



Yang-Mills instantons on real Killing spinor manifolds

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Nonlinear supersymmetry in the quantum Calogero model

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- Integrals for integer coupling
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Some history

- **1923 Burchnall & Chaundy:**
odd-order ordinary differential operators commuting with 1d Hamiltonian
- **1978 Krichever:**
their existence is tied to algebro-geometric, or ‘finite-gap’, nature of Hamiltonian
- **1989 Dunkl:**
commuting operators combining partial differentials and Coxeter reflections
- **1990 Chalykh & Veselov:**
“commutative rings of partial differential operators and Lie algebras”
1st examples of 2d finite-gap Hamiltonians, construction of intertwiners for $g=2$
- **1991 Heckman:**
uses Dunkl operators to construct intertwiners for any multiplicity $g-1$
- **1990s Berest, Chalykh, Etingof, M. Feigin, Ginzburg, Styrkas, Veselov:**
extension to higher dimension $n-1$ and multiplicity $g-1$, in particular:
construction of Baker-Akhiezer functions, explicit formulæ for add’l charges,
including via Darboux dressing with intertwiners (only for $n=3, g=2$)

Integrals for generic coupling

A_{n-1} Calogero Hamiltonian:
$$H = \frac{1}{2} \sum_i p_i^2 + \sum_{i < j} \frac{g(g-1)}{(x^i - x^j)^2} \quad i, j = 1, 2, \dots, n$$

$[x^i, p_j] = i \delta^i_j$ and define
$$P = \sum_{i=1}^n p_i, \quad X = \frac{1}{n} \sum_{i=1}^n x^i$$

Dunkl operators:
$$\pi_i = p_i + \sum_{j(\neq i)} \frac{i g}{x^i - x^j} s_{ij}, \quad s_{ij} = \text{permutation operators}$$

Liouville charges:
$$I_k = \text{res} \left(\sum_i \pi_i^k \right), \quad k = 1, 2, \dots, n \Rightarrow [I_k, I_\ell] = 0$$

$$I_0 = n\mathbb{1}, \quad I_1 = P, \quad I_2 = 2H, \quad I_3 = \sum_i p_i^3 + 3 \sum_{i < j} \frac{g(g-1)}{(x^i - x^j)^2} (p_i + p_j)$$

important observation (hidden in 'potential-free frame'):
$$I_k(g) = I_k(1-g)$$

'dynamical' conformal symmetry: H is part of $sl(2, \mathbb{R})$ algebra, with other generators

$$D = \frac{1}{2} \sum_i (x^i p_i + p_i x^i) \quad \text{and} \quad K = \frac{1}{2} \sum_i (x^i)^2$$

$$[D, H] = 2iH \quad [D, K] = -2iK \quad [K, H] = iD$$

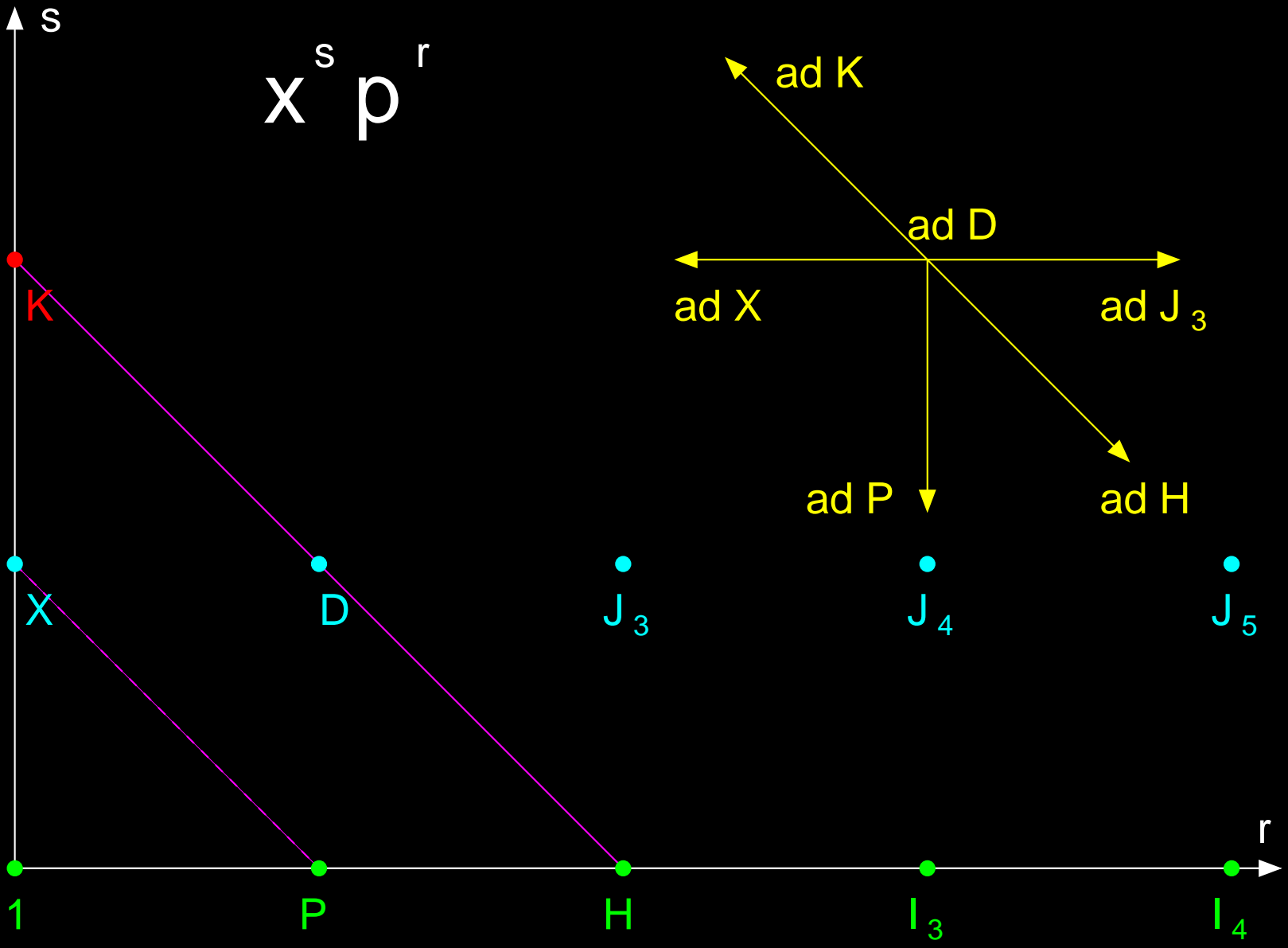
with other Liouville charges: $\frac{1}{i}[D, I_k] = kI_k$, $\frac{1}{i}[K, I_\ell] =: \ell J_\ell$

$$J_1 = nX \quad , \quad J_2 = D \quad , \quad J_3 = \frac{1}{2} \sum_i (x^i p_i^2 + p_i^2 x^i) + g(g-1) \sum_{i < j} \frac{x^i + x^j}{(x^i - x^j)^2}$$

they are 'almost' conserved (linear time dependence) and form a Witt algebra:

$$\frac{1}{i}[D, J_\ell] = (\ell-2)J_\ell \quad \text{and} \quad \frac{1}{i}[H, J_\ell] = -I_\ell$$

$$\frac{i}{k}[I_k, J_\ell] = I_{k+\ell-2} = \frac{i}{\ell}[I_\ell, J_k] \quad \text{and} \quad i[J_k, J_\ell] = (k-\ell)J_{k+\ell-2}$$



universal enveloping algebra provides an infinity of quadratic conserved charges:

$$L_{k,\ell} = \frac{1}{2}\{I_k, J_\ell\} - \frac{1}{2}\{I_\ell, J_k\} = -L_{\ell,k} \quad \Rightarrow \quad [H, L_{k,\ell}] = 0$$

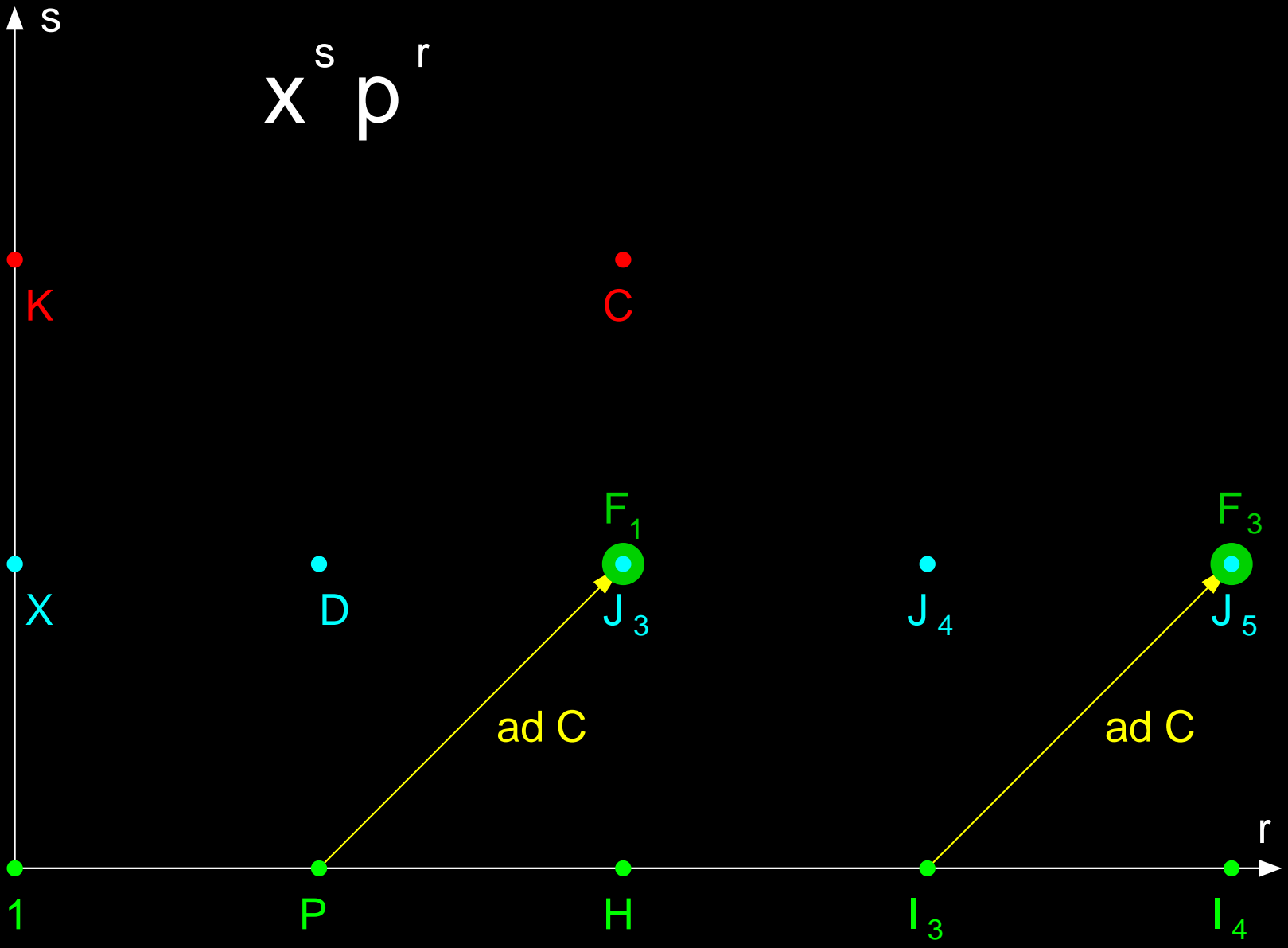
for an algebraically independent set, we take

$$L_{2,\ell} =: F_\ell = \{H, J_\ell\} - \frac{1}{2}\{I_\ell, D\} = \frac{1}{i\ell}[C, I_\ell] \quad \Rightarrow \quad [H, F_\ell] = 0$$

special cases: $F_2 \equiv 0$ and $F_1 = \{H, nX\} - \frac{1}{2}\{P, D\}$

the F_ℓ are also generated by the adjoint action of the $sl(2, \mathbb{R})$ Casimir

$$C = KH + HK - \frac{1}{2}D^2 \quad \Rightarrow \quad [C, H] = [C, D] = [C, K] = 0$$



the F_ℓ are not in involution but obey a quadratic algebra:

$$i[I_k, I_\ell] = 0 \quad , \quad \frac{i}{k}[I_k, F_\ell] = I_{k+\ell-2}I_2 - I_kI_\ell = \frac{i}{\ell}[I_\ell, F_k]$$

$$i[F_k, F_\ell] = (k-\ell)\frac{1}{2}\{F_{k+\ell-2}, I_2\} - (k-2)\frac{1}{2}\{F_k, I_\ell\} + (\ell-2)\frac{1}{2}\{F_\ell, I_k\}$$

special cases: $i[I_1, F_\ell] = I_{\ell-1}I_2 - I_\ell I_1$ but $[I_2, F_\ell] \equiv 2[H, F_\ell] = 0$

however, independent set is $\{I_1, I_2, \dots, I_n; F_1, F_3, \dots, F_n\}$

higher I_k and F_k are order- k polynomials in lower ones

algebra becomes polynomial of order $2n-1$

was already mostly known to

Barucchi & Regge 1977 / Wojciechowski 1983 / Kuznetsov 1996

Integrals for integer coupling

Heckman (1991) constructed intertwiners $g \leftrightarrow g+1$ via Dunkl operators:

$$M(g) I_k(g) = I_k(g+1) M(g) \quad \text{for} \quad M(g) = \text{res} \left(\prod_{i < j} (\pi_i - \pi_j)(g) \right)$$

$$M(g)^* I_k(g+1) = I_k(g) M(g)^* \quad \text{for} \quad M(g)^* = \text{res} \left(\prod_{i < j} (\pi_i - \pi_j)(-g) \right)$$

$$M(1-g) I_k(g) = I_k(g-1) M(1-g) \quad \Leftarrow \quad M(g)^* = M(-g)$$

immediate consequence:

$$\left[M(g)^* M(g), I_k(g) \right] = 0 \quad \text{and} \quad \left[M(g) M(g)^*, I_k(g+1) \right] = 0$$

new conserved charge? no, because it is a polynomial in the Liouville charges:

$$M(g)^* M(g) = M(-g) M(-g)^* = \mathcal{R}(I(g)) =: \mathcal{R}(g)$$

coefficients of $\mathcal{R}(I)$ do not depend on $g \rightarrow$ evaluate for $g=0$:

$$\mathcal{R}(0) = M(0)^* M(0) = \prod_{i < j} (p_i - p_j)^2$$

$$= \begin{vmatrix} 1 & p_1 & \dots & p_1^{n-1} \\ 1 & p_2 & \dots & p_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & p_n & \dots & p_n^{n-1} \end{vmatrix}^2 = \begin{vmatrix} 1 & 1 & \dots & 1 \\ p_1 & p_2 & \dots & p_n \\ \vdots & \vdots & & \vdots \\ p_1^{n-1} & p_2^{n-1} & \dots & p_n^{n-1} \end{vmatrix} \cdot \begin{vmatrix} 1 & p_1 & \dots & p_1^{n-1} \\ 1 & p_2 & \dots & p_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & p_n & \dots & p_n^{n-1} \end{vmatrix}$$

$$= \left| \left(\sum_k p_k^{i+j-2} \right)_{ij} \right| = \det(I_{i+j-2}(0))_{ij}$$

hence:

$$\mathcal{R}(g) = \det(I_{i+j-2}(g))_{ij}$$

$$n = 2: \quad \mathcal{R} = I_1^2 - 2I_2$$

$$n = 3: \quad \mathcal{R} = I_1^2 I_4 - 2I_1 I_2 I_3 + I_2^3 - 3I_2 I_4 + 3I_3^2$$

$$= \frac{1}{6} (I_1^6 - 9I_1^4 I_2 + 8I_1^3 I_3 + 21I_1^2 I_2^2 - 36I_1 I_2 I_3 - 3I_2^3 + 18I_3^2)$$

so far, $g \in \mathbb{R}$ generic; but $g \in \mathbb{N}$ admits intertwiner with free theory ($g=1$):

$$L(g) = M(g-1)M(g-2) \cdots M(2)M(1) \quad \Rightarrow \quad L(g)I_k(1) = I_k(g)L(g)$$

$$L(g)^* = M(-1)M(-2) \cdots M(2-g)M(1-g) \quad \text{is conjugate intertwiner}$$

$$\Rightarrow \quad L(g)L(g)^* = (\mathcal{R}(g))^{g-1} \quad \text{and} \quad L(g)^*L(g) = (\mathcal{R}(g+1))^{g-1}$$

Darboux dressing of some free $G(1)$ with $[G(1), I_k(1)] = 0$ for some k :

$$G(g) = L(g)G(1)L(g)^* \quad \Rightarrow \quad [G(g), I_k(g)] = 0$$

consistent with involution of Liouville charges:

$$L(g)I_k(1)L(g)^* = (\mathcal{R}(g))^{g-1}I_k(g)$$

large choice of 'naked' $G(\mathbf{1})$: any polynomial in $\{p_i\}$ with constant coefficients
 identical particles \rightarrow observables totally (anti)symmetric under s_{ij}
 totally symmetric \rightarrow Liouville integrals; totally antisymmetric \rightarrow simplest is

$$G(\mathbf{1}) = M(\mathbf{0}) = \prod_{i < j} (p_i - p_j)$$

Darboux dressing:

$$\begin{aligned} Q(g) &= L(g) M(\mathbf{0}) L(g)^* \\ &= M(g-1) M(g-2) \cdots M(1) M(\mathbf{0}) M(-1) \cdots M(2-g) M(1-g) \end{aligned}$$

builds a chain relating $I_k(g) = I_k(1-g)$ back to $I_k(g)$:

$$Q(g) I_k(1-g) = I_k(g) Q(g) \quad \Rightarrow \quad [Q(g), I_k(g)] = 0$$

a conserved charge of weight $\frac{1}{2}n(n-1)(2g-1)$ algebraically independent of $\{I_k, F_\ell\}$

seeming other option $g \in \mathbb{N} + \frac{1}{2}$ fails:

$$M(g-1) \cdots M\left(\frac{3}{2}\right) M\left(\frac{1}{2}\right) M\left(-\frac{1}{2}\right) M\left(-\frac{3}{2}\right) \cdots M(1-g) = \left(\mathcal{R}(g)\right)^{g-\frac{1}{2}}$$

check the square of the new charge:

$$\begin{aligned}
 Q(g)^2 &= M(g-1) \cdots M(3-g) M(2-g) \underbrace{M(1-g) M(g-1)} M(g-2) \cdots M(1-g) \\
 &= M(g-1) \cdots M(3-g) M(2-g) \mathcal{R}(g-1) M(g-2) M(g-3) \cdots M(1-g) \\
 &= M(g-1) \cdots M(3-g) \underbrace{M(2-g) M(g-2)} \mathcal{R}(g-2) M(g-3) \cdots M(1-g) \\
 &= M(g-1) \cdots M(3-g) \left(\mathcal{R}(g-2) \right)^2 M(g-3) \cdots M(1-g) \\
 &\quad \vdots \\
 &= M(g-1) M(1-g) \left(\mathcal{R}(1-g) \right)^{2g-2} = \left(\mathcal{R}(1-g) \right)^{2g-1} = \left(\mathcal{R}(g) \right)^{2g-1}
 \end{aligned}$$

again a polynomial in the Liouville integrals, so formally $Q = \mathcal{R}^{g-\frac{1}{2}}$ for $g \in \mathbb{N}$

it remains to compute the action of $\text{ad}Q$ on the extra charges $F_\ell \dots$

... that is: $i[Q, F_\ell] = H [iQ, J_\ell] + [iQ, J_\ell] H - \frac{1}{2}n(n-1)(2g-1) Q I_\ell$

sidestep: compute $[Q, J_\ell]$

from the observation $[I_k, [iQ, J_\ell]] = [iQ, [I_k, J_\ell]] = k [Q, I_{k+\ell-2}] = 0$

it follows that $i[Q, J_\ell] = (2g-1) Q \mathcal{G}_\ell(I)$

with \mathcal{G}_ℓ being a g -independent polynomial in the I_k of weight $\ell-2$

compute it at $g=0$ (with a few tricks...) to arrive at

$$\mathcal{G}_\ell(I) = \frac{1}{2} \sum_{j=0}^{\ell-2} I_{\ell-2-j} I_j - \frac{1}{2}(\ell-1) I_{\ell-2}$$

the first few polynomials:

$$\begin{aligned} \mathcal{G}_1 &= 0 & \mathcal{G}_2 &= \frac{1}{2}n(n-1) & \mathcal{G}_3 &= (n-1)P & \mathcal{G}_4 &= (n-\frac{3}{2})2H + \frac{1}{2}P^2 \\ \mathcal{G}_5 &= (n-2)I_3 + 2HP & \mathcal{G}_6 &= (n-\frac{5}{2})I_3 + I_3P + \frac{1}{2}4H^2 \end{aligned}$$

$$\Rightarrow \quad i[Q, F_\ell] = (2g-1)Q \left(2G_\ell H - \frac{1}{2}n(n-1)I_\ell \right) =: (2g-1)Q C_\ell(I)$$

with C_ℓ being a g -independent polynomial in the I_k of weight ℓ

the first few polynomials:

$$C_1 = -\frac{1}{2}n(n-1)P \quad C_2 = 0 \quad C_3 = (n-1)\left(2HP - \frac{n}{2}I_3\right)$$

$$C_4 = (4n-6)H^2 + HP^2 - \frac{1}{2}n(n-1)I_4$$

the complete nonlinear (\mathbb{Z}_2 graded) algebra of $2n$ conserved charges:

$$[I_k, I_\ell] = 0 \quad i[I_k, F_\ell] = A_{k,\ell}(I) \quad i[F_k, F_\ell] = B_{k,\ell}(I, F)$$

$$[Q, I_\ell] = 0 \quad i[Q, F_\ell] = (2g-1)Q C_\ell(I) \quad \{Q, Q\} = 2(\mathcal{R}(I))^{2g-1}$$

with right-hand sides:

$$A_{k,\ell}(I) = k \left(I_{k+\ell-2} I_2 - I_k I_\ell \right)$$

$$B_{k,\ell}(I, F) = (k-\ell) \frac{1}{2} \{F_{k+\ell-2}, I_2\} - (k-2) \frac{1}{2} \{F_k, I_\ell\} + (\ell-2) \frac{1}{2} \{F_\ell, I_k\}$$

$$C_\ell(I) = \frac{1}{2} \sum_{j=0}^{\ell-2} I_{\ell-2-j} I_j I_2 - \frac{1}{2} (\ell-1) I_{\ell-2} I_2 - \frac{1}{2} n(n-1) I_\ell$$

Two particles

$$x \equiv x^{12} := x^1 - x^2, \quad 2p \equiv p_{12} := p_1 - p_2, \quad \pi_{12} := \pi_1 - \pi_2 = 2 \left(p + \frac{ig}{x} s_{12} \right)$$

$$P = p_1 + p_2$$

$$H = \frac{1}{4}P^2 + \frac{1}{4}\text{res}(\pi_{12}^2) = \frac{1}{4}P^2 + \left(p - \frac{ig}{x} \right) \left(p + \frac{ig}{x} \right) = \frac{1}{4}P^2 + p^2 + \frac{g(g-1)}{x^2}$$

$$F_1 = (x^2 p_1 - x^1 p_2)(p_1 - p_2) + \frac{1}{2}(p_1 + p_2) + 2g(g-1) \frac{x^1 + x^2}{(x^1 - x^2)^2}$$

$$C = \frac{1}{2}(x^2 p_1 - x^1 p_2)^2 - \frac{1}{2} + g(g-1) \frac{(x^1)^2 + (x^2)^2}{(x^1 - x^2)^2}$$

remove center of mass:

$$I_k(g) \Big|_{P=X=0} =: \tilde{I}_k(g) \quad \Rightarrow \quad \tilde{H}(g) = H - \frac{1}{4}P^2 = p^2 + \frac{g(g-1)}{x^2}$$

for integer coupling:

$$M(g) = 2 \operatorname{res}\left(p + \frac{ig}{x} s_{12}\right) = 2\left(p + \frac{ig}{x}\right) = 2x^g p x^{-g}$$

$$M(-g)M(g) = -4\widetilde{H}(g) \quad \text{and} \quad \left(p + \frac{ig}{x}\right)\left(p^2 + \frac{g(g-1)}{x^2}\right) = \left(p^2 + \frac{g(g+1)}{x^2}\right)\left(p + \frac{ig}{x}\right)$$

$$Q(g) = M(g-1) \cdots M(1-g) \quad \Rightarrow$$

$$Q(1) = 2p$$

$$Q(2) = 8\left(p^3 + \frac{3}{x^2}p + \frac{3i}{x^3}\right)$$

$$Q(3) = 32\left(p^5 + \frac{15}{x^2}p^3 + \frac{45i}{x^3}p^2 - \frac{45}{x^4}p\right)$$

$$Q(4) = 128\left(p^7 + \frac{42}{x^2}p^5 + \frac{210i}{x^3}p^4 - \frac{315}{x^4}p^3 + \frac{630i}{x^5}p^2 - \frac{2835}{x^6}p - \frac{2835i}{x^7}\right)$$

nontrivial commutators:

$$i[P, F_1] = 4\widetilde{H} \quad , \quad i[Q, F_1] = -(2g-1)QP \quad , \quad Q^2 = (-4\widetilde{H})^{2g-1}$$

Three particles

$$x^{ij} := x^i - x^j, \quad p_{ij} := p_i - p_j, \quad \pi_{ij} = p_{ij} + \frac{2ig}{x^{ij}} s_{ij} - \frac{ig}{x^{jk}} s_{jk} - \frac{ig}{x^{ki}} s_{ki}$$

$$P = p_1 + p_2 + p_3$$

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + g(g-1) \left(\frac{1}{(x^{12})^2} + \frac{1}{(x^{23})^2} + \frac{1}{(x^{31})^2} \right)$$

$$I_3 = (p_1^3 + p_2^3 + p_3^3) + 3g(g-1) \left(\frac{p_1 + p_2}{(x^{12})^2} + \frac{p_2 + p_3}{(x^{23})^2} + \frac{p_3 + p_1}{(x^{31})^2} \right)$$

$$F_1 = 2(x^1 + x^2 + x^3)H - (x^1 p_1 + x^2 p_2 + x^3 p_3 - i)P$$

$$F_3 = 2J_3 H - (x^1 p_1 + x^2 p_2 + x^3 p_3 - 2i)I_3$$

$$J_3 = p_1 x^1 p_1 + p_2 x^2 p_2 + p_3 x^3 p_3 + g(g-1) \left(\frac{x^1 + x^2}{(x^{12})^2} + \frac{x^2 + x^3}{(x^{23})^2} + \frac{x^3 + x^1}{(x^{31})^2} \right)$$

$$C = \frac{1}{2} \sum_{i < j} (x^j p_i - x^i p_j)^2 - \frac{3}{8} + g(g-1) \sum_i (x^i)^2 \sum_{j < k} \frac{1}{(x^{jk})^2}$$

$$M(g) = \text{res}\left(\pi_{12}(g)\pi_{23}(g)\pi_{31}(g)\right) \quad \Delta = x^{12}x^{23}x^{31}$$

$$= \Delta^g \left(p_{12} p_{23} p_{31} - \frac{ig}{x^{12}} p_{12}^2 - \frac{ig}{x^{23}} p_{23}^2 - \frac{ig}{x^{31}} p_{31}^2 \right. \\ \left. + \frac{2g}{(x^{12})^2} p_{12} + \frac{2g}{(x^{23})^2} p_{23} + \frac{2g}{(x^{31})^2} p_{31} \right) \Delta^{-g}$$

$$= p_{12} p_{23} p_{31} + \frac{2ig}{x^{12}} p_{23} p_{31} + \frac{2ig}{x^{23}} p_{31} p_{12} + \frac{2ig}{x^{31}} p_{12} p_{23} \\ - \frac{4g^2}{x^{12}x^{23}} p_{31} - \frac{4g^2}{x^{23}x^{31}} p_{12} - \frac{4g^2}{x^{31}x^{12}} p_{23} + \frac{g(g-1)}{(x^{12})^2} p_{12} + \frac{g(g-1)}{(x^{23})^2} p_{23} + \frac{g(g-1)}{(x^{31})^2} p_{31} \\ - \frac{6ig^2(g+1)}{x^{12}x^{23}x^{31}} + 2ig(g-1)(g+2) \left(\frac{1}{(x^{12})^3} + \frac{1}{(x^{23})^3} + \frac{1}{(x^{31})^3} \right)$$

$$M^*M = 3I_3^2 - 12I_3HP + \frac{4}{3}I_3P^3 - 4H^3 + 14H^2P^2 - 3HP^4 + \frac{1}{6}P^6$$

$$\begin{aligned}
Q(2) &= \frac{1}{6} p_{12}^3 p_{23}^3 p_{31}^3 \\
&+ \frac{3}{(x^{12})^2} (p_{12}^3 p_{23}^2 p_{31}^2 + 2 p_{12} p_{23}^3 p_{31}^3) \\
&+ \frac{12i}{(x^{12})^3} (p_{12}^2 p_{23}^3 p_{31} + p_{23}^3 p_{31}^3 + 4 p_{12}^2 p_{23}^2 p_{31}^2) \\
&- \left(\frac{12}{(x^{12})^4} - \frac{24}{(x^{12})^2 (x^{31})^2} \right) p_{12}^3 p_{23}^2 \\
&+ \left(\frac{264}{(x^{23})^4} - \frac{180}{(x^{12})^4} - \frac{168}{(x^{12})^2 (x^{23})^2} \right) p_{12} p_{23}^2 p_{31}^2 \\
&+ i \left(\frac{1440}{(x^{12})^5} - \frac{720}{(x^{12})^3 (x^{31})^2} - \frac{720}{(x^{12})^2 (x^{31})^3} \right) p_{12}^3 p_{23} \\
&+ i \left(\frac{1080}{(x^{12})^5} - \frac{360}{(x^{31})^5} - \frac{360}{(x^{12})^3 (x^{31})^2} - \frac{1080}{(x^{12})^2 (x^{31})^3} \right) p_{12}^2 p_{23}^2 \\
&+ \left(\frac{4200}{(x^{12})^6} + \frac{3360}{(x^{23})^6} - \frac{1920}{(x^{12})^3 (x^{23})^3} + \frac{1200}{(x^{23})^3 (x^{31})^3} + \frac{2880}{(x^{12})^2 (x^{31})^4} \right) p_{12}^3 \\
&- \frac{4320}{(x^{12})^2 (x^{31})^4} p_{12}^2 p_{23} - \frac{5760}{(x^{12})^2 (x^{31})^4} p_{12} p_{23} p_{31} \\
&+ i \left(\frac{25200}{(x^{12})^7} - \frac{10080}{(x^{23})^7} - \frac{7200}{(x^{12})^5 (x^{23})^2} - \frac{5760}{(x^{12})^4 (x^{23})^3} + \frac{10080}{(x^{12})^3 (x^{23})^4} - \frac{1440}{(x^{12})^2 (x^{23})^5} \right) p_{12}^2 \\
&- \left(\frac{90720}{(x^{12})^8} + \frac{198720}{(x^{12})^7 (x^{23})} - \frac{129600}{(x^{12})^6 (x^{23})^2} + \frac{34560}{(x^{12})^5 (x^{23})^3} - \frac{17280}{(x^{12})^3 (x^{23})^5} \right) p_{12} \\
&- \frac{181440i}{(x^{12})^9} - \frac{60480i}{(x^{12})^7 x^{23} x^{31}} + \text{all permutations in } (123)
\end{aligned}$$

nontrivial commutators:

$$i[P, F_1] = 6H - P^2$$

$$i[I_3, F_1] = 12H^2 - 3I_3P$$

$$i[P, F_3] = 4H^2 - I_3P$$

$$i[I_3, F_3] = -3I_3^2 + 8I_3HP + 12H^3 - 12H^2P^2 + HP^4$$

$$i[F_1, F_3] = \frac{1}{2}(F_1I_3 + I_3F_1 + F_3P + PF_3)$$

$$i[Q, F_1] = -3(2g-1)QP$$

$$i[Q, F_3] = -3(2g-1)Q\left(I_3 - \frac{4}{3}HP\right)$$

$$Q^2 = \left(3I_3^2 - 12I_3HP + \frac{4}{3}I_3P^3 - 4H^3 + 14H^2P^2 - 3HP^4 + \frac{1}{6}P^6\right)^{2g-1}$$

Four particles

Dunkl operators:

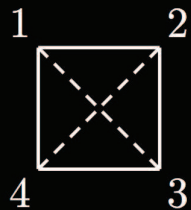
$$\pi_{12} = p_{12} + \frac{2ig}{x_{12}}s_{12} + \frac{ig}{x_{13}}s_{13} - \frac{ig}{x_{23}}s_{23} + \frac{ig}{x_{14}}s_{14} - \frac{ig}{x_{24}}s_{24} \quad \text{and permutations}$$

conserved for any g value: $P, H, I_3, I_4, F_1, F_3, F_4$

conformal Casimir:
$$C = \frac{1}{2} \sum_{i < j} (x^j p_i - x^i p_j)^2 - \frac{3}{8} + g(g-1) \sum_i (x^i)^2 \sum_{j < k} \frac{1}{(x^{jk})^2}$$

basic intertwiner:

$$M(g) = \text{res}(\pi_{12}(g)\pi_{31}(g)\pi_{14}(g)\pi_{23}(g)\pi_{24}(g)\pi_{34}(g))$$



$$= \frac{1}{x^{31} x^{24}} p_{12} p_{23} p_{34} p_{14}$$

$$\begin{aligned}
 M(g) = & \frac{1}{24} \square + \frac{g}{2} \square + \frac{g^2}{2} \square + 2g^2 \square + g(g-1) \square \\
 & - g^2(g+1) \square + 2g^2(g-1) \left(\square + \square - \square \right) - g(g-1)(g+2) \square \\
 & - 4g^3 \left(\frac{1}{3} \square + \square \right) + g(g-1) \left(2 \square - \square \right) \\
 & + 2g^4 \square + 2g^2(g-1)(g+2) \left(2 \square - \square + \square \right) - g^2(g-1)(5g+3) \square \\
 & + 4g^3(g-1) \left(\square + 2 \square \right) + g^2(g-1)^2 \left(\square + \frac{1}{2} \square \right) + 6g^3(g+1) \square \\
 & + 2g^2(g-1) \square - \frac{1}{2}g(g-1)(g^2 - 5g - 18) \square \\
 & - \frac{9}{2}g^3(g+1)^2 \square + 5g^2(g-1)(g+1)(g+2) \left(\square - \frac{1}{2} \square + \frac{1}{2} \square - \frac{1}{2} \square \right) \\
 & + 8g^3(g-1)(g+2) \left(\square - \square \right) - g(g-3)(g-1)(g+2)(g+4) \square \\
 & + 2g^2(g-1)^2 \left((g+2) \square + \frac{(g+1)}{3} \square \right) - 2g^2(g-1)(g+2)(3g+1) \square \\
 & - g^3(g-1)(g+1) \square - g^2(g-1)(g^2 - 5g - 18) \square - 4g^3(g-1)(4g+5) \square \\
 & + \frac{1}{3}g^3(g+1)(5g^2 + 5g + 8) \square - 2g^3(g-1)(g+1)(g+2) \left(4 \square + \square \right) \\
 & + g^2(g-1)^2(g+2)^2 \square - g^2(g-3)(g-1)(g+2)(g+4) \square + \text{all permutations in } (1234).
 \end{aligned}$$

$$\begin{aligned}
M^*M = \frac{1}{576} & \left(P^{12} - 48P^{10}H + 840P^8H^2 - 6368P^6H^3 + 19344P^4H^4 \right. \\
& - 21888P^2H^5 + 4608H^6 + 40P^9I_3 - 1296P^7HI_3 \\
& + 12576P^5H^2I_3 - 33344P^3H^3I_3 + 24576PH^4I_3 \\
& + 544P^6I_3^2 - 9024P^4HI_3^2 + 16896P^2H^2I_3^2 - 4352H^3I_3^2 \\
& + 2496P^3I_3^3 - 1152PHI_3^3 - 192I_3^4 - 36P^8I_4 \\
& + 1152P^6HI_4 - 10800P^4H^2I_4 + 24768P^2H^3I_4 \\
& - 11520H^4I_4 - 1008P^5I_3I_4 + 16416P^3HI_3I_4 \\
& - 25344PH^2I_3I_4 - 7200P^2I_3^2I_4 + 2304HI_3^2I_4 + 468P^4I_4^2 \\
& \left. - 7488P^2HI_4^2 + 9216H^2I_4^2 + 6912PI_3I_4^2 - 2304I_4^3 \right)
\end{aligned}$$

nontrivial commutators:

$$\begin{aligned}
 i[P, F_1] &= 8H - P^2, & i[I_3, F_1] &= 12H^2 - 3I_3P, & i[P, F_3] &= 4H^2 - I_3P \\
 i[I_3, F_3] &= 6I_4H - 3I_3^2, & i[P, F_4] &= 2I_3H - I_4P, & i[I_4, F_1] &= 8I_3H - 4I_4P \\
 i[I_3, F_4] &= -3I_4I_3 + \frac{15}{2}I_4HP + 10I_3H^2 - 5I_3HP^2 - 15H^3P + 5H^2P^3 - \frac{1}{4}HP^5 \\
 i[I_4, F_3] &= -4I_4I_3 + 10I_4HP + \frac{40}{3}I_3H^2 - \frac{20}{3}I_3HP^2 - 20H^3P + \frac{20}{3}H^2P^3 - \frac{1}{3}HP^5 \\
 i[I_4, F_4] &= -4I_4^2 + 12I_4H^2 + 6I_4HP^2 + \frac{8}{3}I_3^2H - \frac{16}{3}I_3HP^3 - 8H^4 - 12H^3P^2 + 6H^2P^4 - \frac{1}{3}HP^6
 \end{aligned}$$

$$i[F_1, F_3] = \frac{1}{2}(F_1I_3 + I_3F_1 + F_3P + PF_3)$$

$$i[F_1, F_4] = \frac{1}{2}(F_1I_4 + I_4F_1 + 2F_4P + 2PF_4 - 12F_3H)$$

$$\begin{aligned}
 i[F_3, F_4] &= F_4I_3 + I_3F_4 - 2F_4HP - \frac{1}{2}F_3I_4 - \frac{1}{2}I_4F_3 - F_3H(2H - P^2) - \frac{1}{2}I_4HF_1 + H^3F_1 \\
 &\quad + \frac{1}{3}I_3H(F_1P + PF_1) - \frac{1}{3}H^2(F_1P^2 + PF_1P + P^2F_1) + \frac{1}{60}H(F_1P^4 + \dots + P^4F_1)
 \end{aligned}$$

$$i[Q, F_1] = -6(2g-1)QP$$

$$i[Q, F_3] = -6(2g-1)Q(I_3 - HP)$$

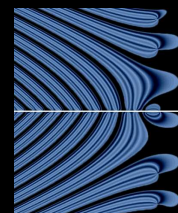
$$i[Q, F_4] = -6(2g-1)Q\left(I_4 - \frac{5}{3}H^2 - \frac{1}{6}HP^2\right)$$

$$Q^2 = (M^*M)^{2g-1}$$

Summary

- for generic coupling g , rank- n Calogero models are superintegrable
- n conserved Liouville charges $\{I_1=P, I_2=2H, I_3, \dots, I_n\}$ in involution
- $\text{ad}C$ produces $n-1$ additional charges $\{F_1, F_3, F_4, \dots, F_n\}$ not in involution
- they obey a nonlinear commutator algebra (polynomial of order $2n-1$)
- Dunkl operators allow one to construct intertwiners $g \leftrightarrow g+1$
- for integer coupling g , chain of intertwiners link with free theory ($g=1$)
- Vandermonde $\prod_{i<j}(p_i-p_j)$ gets mapped to new 'odd' conserved charge Q
- Q is of weight $\frac{1}{2}n(n-1)(2g-1)$ and squares to a polynomial in the I_k
- Q extends commutator algebra to a Z_2 graded one \rightarrow nonlinear 'supersymmetry'
- explicitly worked out the cases $n=2, 3, 4$ (partially for the first time)

DFG



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