O(N) Models, RG and AdS/CFT

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Talk at Recent Developments in String Theory

Ascona

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Based mainly on

- S. Giombi, IK, arXiv:1308.2337
- S. Giombi, IK, B. Safdi, arXiv:1401.0825
- L. Fei, S. Giombi, IK, arXiv:1404.1094

Vectorial AdS/CFT

- Look for AdS duals of CFT's where dynamical fields are in the fundamental of O(N) or U(N) rather than in the adjoint. IK, Polyakov
- Wilson-Fisher O(N) critical points in d=3:

$$S = \int d^d x \left(\frac{1}{2} (\partial \phi^i)^2 + \frac{\lambda}{4} (\phi^i \phi^i)^2 \right)$$

- Even simpler: the O(N) singlet sector of the free theory.
- Conserved currents of even spin

$$J_{(\mu_1 \cdots \mu_s)} = \phi^a \partial_{(\mu_1} \cdots \partial_{\mu_s)} \phi^a + \ldots$$

All Spins All the Time

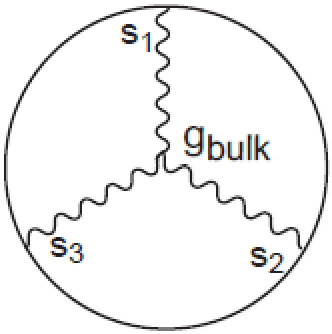
- Similarly, can consider the U(N) singlet sector in the d-dimensional free theory of N complex scalars. There are conserved currents of all integer spin.
- The dual AdS_{d+1} description must consist of massless gauge fields of all integer spin, coupled together.

$$\text{Spectrum}: \qquad s = 1, 2, 3, \ldots, \infty \qquad \text{gauge fields}$$
$$s = 0, \quad m^2 = -2(d-2) \quad \text{scalar}$$

- Vasiliev and others have constructed the classical EOM for some such interacting theories.
- Complicated. No known action principle.

Matching of 3-pt functions

- n-point functions of the currents do not vanish in the free CFT. This requires the bulk theory to be interacting.
- At leading order in N, the classical EOM may be used to calculate the 3-pt functions Giombi, Yin



$$g_{\text{bulk}} \sim \frac{1}{\sqrt{N}}$$

- The inverse Newton constant is quantized
Maldacena, Zhiboedov

$$G_N^{-1} \propto N$$

Sphere Free Energy

- Compare the free energy on the d-sphere at the boundary of Euclidean AdS with the bulk calculation $Z_{\rm bulk} = e^{-\frac{1}{G_N}F^{(0)} - F^{(1)} - G_N F^{(2)} + \ldots}$
- Cannot determine the leading classical piece (no known action), but focus on the one-loop correction.
- In the free CFT, $F = -\log Z_{S^d} = N F_{\rm free\ scalar}$
- For example, in d=3 U(N) singlet CFT

$$N\left(\frac{\log 2}{4} - \frac{3\zeta(3)}{8\pi^2}\right)$$

Conformal Scalar on S^d

- In any dimension

$$F_S = -\log|Z_S| = \frac{1}{2}\log\det\left[\mu_0^{-2}\mathcal{O}_S\right] \qquad \mathcal{O}_S \equiv -\nabla^2 + \frac{d-2}{4(d-1)}R$$

- The eigenvalues and degeneracies are

$$\lambda_n = \left(n + \frac{d-1}{2}\right)^2 - \frac{1}{4} \qquad n \geq 0 \qquad m_n = \frac{(2n+d-1)(n+d-2)!}{(d-1)!n!}$$

$$F_S = \frac{1}{2}\sum_{n=0}^{\infty} m_n \left[-2\log(\mu_0 a) + \log\left(n + \frac{d}{2}\right) + \log\left(n - 1 + \frac{d}{2}\right)\right]$$

- Using zeta-function regularization in d=3,

$$F_B = -\frac{1}{2}\frac{d}{ds}\left[2\zeta(s-2,1/2) + \frac{1}{2}\zeta(s,1/2)\right]\Big|_{s=0} = \frac{1}{16}\left(2\log 2 - \frac{3\zeta(3)}{\pi^2}\right) \approx .0638$$

- Check cancellation of the $\mathcal{O}(N^0)$ term in F_{bulk}

$$Z_{1-\text{loop}} = \frac{1}{\left[\det\left(-\nabla^2-2\right)\right]^{\frac{1}{2}}} \prod_{s=1}^{\infty} \frac{\left[\det_{s-1}^{STT}\left(-\nabla^2+s^2-1\right)\right)\right]^{\frac{1}{2}}}{\left[\det_{s}^{STT}\left(-\nabla^2+s(s-2)-2\right)\right]^{\frac{1}{2}}}$$

$$F^{(1)}_{(\Delta,s)} = -\frac{1}{2}\zeta'_{(\Delta,s)}(0) - \frac{1}{2}\zeta_{(\Delta,s)}(0)\log\left(\ell^2\Lambda^2\right)$$

$$\zeta_{(\Delta,s)}(0) = \frac{1}{24}(2s+1)\left[\nu^4 - \left(s+\frac{1}{2}\right)^2\left(2\nu^2+\frac{1}{6}\right) - \frac{7}{240}\right], \qquad \nu \equiv \Delta - \frac{3}{2}$$

$$\begin{aligned} F^{(1)}\Big|_{\log-\text{div}} &= -\frac{1}{2}\left(\zeta_{(1,0)}(0) + \sum_{s=1}^{\infty}\left(\zeta_{(s+1,s)}(0) - \zeta_{(s+2,s-1)}(0)\right)\right)\log\left(\ell^2\Lambda^2\right) \\ &= \left(\frac{1}{360} + \sum_{s=1}^{\infty}\left(\frac{1}{180} - \frac{s^2}{24} + \frac{5s^4}{24}\right)\right)\log\left(\ell^2\Lambda^2\right) \end{aligned}$$

- The UV log divergence cancels using standard zeta function regularization. This is evidence for one-loop finiteness of the Vasiliev theory in AdS_4.

- The finite part for each spin Camporesi, Higuchi

$$\zeta'_{(\Delta,s)}(0) = \frac{1}{3}(2s+1)\left[\frac{\nu^4}{8} + \frac{\nu^2}{48} + c_1 + \left(s+\frac{1}{2}\right)^2 c_2 + \int_0^{\nu} dx \left[\left(s+\frac{1}{2}\right)^2 x - x^3\right]\psi(x+\frac{1}{2})\right]$$

- Sum over spins vanishes using the Hurwitz-Lerch function to regularize:

$$\Phi(z,s,v) = \frac{1}{\Gamma(s)}\int_0^{\infty} dt \frac{t^{s-1}e^{-vt}}{1-ze^{-t}} = \sum_{n=0}^{\infty}(n+v)^{-s}z^n$$

- Perfect agreement with the CFT where the $\mathcal{O}(N^0)$ term vanishes!
- In the minimal Vasiliev theory with only even spins we encounter a surprise. The log divergence cancels, but the finite part does NOT vanish.

Free O(N) Model

- The sum over even spins in AdS gives

$$F^{(1)}_{\rm min} = \frac{\log 2}{8} - \frac{3\zeta(3)}{16\pi^2}$$

- This is exactly the F-value of a real massless scalar in 3 dimensions! Giombi, IK
- This suggests a shift in the identification of the quantized Vasiliev coupling: $N \to N-1$
- We conjecture that the classical term is

$$\frac{1}{G_N} F^{(0)}_{\rm min} = (N-1)\left(\frac{\log 2}{8} - \frac{3\zeta(3)}{16\pi^2}\right)$$

- Then the sum of the classical and one loop terms in the minimal (even spin) Vasiliev theory would agree with the CFT of N real scalars.

Even Boundary Dimensions

- In even d, the CFT sphere free energy is UV logarithmically divergent, the coefficient being related to the Weyl *a-anomaly.*
- In the bulk, this logarithmic divergence is reflected in the IR divergence of the AdS_{d+1} volume for odd d+1

$$\int \mathrm{vol}_{\mathrm{AdS}_{d+1}} = \frac{2(-\pi)^{d/2}}{\Gamma\left(1+\frac{d}{2}\right)} \log R$$

- The coefficient of log R in the bulk free energy is dual to the a-anomaly coefficient on the CFT side.
- There is no UV divergence in the bulk in this case (in odd dimensional spacetime, $\zeta(0)=0$ identically).

Anomaly Matching

- If the CFT is free, the a-anomaly should be Na_{scalar} without 1/N corrections.
- The results we find are consistent with the general picture: Giombi, IK, Safdi

$$F^{(1)} = 0$$

$$F^{(1)}_{\text{min HS}} = F^{\text{conf. scalar}}_{S^d} = a_{\text{scalar}} \log R$$

where a_{scalar} is the a-anomaly coefficient of one real conformal scalar in d-dimensions (e.g. a_{scalar}=1/90, -1/756, 23/113400… in d=4,6,8…).

Example: d=4

- For the Vasiliev theory in AdS_5 with all integer spins, the one-loop bulk free energy

$$
\begin{aligned}
F^{(1)} &= -\frac{\log R}{360}\sum_{s=1}^{\infty} s^2(1+s)^2\left(3+14s(1+s)\right) \\
&= -\left(\frac{1}{18}\zeta(-3)+\frac{7}{60}\zeta(-5)\right)\log R = 0
\end{aligned}
$$

- For the minimal theory with even spins only

$$
\begin{aligned}
F^{(1)}_{\text{min HS}} &= -\frac{\log R}{360}\sum_{s=2,4,\ldots}^{\infty} s^2(1+s)^2\left(3+14s(1+s)\right) \\
&= -\left(\frac{4}{9}\zeta(-3)+\frac{56}{15}\zeta(-5)\right)\log R = +\frac{1}{90}\log R
\end{aligned}
$$

- The +1/90 is the a-anomaly coefficient of a real scalar in d=4.
- For the even spin theory in any d, $G_N \sim \frac{1}{N-1}$

Interacting CFT's

- A scalar operator $\mathcal{O}(x^\mu)$ in d-dimensional CFT is dual to a field $\Phi(z, x^\mu)$ in AdS_{d+1} which behaves near the boundary as z^Δ
- There are two choices $$\Delta_\pm = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 + m^2}$$
- If we insist on unitarity, then Δ_- is allowed only in the Breitenlohner-Freedman range

IK, Witten

$$-(d/2)^2 < m^2 < -(d/2)^2 + 1$$

- Flow from a large N CFT where $\mathcal{O}(x^\mu)$ has dimension Δ_- to another CFT with dimension Δ_+ by adding a double-trace operator. Witten; Gubser, IK
- Can flow from the free d=3 scalar model in the UV to the Wilson-Fisher interacting one in the IR. The dimension of scalar bilinear changes from 1 to 2 +O(1/N). The dual of the interacting theory is the Vasiliev theory with Δ=2 boundary conditions on the bulk scalar.
- The 1/N expansion is generated using the Hubbard-Stratonovich auxiliary field.

$$S = \int d^d x \left(\frac{1}{2} (\partial \phi^i)^2 + \frac{1}{2} \sigma \phi^i \phi^i - \frac{\sigma^2}{4\lambda} \right)$$

- In 2<d<4 the quadratic term may be ignored in the IR:

$$Z = \int D\phi D\sigma \, e^{-\int d^d x \left(\frac{1}{2}(\partial\phi^i)^2 + \frac{1}{2\sqrt{N}}\sigma\phi^i\phi^i\right)}$$
$$= \int D\sigma \, e^{\frac{1}{8N}\int d^d x d^d y \, \sigma(x)\sigma(y) \, \langle\phi^i\phi^i(x)\phi^j\phi^j(y)\rangle_0 + \mathcal{O}(\sigma^3)}$$

- Induced dynamics for the auxiliary field endows it with the propagator

$$\langle\sigma(p)\sigma(-p)\rangle = 2^{d+1}(4\pi)^{\frac{d-3}{2}}\Gamma\left(\frac{d-1}{2}\right)\sin(\frac{\pi d}{2})\,(p^2)^{2-\frac{d}{2}} \equiv \tilde{C}_\sigma(p^2)^{2-\frac{d}{2}}$$

$$\langle\sigma(x)\sigma(y)\rangle = \frac{2^{d+2}\Gamma\left(\frac{d-1}{2}\right)\sin(\frac{\pi d}{2})}{\pi^{\frac{3}{2}}\Gamma\left(\frac{d}{2}-2\right)}\frac{1}{|x-y|^4} \equiv \frac{C_\sigma}{|x-y|^4}$$

- The 1/N corrections to operator dimensions are calculated using this induced propagator. For example,

$$\Delta_\phi = \frac{d}{2} - 1 + \frac{1}{N}\eta_1 + \frac{1}{N^2}\eta_2 + \dots$$

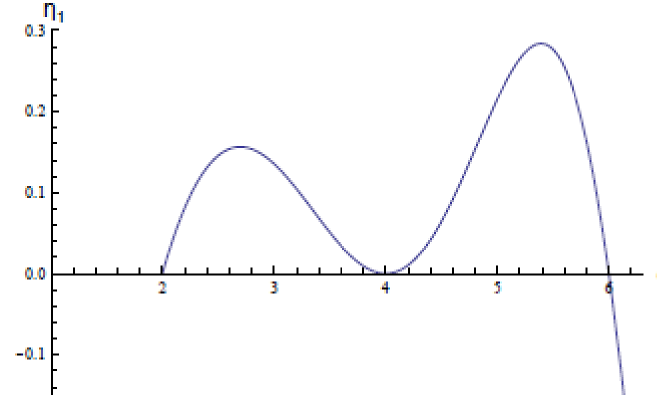
- For the leading correction need

$$\frac{1}{N}\int \frac{d^d q}{(2\pi)^d} \frac{1}{(p-q)^2} \frac{\tilde{C}_\sigma}{(q^2)^{\frac{d}{2}-2+\delta}}$$

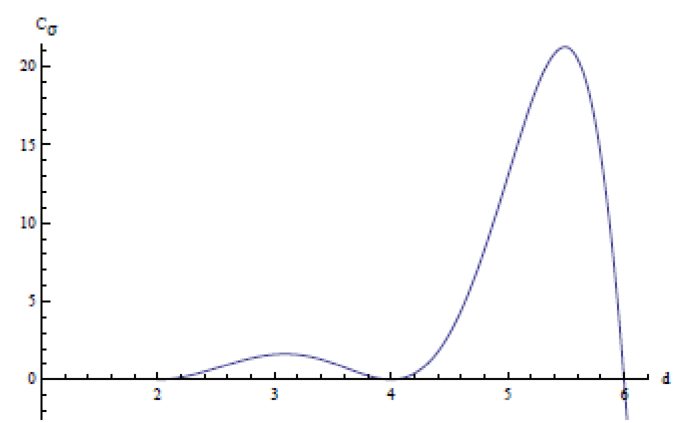
- δ is the regulator later sent to 0.

$$\eta_1 = \frac{\tilde{C}_\sigma (d-4)}{(4\pi)^{\frac{d}{2}} d\Gamma(\frac{d}{2})} = \frac{2^{d-3}(d-4)\Gamma\left(\frac{d-1}{2}\right)\sin\left(\frac{\pi d}{2}\right)}{\pi^{\frac{3}{2}}\Gamma\left(\frac{d}{2}+1\right)}$$

- When the leading correction is negative, the large N theory is non-unitary.
- It is positive not only for 2<d< 4, but also for 4<d<6.



- The 2-point function coefficient C_σ is similar



Gross-Neveu CFT

- Multiple Dirac fermions with action

$$\mathcal{S}(\bar{\psi},\psi) = -\int \mathrm{d}^d x \left[\bar{\psi}\cdot\partial\!\!\!/\psi + \tfrac{1}{2N} G\left(\bar{\psi}\cdot\psi\right)^2\right]$$

- In 2 < d < 4 there is a UV fixed point, at least for large N.
- In $d = 4 - \varepsilon$ can also be described as an IR fixed point of the Gross-Neveu-Yukawa model

Zinn-Justin, Moshe; Hasenfratz et al

$$\mathcal{S}(\bar{\psi},\psi,\sigma) = \int \mathrm{d}^d x \left[-\bar{\psi}\cdot\left(\partial\!\!\!/ + g\Lambda^{\varepsilon/2}\sigma\right)\psi + \tfrac{1}{2}\left(\partial_\mu\sigma\right)^2 + \tfrac{1}{2}m^2\sigma^2 + \frac{\lambda}{4!}\Lambda^{\varepsilon}\sigma^4\right]$$

- The beta functions are

$$\beta_\lambda = -\varepsilon\lambda + \frac{1}{8\pi^2}\left(\frac{3}{2}\lambda^2 + N\lambda g^2 - 6Ng^4\right)$$

$$\beta_{g^2} = -\varepsilon g^2 + \frac{N+6}{16\pi^2}g^4,$$

- IR stable fixed point Moshe, Zinn-Justin

$$g_*^2 = \frac{16\pi^2\varepsilon}{N+6}, \quad \lambda_* = 16\pi^2 R\varepsilon \qquad R = \frac{24N}{(N+6)\left[(N-6)+\sqrt{N^2+132N+36}\right]}$$

- In d=3 the U(N) singlet sector of the large N model has been conjectured to be dual to type B Vasiliev theory in AdS_4 with the alternate boundary conditions. Leigh, Petkou; Sezgin, Sundell

Towards Interacting 5-d O(N) Model

- Scalar large N model with $\frac{\lambda}{4}(\phi^i\phi^i)^2$ interaction has a good UV fixed point for 4<d<6. Parisi
- In $4+\epsilon$ dimensions $\beta_\lambda = \epsilon\lambda + \frac{N+8}{8\pi^2}\lambda^2 + \ldots$
- So, the UV fixed point is at a negative coupling

$$\lambda_* = -\frac{8\pi^2}{N+8}\epsilon + O(\epsilon^2)$$

- At large N, conjectured to be dual to Vasiliev theory in AdS_6 with Δ_- boundary condition on the bulk scalar. Giombi, IK, Safdi
- Check of 5-dimensional F-theorem $-F = \log Z_{S^5}$

$$F^{(1)}_{\rm UV} - F^{(1)}_{\rm IR} = -\frac{3\zeta(5) + \pi^2\zeta(3)}{96\pi^4} \approx -0.0016$$

Perturbative IR Fixed Points

- Work in $d = 6 - \epsilon$ with O(N) symmetric cubic scalar theory $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{g_1}{2}\sigma(\phi^i\phi^i) + \frac{g_2}{6}\sigma^3$
- The beta functions Fei, Giombi, IK

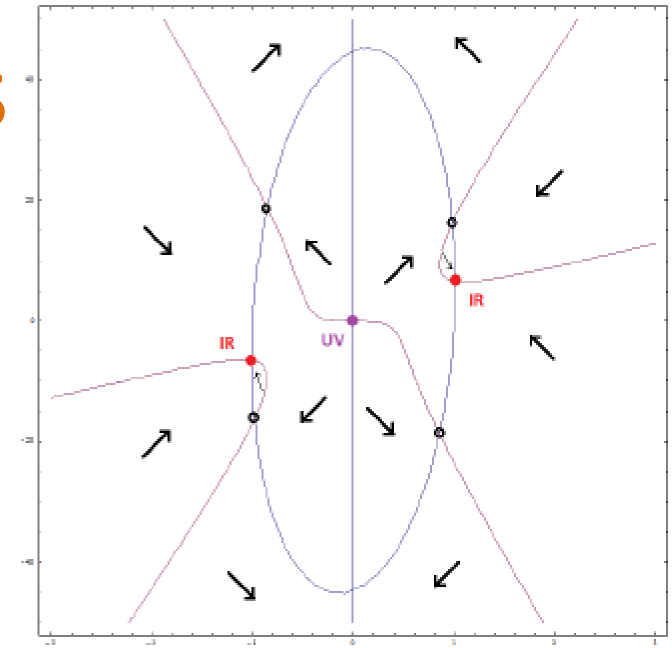
$$\beta_1 = -\frac{\epsilon g_1}{2} + \frac{(N-8)g_1^3 - 12g_1^2 g_2 + g_1 g_2^2}{12(4\pi)^3}$$

$$\beta_2 = -\frac{\epsilon g_2}{2} + \frac{-4Ng_1^3 + Ng_1^2 g_2 - 3g_2^3}{4(4\pi)^3}$$

- For large N, the IR stable fixed point is at real couplings

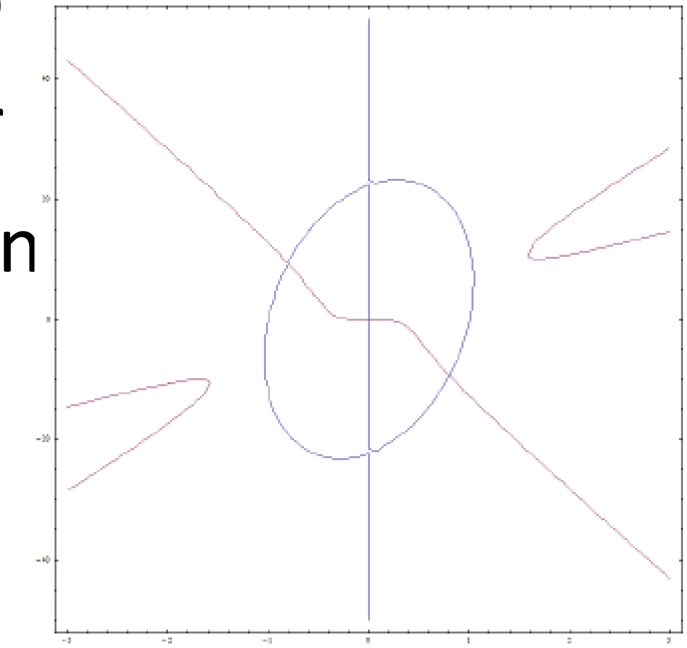
$$g_{1*} = \sqrt{\frac{6\epsilon(4\pi)^3}{N}} \qquad g_{2*} = 6g_{1*}$$

RG Flows



- Here is the flow pattern for N=2000

- The IR stable fixed points go off to complex couplings for N < 1039. Large N expansion breaks down very early!



- The dimension of sigma is $\Delta_\sigma = 2 - \frac{\epsilon}{2} + \frac{Ng_1^2 + g_2^2}{12(4\pi)^3}$
- At the IR fixed point this is $2 + 40\frac{\epsilon}{N}$
- Agrees with the large N result for the O(N) model in d dimensions:

Petkou (1995) $$2 + \frac{4}{N}\frac{\Gamma(d)}{\Gamma(d/2-1)\Gamma(1-d/2)\Gamma(d/2)\Gamma(d/2+1)}$$

- For N=0, the fixed point at imaginary coupling may lead to a description of the Lee-Yang edge singularity in the Ising model. Michael Fisher (1978)
- For N=0, Δ_σ is below the unitarity bound $2 - \frac{\epsilon}{2}$
- For N>1039, the fixed point at real couplings is consistent with unitarity in $d = 6 - \epsilon$

Critical N

- What is the critical value of N in d=5 below which the unitary fixed point disappears?
- A two-loop calculation gives Fei, Giombi, IK, Tarnopolsky (in preparation) $N_{crit} = 1038.266 - 609.8205\epsilon$
- 1/N Expansion of γ_ϕ in d=5 $\frac{0.216152}{N} - \frac{4.342}{N^2} - \frac{121.673}{N^3} + \dots$

suggests $N_{crit} \sim 35$

- In the $d = 4 - \epsilon$ Wilson-Fisher fixed point

$$\gamma_\phi = \frac{N+2}{4(N+8)^2}\epsilon^2 + \frac{N+2}{16(N+8)^4}\left(-N^2+56N+272\right)\epsilon^3 + \frac{N+2}{64(N+8)^6}\left(-5N^4-230N^3+1124N^2+17920N+46144-384\zeta(3)(5N+22)(N+8)\right)\epsilon^4$$

- Setting $\epsilon = -1$ gives $N_{crit} \sim 8$

Conformal Bootstrap

- It is interesting to study the d=5 theory directly for finite N using, for example, the conformal bootstrap.
- The first bootstrap results look encouraging. There is evidence for a minimum of C_J and C_T which for large N matches with the O(N) model results.

Nakayama, Ohtsuki

(Meta) Stability?

- Since the UV lagrangian is cubic, does the theory make sense non-perturbatively?
- When the CFT is studied on S^d or $R \times S^{d-1}$ the conformal coupling of scalar fields to curvature renders the perturbative vacuum meta-stable.
- This suggests that the dual Vasiliev theory is metastable, but only for the $\Delta\_$ boundary conditions.

Conclusions

- Vasiliev theories in AdS_{d+1} are one-loop finite theories of Quantum Gravity.
- Provided one-loop evidence for dualities with U(N) and O(N) singlet sectors of scalar field theories. In the O(N) case $G_N \sim \frac{1}{N-1}$
- Found a new description of the UV fixed points of the scalar O(N) model in $4<d<6$ valid for sufficiently large N.
- The (meta) stability of these theories deserves further investigations.