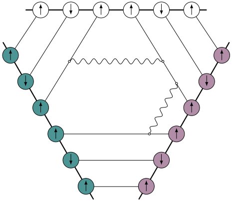
Spin Chains and Three-Point Functions in $\mathcal{N} = 4$ Super Yang–Mills Theory

Florian Loebbert
Humboldt University Berlin



work with
Yunfeng Jiang,
Ivan Kostov,
Didina Serban

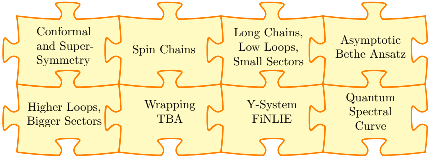
[JHEP 1404 (2014) 019]

Planar $\mathcal{N} = 4$ super Yang–Mills Theory

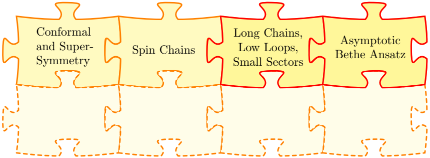
Conformal Field Theory $\rightarrow$ We want Spectrum and Three-Point Functions!

Use Integrability [Review 2012 Beisert et al.] [Kazakov's Talk]

1. Spectrum (almost solved):



2. Three-Point Functions:



- Simplest subsector: Complex scalar fields X, Z in $\mathfrak{su}(2)$ sectors.
- Study one-loop correction (previously obtained in [Gromov Vieira, 12]).
- Compare to string theory result.

Asymptotic Spectrum: $\mathfrak{su}(2)$-Sector

Higher Loops: Dilatation Operator $\mathcal{Q}_2(g^2) = H_2 + g^2 \ldots$

↖ 't Hooft coupling

Spin Chains (cyclic): $\leftrightarrow$ Gauge invariant states:

$$|{\downarrow\uparrow\uparrow}\ldots{\downarrow}\rangle(x) \quad \leftrightarrow \quad \mathcal{O}(x) = \mathrm{Tr}(XZZ\ldots X)(x).$$

Excitations: Characterized by sets of rapidities $\mathbf{u} = \{u_1, u_2, \ldots, u_M\}$:

$$\frac{x(u_j + \frac{i}{2})}{x(u_j - \frac{i}{2})} = e^{ip_j} \quad \rightarrow \quad |\mathbf{u}\rangle \sim |\ldots\uparrow\uparrow\uparrow \overset{e^{ip_j}}{\downarrow} \uparrow\uparrow\uparrow\ldots\uparrow\uparrow\uparrow \overset{e^{ip_k}}{\downarrow} \uparrow\uparrow\uparrow\ldots\ldots\rangle$$

Integrability : Tower of commuting charges: $\mathcal{Q}_r(g)$ with $Q_2(g) = \mathcal{D}(g)$

$\Rightarrow$ Dilatation Operator diagonalized by Bethe Ansatz [Minahan Zarembo, 02]

$$\left(\frac{x(u_k + \frac{i}{2})}{x(u_k - \frac{i}{2})}\right)^L = \prod_{\substack{j\neq k\\ j=1}}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}\, e^{2i\phi(u_k,u_j)}$$

BDS: [Beisert, Dippel Staudacher, 04]

Dressing: [Arutyunov, Beisert Eden, Frolov, Hernández López, Staudacher, ...]

- Valid up to wrapping order!
- Dressing Phase starts at four loops

Two Perturbative Definitions of Higher-Loop Spin Chains

I. Deformations using Boost Operators [Bargheer, Beisert FL, 08/09]

Start with Heisenberg (XXX) spin chain with local charges H_r:

$$\text{One loop: } \mathcal{Q}_r(0) \equiv H_r \qquad \text{with} \qquad [H_r, H_s] = 0 \qquad r, s = 2, 3, \ldots$$

Construct higher-loop integrable charges:

$$\frac{d}{dg}\mathcal{Q}_r(g) = \tau_s[\mathcal{B}_s(g), \mathcal{Q}_r(g)] \quad \Rightarrow \quad \mathcal{Q}_r(g) = T_{\text{Boost}}(g) H_r T^{-1}_{\text{Boost}}(g) + \text{wrapping}$$

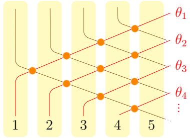
first moment of local charge $\mathcal{Q}_s$

II. Inhomogeneous Spin Chains [Beisert, Dippel Staudacher, 04][Serban Staudacher, 05][Jiang,Kostov FL,Serban,14]

$$\prod_{j=1}^{L} \frac{u_k - \theta_j(g) + \frac{i}{2}}{u_k - \theta_j(g) - \frac{i}{2}} = \prod_{j=1; j\neq k}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}, \qquad \theta_j^{\text{BDS}}(g) = 2g \sin\frac{2\pi j}{L}$$

Corner Transfer Matrix

Works at leading orders, no general proof!

$T_\theta(u) =$ 

$$\mathcal{Q}_2(\theta) = T_\theta(0)\, H_2\, T_\theta^{-1}(0) + \text{wrapping}$$

▶ T_{Boost} and T_θ singular on periodic chains → **no similarity transformations**

S-Operator

Two singular transformations T_{Boost} and T_θ generate the same singularity $\to$ twist of the Bethe equations!

$$\mathcal{Q}_2(g) \simeq T_{\text{Boost}}(g) H_2 T^{-1}_{\text{Boost}}(g) \qquad \mathcal{Q}_2(\theta) \simeq T_\theta(0)\, H_2\, T^{-1}_\theta(0)$$

$\Rightarrow$ Inhomogeneous and Boost-Deformed Chains are related by similarity transformation up to wrapping order: [Bargheer Beisert, FL, 09][Jiang, Kostov FL, Serban, 14]

Unitary S-Operator $\quad S = T_{\text{Boost}} \times T_\theta^{-1} \quad \Rightarrow \quad \mathcal{Q}_2(g) = S\,\mathcal{Q}_2(\theta) S^{-1}$

- S is well-defined on periodic spin chains as opposed to T_{Boost} and T_θ!
- S defines *exact morphism* of the Yangian algebra/monodromy matrix:

Comparison with *Theta-morphism* of [Gromov Vieira, 12]:
(Exact)S-morphism = (Approximate)Theta-morphism + Cross-terms

Three-Point Functions

Correlator of three eigenstates of the dilatation operator in three $\mathfrak{su}(2)$ sectors:

States:	$\mathcal{O}_1(x_1)$	$\mathcal{O}_2(x_2)$	$\mathcal{O}_3(x_3)$
Made of Scalars:	Z, X	$\bar{Z}, \bar{X}$	$Z, \bar{X}$
Bethe state:	$\lvert\mathbf{u}_1\rangle$	$\lvert\mathbf{u}_2\rangle$	$\lvert\mathbf{u}_3\rangle$

Conformal symmetry:

$$\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}(g^2)}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}}$$

Integrability:

$$C_{123}(g^2) = \frac{\langle\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\rangle}{(\langle\mathbf{u}_1|\mathbf{u}_1\rangle\langle\mathbf{u}_2|\mathbf{u}_2\rangle\langle\mathbf{u}_3|\mathbf{u}_3\rangle)^{1/2}}$$ [Escobedo,Gromov Sever,Vieira,10]

Scalar Products (of one on-shell and one off-shell Bethe state):

$${}_{\text{loop}}\langle\mathbf{u}| = \langle\mathbf{u},\boldsymbol{\theta}|S^{-1} \qquad\qquad |\mathbf{u}\rangle_{\text{loop}} = S|\mathbf{u},\boldsymbol{\theta}\rangle$$

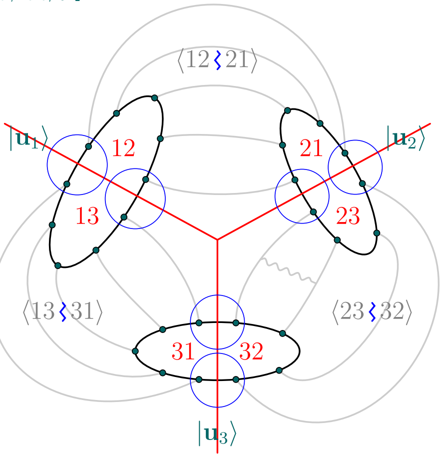
$\Rightarrow$ Use *Slavnov-Determinant-Formula* for inhomogeneous scalar products:

$${}_{\text{loop}}\langle\mathbf{u}|\mathbf{v}\rangle_{\text{loop}} = \langle\mathbf{u},\boldsymbol{\theta}|\mathbf{v},\boldsymbol{\theta}\rangle \simeq A_{\mathbf{u}\cup\mathbf{v},\boldsymbol{\theta}}, \qquad A_{\mathbf{u},\boldsymbol{\theta}} = \mathrm{Det}(\mathbb{I}-K)$$ [Jiang, Kostov FL, Serban, 14]

$$K_{jk} = \frac{iE_j}{u_j-u_k+i}, \qquad E_j = \frac{Q_\theta(u_j-\frac{i}{2})}{Q_\theta(u_j+\frac{i}{2})}\prod_{k=1;k\neq j}^{M}\frac{u_j-u_k+i}{u_j-u_k}, \qquad Q_\theta(u) = \prod_{j=1}^{L}(u-\theta_j).$$

Find Structure Constants $\langle \mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3} \rangle$

General idea: [Escobedo,Gromov Sever,Vieira,10]:

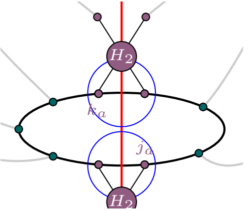


Tree-level: [Slavnov 89][Escobedo, Gromov Sever,Vieira, 10][Foda 11]

- Solved → Slavnov-determinants

One Loop: Two Types of Loop-Insertions

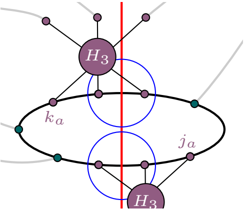
1. To the correlator: [Okuyama Tseng, 04][Roiban Volovich, 04][Alday,David Gava,Narain, 05]



Insertion of the Heisenberg-Hamiltonian (one-loop Dilatation Operator) at the splitting points:

$$\mathbb{I}_a = 1 - g^2(H_{2,k_a} + H_{2,j_a}), \quad {\scriptstyle a=1,2,3}$$

2. To the eigenstates: [Jiang,Kostov FL,Serban,14]



Insertion from the S-operator at the splitting points:

$$\delta S_a = 1 - g^2(H_{3,k_a} + H_{3,j_a}), \quad {\scriptstyle a=1,2,3}$$

Combine Things

Skip some nontrivial steps: [Escobedo, Gromov, Sever, Vieira, 10] [Foda 11]

$$\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \rangle = \sum_{i_1,\dots,i_{L_{12}}=\uparrow,\downarrow} \langle \mathbf{u}_2, \boldsymbol{\theta}_2 | \delta S_2^{-1}\, \mathbb{I}_2 \,| i_1 \dots i_{L_{12}} \underbrace{\uparrow \dots \uparrow}_{L_{23}} \rangle$$

$$\times \langle i_1 \dots i_{L_{12}} \underbrace{\downarrow \dots \downarrow}_{L_{13}} | \,\mathbb{I}_1\, \delta S_1 | \mathbf{u}_1, \boldsymbol{\theta}_1 \rangle$$

$$\times \langle \underbrace{\uparrow \dots \uparrow}_{L_{23}} \underbrace{\downarrow \dots \downarrow}_{L_{13}} | \mathbb{I}_3\, \delta S_3 | \mathbf{u}_3, \boldsymbol{\theta}_3 \rangle$$

Two steps to get simple form: [Jiang, Kostov, FL, Serban, 14]

1. Rewrite insertions $\mathbb{I}_a$ and δS_a in terms of derivatives $\partial_k = \partial / \partial \theta_k$
2. Fix θ to coupling dependent BDS-values $\theta_{a,\ell}^{\text{BDS}}(g) = 2g \sin \frac{2\pi\ell}{L_a}$

$$\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \rangle = F_{123}(\boldsymbol{\theta}) + g^2 \delta F_{123}(\boldsymbol{\theta}) = (1 + g^2 \hat{\Delta}) F_{123}(\boldsymbol{\theta}) + \mathcal{O}(g^3) \Big|_{\theta = \theta^{\text{BDS}}}$$

$$F_{123}(\boldsymbol{\theta}) = A_{\mathbf{u}_2 \cup \mathbf{u}_1, \boldsymbol{\theta}_{12}} + A_{\mathbf{u}_3, \boldsymbol{\theta}_{13}} \qquad \hat{\Delta} = \hat{\Delta}_{12} + \hat{\Delta}_{03} \qquad \delta E_r^{ab} = E_r^b - E_r^a$$

$$\hat{\Delta}_{ab} A_{\mathbf{u}_a \cup \mathbf{v}_b, \boldsymbol{\theta}_{bc}} = \left(\partial_1^b \partial_2^b - i \delta E_2^{ab}\, \partial_1^b + i \delta E_3^{ab} - \tfrac{1}{2} (\delta E_2^{ab})^2 \right) A_{\mathbf{u}_a \cup \mathbf{v}_b, \boldsymbol{\theta}_{bc}}$$

- Agrees with [Gromov, Vieira, 12], but simpler → Can take thermodynamical limit.

Comparison With String Computation

Thermodynamical Limit (Gauge Theory):

- State of length L with M excitations.
- Take $L \to \infty$, $M \to \infty$, $\frac{M}{L}$ finite, $\lambda' = \frac{g^2}{L^2} \ll 1$.

Frolov–Tseytlin Limit (String Theory): [Frolov Tseytlin, 02]

- Rotating string on S^3 with angular momentum J
- $g^2 \to \infty$, $J \to \infty$, $\lambda' = \frac{g^2}{J^2} \ll 1$

Spectrum: Known that first two orders in λ' agree in Gauge & String theory.

Three-point functions?: One-loop 3pt-function requires two-loop eigenfunction of dilatation operator.
$\to$ Expect match at first order in λ'.

Thermodynamical vs Frolov–Tseytlin Limit

Gauge Theory: [Jiang,Kostov FL,Serban, 14][Kostov 12²]

suppressed by $1/L$ since localized at splitting points

$$\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \rangle_{\text{Gauge}} = F_{123}(\boldsymbol{\theta}) + g^2 \delta F_{123}(\boldsymbol{\theta}) + \mathcal{O}(g)^4$$

$$\simeq \left[\oint \frac{du}{2\pi} \mathrm{Li}_2 \left[e^{ip_1(u)+ip_2(u)-iq_3(u)} \right] \right]_1 + \Big[\sim \Big]_2$$

String Theory: [Kazama Komatsu, 13] also: [Janik, 11 Wereszczynski]

$$\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \rangle_{\text{String}} \simeq \Big[\sim \Big]_1 + \Big[\sim \Big]_2 \quad \rightarrow \quad \text{Integrand agrees} \quad \checkmark$$

$$+ \Big[\sim \Big]_3 + \Big[\sim \Big]_4 \quad \rightarrow \quad \text{Should vanish for agreement.} \quad ?$$

Do contours match for first term? Does the second term vanish?

Last Slide Before Dinner

Summary:

Asymptotic Spectrum:
Inhomogeneous Bethe Ansatz $\to$ Fix $\theta = \theta(g)$

Structure Constants (1 Loop):
Inhomogeneous Correlators $\to$ Fix $\theta = \theta(g)$ & Act with $\hat{\Delta}$ on splitting points
$\to$ *Matches string theory result in Frolov–Tseytlin limit.*

Future Puzzles:

Two loops: Use above method $\to$ Recursion for $\hat{\Delta}$?

Higher Loops: How to include Dressing Phase?

Asymptotic Spin Chain		Generator	Relation
Inhomogeneous Chain		T_θ	$S = T_{\text{Boost}} \times T_\theta^{-1}$
$\mathcal{N} = 4$ SYM $\mathfrak{su}(2)$	Boost	T_{Boost}	
	Bilocal	T_{Bilocal}	$S = T_{\text{Bilocal}} \times$?
?		?	

Thank You!