3,5} + \zeta_{3,5,3}}{5}) \\
& -96\,g^{18}\,(10568224 - 11884608\,\zeta_3 + 148896\,\zeta_3\,\zeta_5 - 177768\,\zeta_3\,\zeta_5^2 - 354384\,\zeta_3\,\zeta_7 - 1244484\,\zeta_3\,\zeta_9 + 2901096\,\zeta_{11}\,\zeta_3 \\
& \quad + 533952\,\zeta_3^2 + 284904\,\zeta_3^2\,\zeta_5 - 229824\,\zeta_3^2\,\zeta_7 + 209952\,\zeta_3^3 - 5993280\,\zeta_5 + 963954\,\zeta_5\,\zeta_7 + 2553120\,\zeta_5\,\zeta_9 - 576000\,\zeta_5^2 \\
& \quad + 2324196\,\zeta_7 + 1184274\,\zeta_7^2 + 2573892\,\zeta_9 + 355266\,\zeta_{11} + 2644434\,\zeta_{13} - 15810795\,\zeta_{15} \\
& \quad + 163296\,\frac{\zeta_{11} - \zeta_3\,\zeta_{3,5} + \zeta_{3,5,3}}{5} - 13608\,(\zeta_3\,\zeta_{3,7} - \zeta_{3,7,3} + \zeta_3^2\,\zeta_5 - \zeta_5\,\zeta_{5,3} + \zeta_{5,3,5}))
\end{aligned}$$

Bajnok,Janik
Leurent,Serban,Volin
Bajnok,Janik,Lukowski
Lukowski,Rej,
Velizhanin,Orlova
Leurent, Volin

Leurent, Volin
(8 loops from FiNLIE)

Volin
(9-loops from spectral curve)

- Confirmed up to 5 loops by direct graph calculus (6 loops promised)

Fiamberti,Santambrogio,Sieg,Zanon
Velizhanin
Eden,Heslop,Korchemsky,Smirnov,Sokatchev

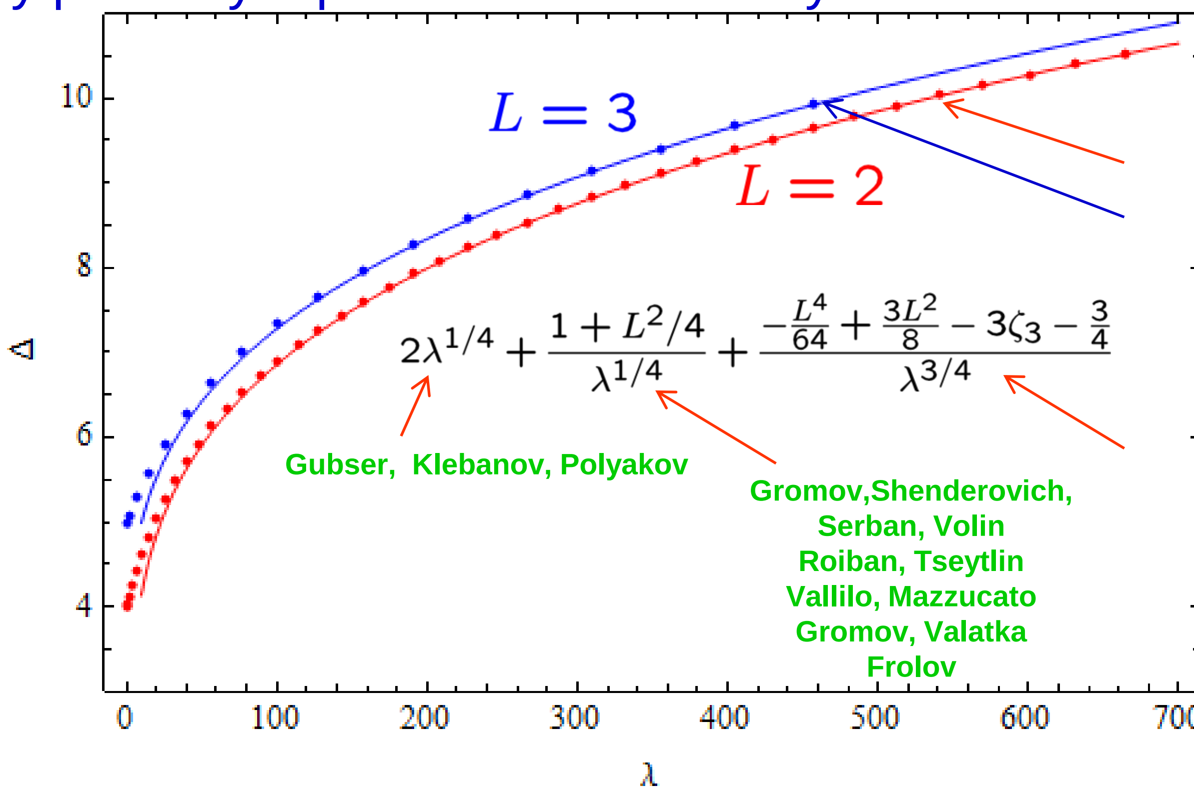
AdS string quasiclassics and numerics in SL(2) sector: twist-L operators of spin S $\mathrm{Tr}\,\mathcal{D}^S Z^L$

- 3 leading strong coupling terms were calculated for any S and L
  - Numerics from Y-system, TBA, FiNLIE, at any coupling:

$S = 2, \quad L = 2, \quad n = 1$ - for Konishi operator

$S = 2, \quad L = 3, \quad n = 1$ - and twist-3 operator

They perfectly reproduce the TBA/Y-system or FiNLIE numerics



$$2\lambda^{1/4} + \frac{1 + L^2/4}{\lambda^{1/4}} + \frac{-\frac{L^4}{64} + \frac{3L^2}{8} - 3\zeta_3 - \frac{3}{4}}{\lambda^{3/4}}$$

Y-system numerics

Gromov,V.K.,Vieira

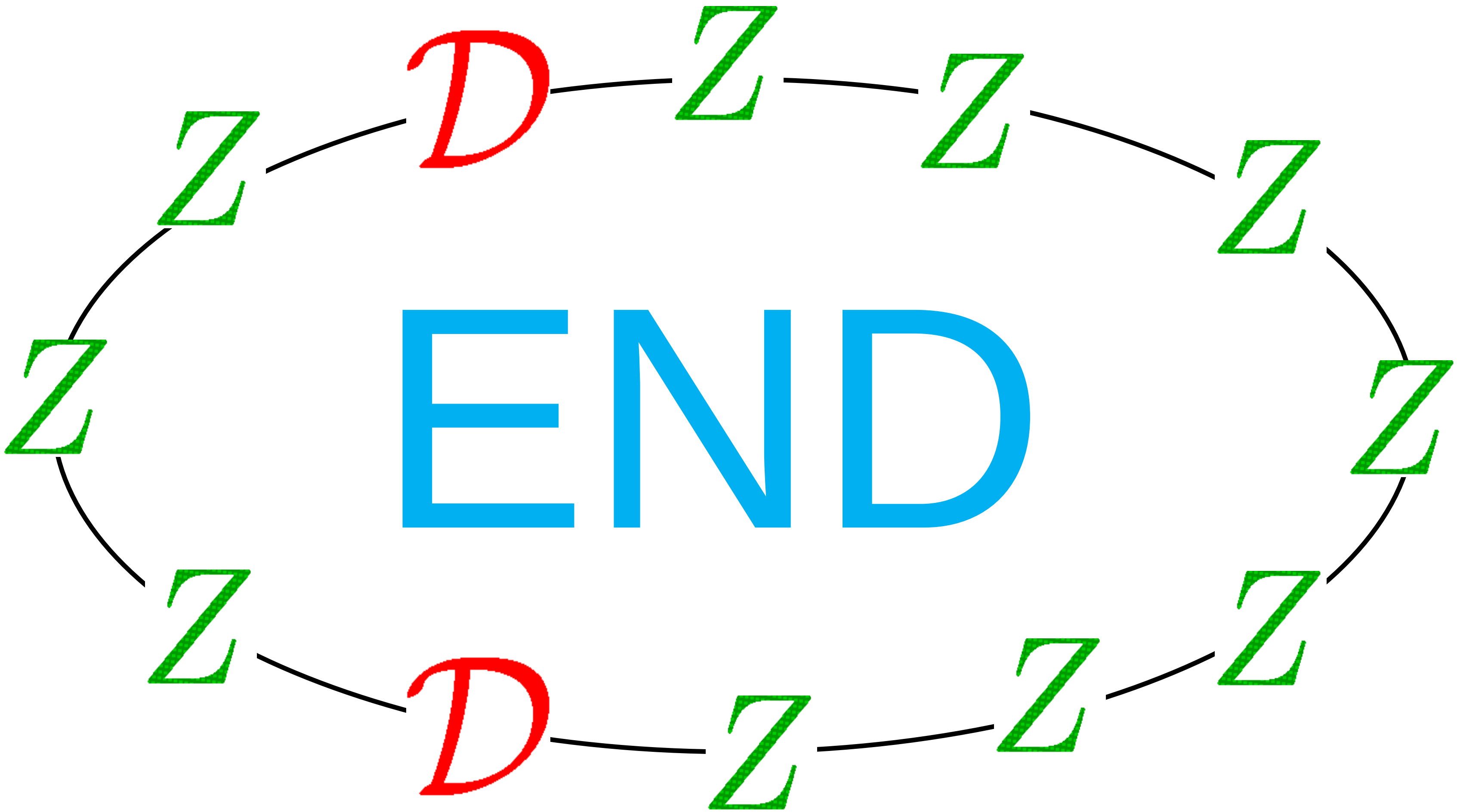
Frolov

Gromov,Valatka

$\lambda = 16\pi^2 g^2$

Gromov, Valatka,

- AdS/CFT Integrability passes all known tests!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