Mock Moonshine

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Thirty years ago some very surprising relations between the Fourier coefficients of the Jacobi Hauptmodul or the J-function and representations of the largest finite simple sporadic group, the Monster were discovered. Precise formulation of these relations is now called the 'Monstrous Moonshine'. It has given rise to a large body of new mathematics. The moonshine correspondence has been extended to other groups revealing unexpected relations to conformal field theory, gravity and string theory in physics and to Ramanujan's mock theta functions and its extensions in mathematics. We call these results which have been discoverd in the last 30 years ' Mock Moonshine'. We will survey some recent developments related to Mock Moonshine and indicate directions for future research.

List of Topics

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2. Modular objects

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6. Mock Moonshine and Field Theories

The following historical material is taken from [4]. Recall that a group is called **simple** if it has no proper non-trivial normal subgroup. Thus an abelian group is simple if and only if it is isomorphic to one of the groups $\mathbf{Z}_p,$ for p a prime number. This is the simplest example of an infinite family of finite simple groups. Another infinite family of finite simple groups is the family of alternating groups $A_n, n > 4$ that we study in the first course in algebra. These two families were known in the 19th century. The last of the families of finite groups, called groups of Lie type were defined by Chevalley in the mid 20th century. This work led to the classification of all infinite families of finite simple groups. However, it was known that there were finite simple groups which did not belong to any of these families. Such groups are called **sporadic groups**.

The first sporadic group was constructed by Mathieu in 1861. In fact, he constructed five sporadic groups, now called Mathieu groups. There was an interval of more than 100 years before the sixth sporadic group was discovered by Janko in 1965. By 1981, nineteen new sporadic groups were discovered bringing the total to 25. The existence of the 26th and the largest of these groups was conjectured independently by Fischer and Griess in 1973. The construction of this "Friendly Giant" (now called the Monster) was announced by Griess in 1981 and its uniqueness was soon proved by Norton. So the classification of finite simple groups was complete. It ranks as the greatest achievement of twentieth century mathematics. Hundreds of mathematicians contributed to it. The various parts of the classification proof together fill thousands of pages. The project to organize all this material and to prepare a flow chart of the proof was started by Gorenstein and is expected to continue for years to come.

McKay correspondence is an observation about the closeness of the coefficients of Jacobi's Hauptmodul and the character degrees of representations of the Monster. To understand this as well as the full moonshine conjectures we need to know the classical theory of modular forms and functions. We now discuss parts of this theory needed in our work.

The **modular group** $\Gamma = \mathbf{SL}(2,\mathbf{Z})$ acts on the upper half plane $\mathbf{H}$ by fractional linear transformations as follows:

$$Az = \frac{az+b}{cz+d},\ \ z \in \mathbf{H}, A \in \Gamma\ . \qquad (1)$$

A function f that is analytic on $\mathbf{H}$ and at ∞ is called a **modular form of weight** k if it satisfies the following conditions:

$$f(Az) = (cz+d)^k f(z)\ \ ,\forall A \in \Gamma.$$

Jacobi's **Hauptmodul** J for the modular group Γ is a modular form of weight 0, i.e a **modular function**. The first few terms in the Fourier expansion of the function J are given by

$$J = q^{-1} + 196884q + 21493760q^2 + \ldots\ . \quad (2)$$

We note that the quotient space $\mathbf{H}/\Gamma$ under the action of Γ on the upper half plane $\mathbf{H}$ is a surface of genus zero. The fact that the coefficients of the J-**function** are integers has had several arithmetic applications, for example in the theory of complex multiplication and class field theory. The p-adic properties and congruences of these coefficients have been studied extensively. The positivity of the coefficients suggests (according to an old fock lore) that we should ask if they are the ranks of representations of some interesting group. As we will see, this in fact, turns out to be the case here.

As evidence for the existence of the largest sporadic simple group F_1 predicted in 1973 by Fischer and Griess mounted, several scientists conjectured that this exceptional group should have relations with other areas of mathematics and should even appear in some natural phenomena. The results that have poured in since then seem to justify this early assessment. Some strange coincidences noticed first by McKay and Thompson were investigated by Conway and Norton. They called these unbelievable set of conjectures "**Monstrous Moonshine**" and the Fischer-Griess group F_1 the **Monster** and denoted it by $\mathbb{M}$. The existence and uniqueness of the Monster was the last piece in the classification of finite simple groups. This classification is arguably, the greatest achievement of 20th century mathematics.

Graded algebraic structures appear naturally in many mathematical and physical theories. V_0 (resp. V_1) **even or Bosonic** (resp. **odd or Fermionic**). In physics a $\mathbf{Z}_2$-graded space is called a **superspace**. In 1984, Ed Witten (Fields medal, Kyoto ICM 1990) used supersymmetric quantum mechanics to obtain topological invariants of a manifold. This work is an early example of what I have called 'Physical Mathematics'. Other algebraic structures (such as Lie, commutative etc.) have their graded counterparts. It was **Grassmann** (1809 - 1877) who first defined the structure of an **exterior algebra** associated to a finite dimensional vector space. Grassmann's work was well ahead of his time and did not receive recognition for a long time. In the preface to his 1862 book he wrote: *there will come a time when these ideas, perhaps in a new form will enter into contemporary developments*. Indeed, Grassmann's expectation has come to fruition and his work has found many applications in mathematics and physics.

We define the quantum dimension $\dim_q V$ of a graded vector space V , graded by $\mathbf{Z}$ as a power series in the formal variable q. In particular,

$$\dim_q V = \sum_{n \in \mathbf{Z}} q^n (\dim(V_n)\ , q = e^{(2\pi i z)}, z \in \mathbf{C}$$

can be regarded as the Fourier expansion of a complex function. The monster Lie algebra is the simplest example of a Lie algebra of physical states of a chiral string on a 26-dimensional orbifold. This algebra can be defined by using the infinite dimensional graded representation $V^\natural$ of the Monster. Its quantum dimension is related to Jacobi's Hauptmodul J. It was the coefficient 196884 in the formula for J that attracted John McKay's attention. This number is very close to 196883, the character degree of the smallest non-trivial irreducible representation of the Monster. McKay communicated his observation to Thompson. We summarize below Thomson's observations on the numerology between the Monster and the Jacobi modular function J.

Let $c(n)$ denote the coefficient of the n-th term in $J(q)$ and let χ_n be the n-th irreducible character of the Monster group $\mathbf{M}$. Then the character degree $\chi_n(1)$ is the dimension of the n-th irreducible representation of $\mathbf{M}$. We list below the first few values of $c(n)$ and $\chi_n(1)$.

A strange correspondence

n	$c(n)$	$\chi_n(1)$
1	1	1
2	$196,884$	$196,883$
3	$21,493,760$	$21,296,876$
4	$864,299,970$	$842,609,326$
5	$20,245,856,256$	$18,538,750,076$

These observations led to Conway and Norton's Monstrous Moonshine or Moonshine Conjectures which we now state.

1. For each $g \in \mathbf{M}$ there exists a function $T_g(z)$ with normalized Fourier series expansion given by

$$T_g(z) = q^{-1} + \sum_{1}^{\infty} c_g(n) q^n \ . \qquad (3)$$

There exists a sequence H_n of representation of $\mathbf{M}$, called the **head representations** such that $c_g(n) = \chi_n(g)$, where χ_n is the character of H_n.

2. For each $g \in \mathbf{M}$, there exists a Hauptmodul J_g for some modular group of genus zero, such that $T_g = J_g$.

3. Let $[g]$ denote the set of all elements in $\mathbf{M}$ that are conjugate to g^i, $i \in \mathbf{Z}$. Then T_g depends only on the class $[g]$. However, $[g]$ is not the usual conjugacy class. There are 194 conjugacy classes of $\mathbf{M}$ but only 171 distinct McKay-Thompson series.

Conway and Norton calculated all the functions T_g and compared their first few coefficients with the coefficients of known genus zero Hauptmoduls. Such a check turns out to be part of Borcherds (Fields medal, Berlin ICM 1998) proof which he outlined in his lecture at the 1998 ICM in Berlin. The first step was the construction of the **Moonshine Module** which makes essential use of ideas from string theory. The entire book by Frenkel, Lepowsky and Meurman is devoted to the construction of this module, denoted by $V^\natural$. It has the structure of an algebra called the **Moonshine vertex operator algebra** (also denoted by $V^\natural$). They proved that the automorphism group of the infinite dimensional graded algebra $V^\natural$ is the largest of the finite, sporadic, simple groups, namely, the Monster.

After formulating the monstrous moonshine conjecture, Conway and Norton speculated that moonshine phenomenon may also be found for othe sporadic groups. This has been realized for a number of sporadic groups. Computations of several low order coefficients in the Fourier expansion of some modular forms shows that they can be expressed in terms of irreducible representations of sporadic groups. For example, a hauptmodul for the baby monster group of Fischer has properties similar to the monstrous moonshine conjecture, i.e. the hauptmodul can be identified with the quantum or graded dimension of representations of the baby monster. The Fourier expansion of this hauptmodul starts with

$$q^{-1} + 4372q + 96256q^2 + \ldots \qquad .$$

The ranks of the two smallest non-trivial ireducible representations of the baby monster are 4371 and 96256 respectively.

The study of moonshine phenomenon for the largest of the Mathieu groups, $\mathbf{M}_{24}$ by Eguchi, Ooguri and Tachikawa [3] in 2010 showed an unexpected relatiobn with the complex elliptic genus of a $K3$ surface X. In theoretical physics this appears as the partition function of physical states in certain sigma models on X. This observation and further work by Terry Gannon showed that the original moonshine has to be modified and extended when applied to sporadic groups such as the Mathieu group $\mathbf{M}_{24}$. It also suggested strong links to string theory. In further study, Eguchi and coworkers discovered that in the decomposition of the elliptic genus of X into irreducible characters of the $N = 4$ superconformal algebra the following q-series appears.

$$H^{(2)} = 2q^{-1/8}\left(-1 + 45q + 231q^2 + 770q^3 \ldots\right).$$

The coefficients $45, 231, 770, \ldots$ turn out to be the ranks of the first few irreducible representations of the Mathieu group $\mathbf{M}_{24}$. This observation was later extended to the Mathieu moonshine conjecture by construction of the q-series

$$H_g^{(2)}\ , g \in \mathbf{M}_{24} \text{ with } , H_e^{(2)}\ = H^{(2)}\ ,$$

where e is the identity element of $\mathbf{M}_{24}$. However, unlike the q-series appearing in the monstrous moonshine, the series $H_g^{(2)}$, are not modular functions for any modular group. They turn out to be mock modular forms, first introduced by Ramanujan in his well known last letter to Hardy (he called them mock theta functions) dated January 12, 1920. In the rest of the 20th century, a number of people worked on these functions and found several new examples (most of these were already known to Ramanujan and were later found in his lost notebooks).

Precise definition of a mock theta function was given in 2001 by Zwengers. Zwengers fundamental work has now led to infinitely many new examples of mock theta functions. Mock modular forms are now known to appear in the moonshine conjectures of several groups. We would like to call these as examples of Mock moonshine. These sporadic groups arise as subgroups of the group of automorphisms of lattices. Conway discovered his sporadic groups by studying the symmetries of the 24-dimensional Leech lattice which appeared in the study of optimal sphere packing in 24 dimensions. The origin of the sphere packing problem can be traced back to Kepler. Kepler was an extraordinary observer of nature. His observations of snowflakes, honey-combs and the packing of seeds in various fruits led him to his lesser known study of the **sphere packing problem**. The sphere packing problem asks for the densest packing of standard unit spheres in a given Euclidean space.

The answer can be expressed by giving the number of spheres that touch a fixed sphere. For dimensions 1, 2 and 3 Kepler found the answers to be 2, 6 and 12 respectively. The lattice structures on these spaces played a crucial role in Kepler's "proof". The three dimensional problem came to be known as **Kepler's (sphere packing) conjecture**. The slow progress in the solution of this problem led John Milnor to remark that here is a problem that nobody can solve but its answer is known to every schoolboy. It was only solved recently (1998, Tom Hales). The Leech lattice provides the tightest sphere packing in a lattice in 24 dimensions (its proof was announced by Cohn and Kumar in 2004) and the sphere packing problem in most other dimensions is still wide open. In the Leech lattice each 24-dimensional sphere touches 196,560 others.

Symmetries of the Leech lattice contained Mathieu's largest sporadic group and it had a large number of symmetries of order 2. Leech believed that the symmetries of his lattice contained other sporadic groups. Leech was not a group theorist and he could not get other group theorists interested in his lattice. But he did find a young mathematician, who was not a group theorist, to study his work. In 1968, John Conway was a junior faculty member at Cambridge. He quickly became a believer in Leech's ideas. He tried to get Thompson (the great guru of group theorists) interested. Thompson told him to find the size of the group of symmetries and then call him. Conway later remarked that he did not know that he was using a folk theorem which says:

The two main steps in finding a new sporadic group are, find the size of the group of symmetries, and call Thompson.

Conway worked very hard on this problem and soon came up with a number. This work turned out to be his big break. It changed the course of his life and has made him into a world class mathematician. He called Thompson with his number. Thompson called back in 20 minutes and told him that half his number could be a possible size of a new sporadic group and that there were two other new poradic groups associated with it. The four groups that Conway discovered are now denoted by $Co0, Co1, Co2, Co3$ in Conway's honor. Further study by Conway and Thompson showed that the symmetries of the Leech lattice give 12 sporadic groups in all, including all five Mathieu groups, the first set of sporadic groups discovered over a hundred years ago.

Leech lattice belongs to a family of even unimodular positive definite lattices of rank 24 which were studied by Niemeier in 1979. Niemeier showed that Leech lattice is the unique lattice among these 24 with no root vectors, while the other 23 are characterized by their root systems. They are called Niemeier lattices. Recently Cheng, Duncan and Harvey [1] have shown how to associate a finite group G_i and a vector valued mock modular form H_i with each of the 23 Niemeier lattices L_i, where $1 \leq i \leq 23$. Umbral Moonshine Conjecture is the conjecture exhibiting moonshine like phenomena for each of the pairs G_i, H_i. The groups G_i are called Umbral groups. It turns out that the Mathieu group M_{24} is one of the Umbral groups. Mock Moonshine phenomenon was discovered for this group in [3] while studying a non-linear sigma model on a $K3$ surface.

A detailed calculation of all the 23 cases and possible moonshine moduls have been obtained by Cheng and Harisson in [6] by considering the relation between the ADE root systems of the Niemeier lattices and the corresponding singularities of $K3$ surfaces and their resolution. This gives a precise statement of the Umbral Moonshine conjectures. Connection with the geometry of $K3$ surfaces in each of the cases is also discussed there. From physical point of view elliptic genus of a $K3$ surface can be thought of as counting the BPS states of a non-linear sigma model. Modular aspect of the corresponding partition functions was an unexpected surprise when it was first noticed. What is even more surprising is that this case is not isolated.

Inspite of these advances, no connection of these cases to conformal field theory construction similar to the original Monster module was known. This situation has changed with the work of Cheng and co-workers in [5]. They have provided the first examples of mock (modular) moonshine for some sporadic simple groups by explicitly constructing underlying simple, and solvable conformal field theories in each case. Their starting point is the unique, non-trivial 24-dimensional representation of the Conway group $CO0$. They then study the subroups of $Co0$ which leave invariant (pointwise) some 2 and 3 dimensional subspces of this representation space. These subgoups are described in the book 'Sphere packings, Lattices and Groups' by Conway and Sloane. The list includes in particular, the Mathieu groups M_{22} and M_{23}.

They work out these two cases in full and also consider some other groups which arise as subgroups fixing n-planes ($n = 2, 3$). This work is also closely related to the well known ADE classification of some conformal field theories and characters of affine Lie algebras by Capelli, Itzykson and Zuber. Several scientists (including many present here) have also considered partition functions that arise in superstring theory and quantum gravity where modular objects appear. Gauge theory to string theory correspodence leading to generalized Gromov-Witten invariants also has links to modular objects.

Recently, Dabholkar, Murthy, and Zagier [2] have shown that the partition functions counting degeneracies of black holes in quantum gravity in a special $\mathcal{N} = 4$ supersymmetric theory can be expressed in terms of newly discovered modular objects. They also include all of Ramanujan's mock theta functions.

Some of the moonshine conjectures are now theorems. However, their deep significance for mathematics and physics is still emerging. So mathematicians and physicists, young and old, should rejoice at the emergence of a new subject, guaranteed to be rich and varied and deep, with many new questions to be asked and many of the conjectured results yet to be proved. It is indeed quite extraordinary that a new light should be shed on the theory of modular and mock modular forms, one of the most beautiful and extensively studied areas of classical mathematics, by the exotic sporadic groups from the Mathieu groups to the Monster. That the influence of Mock Moonshine goes beyond mathematics, into areas of theoretical physics such as conformal field theory, chiral algebras, string theory and quantum gravity may be taken as strong evidence for "Physical Mathematics", the newly created area of research in mathematics and physics.

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