Ambitwistor strings, the scattering equations and null infinity

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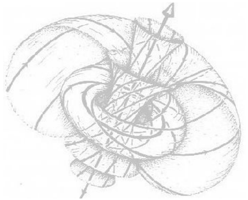
With David Skinner. arxiv:1311.2564, Yvonne Geyer & Arthur Lipstein 1404.6219, 1406.1462.

[Cf. also Cachazo, He, Yuan arxiv:1306.2962, 1306.6575, 1307.2199, 1309.0885]

Ambitwistors and gravity

Ambitwistors: Space of complex null geodesics extends Penrose/Ward's gravity/Yang-Mills twistor constructions to non-self-dual fields.

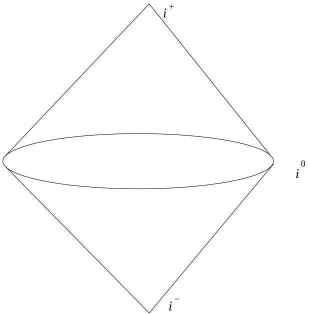


- Penrose's scattering formulae [1972].
- Yang-Mills Witten and Isenberg, et. al. 1978, Witten 1985.
- Conformal and Einstein gravity LeBrun [1983] Baston, M, [1987] LeBrun [1991].

Over last year, things have sped up:

- New gravity and Yang-Mills scattering formulae in all dimensions [Cachazo, He & Yuan 1307.2199, 1309.0885]
- Arise from ambitwistor strings [M. & Skinner 1311.2564]
- Expressed twistorially in 4d Geyer, Lipstein & M 1404.6219.
- Related to $\mathscr{I}$, null geodesic scattering and the BMS group [Geyer, Lipstein & M. 1406.1462].

Conformal scattering theory and tree-level S-matrix



- Pose asymptotic data g_{in} at $\mathscr{I}$.
- Solve for g on M.
- S-matrix $\mathcal{S}[g_{in}] =$ the action $S[g]$ evaluated on g.
- generating function for scattering.

Perturbatively:

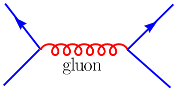
- Take $g_{\mathrm{in}} = \sum_{i=1}^{n} \eta_i g_i|_{\mathscr{I}}$,
- $\mathcal{M}(g_1, \ldots, g_n) =$ Coeff of $\prod_i \eta_i$ in $S_{EG}[g]$

Use Fourier modes for g_j: $g_{j\mu\nu} = \epsilon_{j\mu}\epsilon_{j\nu}e^{ik_j\cdot x}$:

- momentum k_j, $k_j^2 = 0$.
- polarization data satisfies $\quad k \cdot \epsilon = 0\,, \quad \epsilon \sim \epsilon + \alpha k\,.$

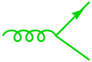
For n-particle scattering $\mathcal{M}(1, \ldots, n) = \mathcal{M}(k_1, \epsilon_1, \ldots, k_n, \epsilon_n)$.

Amplitudes are realized as sums of Feynman integrals.

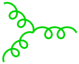


Feynman diagrams are more than pictures. They represent algebraic formulas for the propagation and interaction of particles.

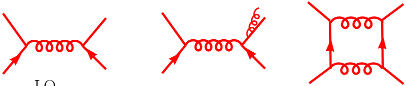
$$\frac{\eta_{\mu\nu}}{p^2}$$



$$g\,\gamma_\mu$$



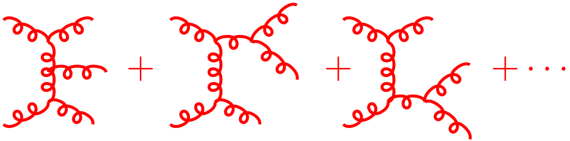
$$g[\,(p_1-p_2)_\rho\,\eta_{\mu\nu}+(p_2-p_3)_\mu\,\eta_{\nu\rho}+(p_3-p_1)_\nu\,\eta_{\rho\mu}\,]$$



Trees $\leftrightarrow$ classical, loops $\leftrightarrow$ quantum.

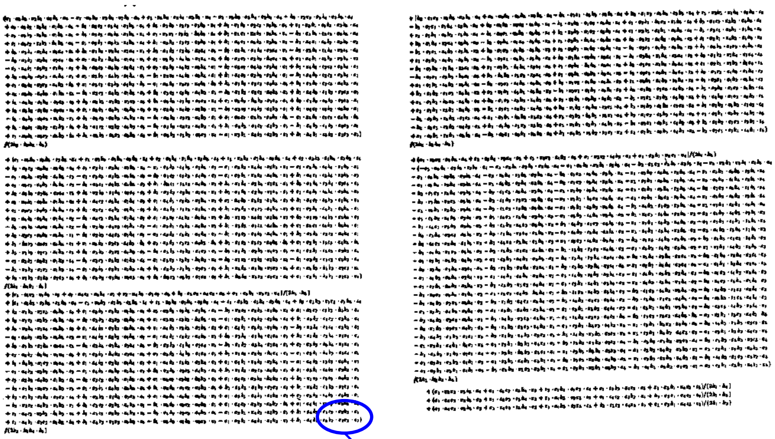
Consider the five-gluon tree-level amplitude of QCD. Enters in calculation of multi-jet production at hadron colliders.

Described by following Feynman diagrams:



If you follow the textbooks you discover a disgusting mess.

Result of a brute force calculation:



$$k_1 \cdot k_4\,\varepsilon_2 \cdot k_1\,\varepsilon_1 \cdot \varepsilon_3\,\varepsilon_4 \cdot \varepsilon_5$$

The scattering equations

Take n null momenta $k_i \in \mathbb{R}^d$, $i = 1, \dots, n$, $k_i^2 = 0$, $\sum_i k_i = 0$,

- define $P : \mathbb{CP}^1 \to \mathbb{C}^d$

$$P(\sigma) := \sum_{i=1}^{n} \frac{k_i}{\sigma - \sigma_i}\,, \qquad \sigma, \sigma_i \in \mathbb{CP}^1\,.$$

- Solve for $\sigma_i \in \mathbb{CP}^1$ with the *n scattering equations*

$$k_i \cdot P(\sigma_i) = \mathrm{Res}_{\sigma_i} P(\sigma) \cdot P(\sigma) = \sum_{j=1}^{n} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0\,.$$

- Then $P(\sigma) \cdot P(\sigma) = 0 \; \forall \sigma$.
- For Mobius invariance $\Rightarrow P \in \mathbb{C}^d \otimes K$, $K = \Omega^{1,0}\mathbb{CP}^1$
- only $n - 3$ scattering equations are independent.
- There are $(n - 3)!$ solutions.

First arose for high energy string scattering [Gross-Mende 1988].
Underpin twistor-string formulae also [Witten 2004].

Formulae for gravity, Yang-Mills and scalar amplitudes.

Scatter n spin s massless particles, momenta k_i, $k_i^2 = 0$,

- polarizations ϵ_{1i} for spin 1, $\epsilon_{1i} \otimes \epsilon_{2i}$ for spin-2

$$k_i \cdot \epsilon_{ri} = 0\,, \qquad \epsilon_{ri} \sim \epsilon_{ri} + \alpha_r k_i\,, \quad r = 1, 2.$$

- Introduce skew $2n \times 2n$ matrices $M_r = \begin{pmatrix} A & C_r \\ -C_r^t & B_r \end{pmatrix}$,

$$A_{ij} = \frac{k_i \cdot k_j}{\sigma_i - \sigma_j},, \quad B_{rij} = \frac{\epsilon_{ri} \cdot \epsilon_{rj}}{\sigma_i - \sigma_j}\,, \quad C_{rij} = \frac{k_i \cdot \epsilon_{rj}}{\sigma_i - \sigma_j}\,, \quad \text{for } i \neq j$$

and $A_{ii} = B_{ii} = 0$, but $C_{rii} = \epsilon_{ri} \cdot P(\sigma_i)$.

Theorem (Cachazo, He, Yuan 2013)

Tree-level gravity amplitude in d-dims are 'sum'

$$\mathcal{M}(1, \ldots, n) = \delta^d \left(\sum_i k_i \right) \int_{\mathbb{CP}^{1\,n}} \frac{Pf'(M_1) Pf'(M_2)}{\mathrm{Vol}\ \mathrm{SL}(2, \mathbb{C})} \prod_i{}' \bar{\delta}(k_i \cdot P(\sigma_i)) d\sigma_i$$

[For YM, replace $Pf'(M_2)$ by Parke-Taylor $\left(\prod_i (\sigma_i - \sigma_{i-1})\right)^{-1}$.]

Complexify real space-time $M_{\mathbb{R}} \rightsquigarrow M$, and null covectors P.

$\mathbb{A} :=$ space of complex null geodesics with scale of P.

- $\mathbb{A} = T^*M|_{P^2=0}/\{D_0\}$ where $D_0 := P \cdot \nabla =$ geodesic spray.
- D_0 has Hamiltonian P^2 wrt symplectic form $\omega = dP_\mu \wedge dx^\mu$.
- Symplectic potential $\theta = P_\mu dx^\mu$, $\omega = d\theta$, descend to $\mathbb{A}$.

Projectivise: $P\mathbb{A} :=$ space of *unscaled* complex light rays.

- On $P\mathbb{A}$, $\theta \in \Omega^1_{P\mathbb{A}} \otimes L$ is a holomorphic contact structure.

Theorem (LeBrun 1983)

The complex structure on $P\mathbb{A}$ determines M and conformal metric g. The correspondence is stable under arbitrary deformations of the complex structure of $P\mathbb{A}$ that preserve θ.

Linearized LeBrun correspondence Baston & M. 1986

θ determines complex structure on $P\mathbb{A}$ via $\theta \wedge d\theta^{d-2}$. So:

$$\text{Deformations of complex structure } \leftrightarrow [\delta\theta] \in H^1_{\bar\partial}(P\mathbb{A}, L)\,.$$

Proposition

For $\delta g_{\mu\nu} = \mathrm{e}^{ik\cdot x}\epsilon_\mu\epsilon_\nu$ on flat space-time

$$\delta\theta = \bar\delta(k\cdot P)\mathrm{e}^{ik\cdot X}(\epsilon\cdot P)^2\,.$$

Delta-function support on $k\cdot P = 0 \Rightarrow$ the scattering equations.

Proof: Penrose gives hamiltonian for null geodesic scattering

$$j = \mathrm{e}^{ik\cdot X}\frac{(\epsilon\cdot P)^2}{k\cdot P}\,.$$

Gives $\delta\theta = \mathcal{L}_{X_j}\theta = \mathrm{d}(\theta(X_j)) + \mathrm{d}\theta(X_j,\cdot) = \bar\partial j\,.$ □

Action for null geodesics

For real space-time $(M_{\mathbb{R}}, g_{\mathbb{R}})$ dimension d:

- **Phase space action:** null geodesic γ, $(X, P) : \mathbb{R} \to T^*M_{\mathbb{R}}$

$$S = \int_{\gamma} (P \cdot dX - eP^2/2)\,,$$

- $e \in \Omega^1(\gamma)$ is 'einbein' and Lagrange multiplier for $P^2 = 0$.
- Gauge freedom $\delta(X, P, e) = (\alpha P, 0, 2d\alpha)$.

Phase space of real null geodesics: $\mathbb{A}_{\mathbb{R}} := T^*M_{\mathbb{R}}|_{P^2=0}/\{\text{gauge}\}$

Complexify: $\gamma \rightsquigarrow \Sigma$, Riemann surface, and $(M_{\mathbb{R}}, g_{\mathbb{R}}) \rightsquigarrow (M, g)$.

Ambitwistor string action:

- $X : \Sigma \to M$, $P \in K \otimes X^* T^* M$

$$S = \int (P \cdot \bar{\partial} X - e\, P^2/2)\,.$$

  with $e \in \Omega^{0,1} \otimes T$, where $K = \Omega^{1,0}_{\Sigma}$ and $T = T^{1,0}\Sigma$.
- e again enforces $P^2 = 0$,
- flat space gauge freedom: $\delta(X, P, e) = (\alpha P, 0, 2\bar{\partial}\alpha)$.

Ambitwistor space: $\mathbb{A} = T^* M|_{P^2=0}/\{\text{gauge}\}$.

Quantizing spin 0 ambitwistor string

To quantize, gauge fix

$$S = \int (P \cdot \bar{\partial} X - e\, P^2/2)\,.$$

with $e = 0$ and ghosts $(\tilde{b}, \tilde{c}) \in (K^2, T)$ plus usual $(b, c) \in (K^2, T)$ for diffeos

$$S_{\text{ghost}} = \int b\bar{\partial}c + \tilde{b}\bar{\partial}\tilde{c}\,.$$

This gives BRST operator

$$Q = \int cT + \tilde{c}P^2\,.$$

We have central charge

$$C = 2d - 26 - 26$$

so to quantize consistently $Q^2 = 0 \Rightarrow d = 26$.

Vertex operators and amplitudes

- Integrated vertex ops = perturbations of action $\leftrightarrow \delta g$.
- Action is $\int \theta = \int P \cdot \bar{\partial} X$ so integrated vertex operator is

$$\mathcal{V}_i = \int_\Sigma \delta\theta(\sigma_i) = \int_\Sigma \bar{\delta}(k_i \cdot P(\sigma_i))\, \mathrm{e}^{ik\cdot X(\sigma_i)} (\epsilon_i \cdot P(\sigma_i))^2 \,.$$

- Quantum consistency implies field equations:

$$\{Q, \mathcal{V}_i\} = 0 \quad \Leftrightarrow \quad k^2 = 0, \quad k^\mu \epsilon_{\mu\nu} = 0\,.$$

- Fixed vertex operators provide Fadeev Popov determinants for fixing remaining gauge symmetries $G = SL(2, \mathbb{C}) \times \mathbb{C}^3$ for Mobius on $\mathbb{CP}^1$ and translations along D_0.

Replace fixed vertex ops by quotient by G to give amplitude as path-integral

$$\mathcal{M}(1, \ldots, n) = \int \frac{D[X, P, \ldots]}{\mathrm{Vol}\ G}\, \mathrm{e}^{iS} \prod_{i=1}^{n} \mathcal{V}_i \,.$$

- Take $e^{ik_i \cdot X(\sigma_i)}$ factors into action to give

$$S = \frac{1}{2\pi} \int_\Sigma P \cdot \bar{\partial} X + 2\pi \sum_i ik \cdot X(\sigma_i) \, .$$

- Gives field equations $\bar{\partial} X = 0$ and,

$$\bar{\partial} P = 2\pi \sum_i ik \delta^2(\sigma - \sigma_i) \, .$$

- Solutions $X(\sigma) = X = \text{const.}$, and

$$P(\sigma) = \sum_i \frac{k_i}{\sigma - \sigma_i} d\sigma \, .$$

Thus path-integral reduces to

$$\mathcal{M}(1, \dots, n) = \delta^d(\sum_i k_i) \int_{(\mathbb{CP}^1)^{n-3}} \frac{\prod_i{}' \bar{\delta}(k_i \cdot P) \, (\epsilon_i \cdot P(\sigma_i))^2}{\text{Vol } G}$$

We see $P(\sigma)$ appearing and scattering equations.

Unfortunately: no good interpretation of these as amplitudes.

Evaluation of amplitude

- Take $e^{ik_i \cdot X(\sigma_i)}$ factors into action to give

$$S = \frac{1}{2\pi}\int_\Sigma P\cdot\bar\partial X + 2\pi\sum_i ik\cdot X(\sigma_i)\,.$$

- Gives field equations $\bar\partial X = 0$ and,

$$\bar\partial P = 2\pi\sum_i ik\delta^2(\sigma-\sigma_i)\,.$$

- Solutions $X(\sigma) = X = \text{const.}$, and

$$P(\sigma) = \sum_i \frac{k_i}{\sigma-\sigma_i}d\sigma\,.$$

Thus path-integral reduces to

$$\mathcal{M}(1,\ldots,n) = \delta^d(\sum_i k_i)\int_{(\mathbb{CP}^1)^{n-3}}\frac{\prod_i{}'\bar\delta(k_i\cdot P)\,(\epsilon_i\cdot P(\sigma_i))^2}{\text{Vol }G}$$

We see $P(\sigma)$ appearing and scattering equations.

Unfortunately: no good interpretation of these as amplitudes.

Spinning light rays and super ambitwistor space

To get Pfaffians include RNS spin vectors Ψ_r^μ, fermions:

$$S[X,P,\Psi] = \int P_\mu \mathrm{d}X^\mu - \frac{e}{2}P_\mu P^\mu + \sum_{r=1}^{2} g_{\mu\nu}\Psi_r^\mu \mathrm{d}\Psi_r^\nu - \chi_r P_\mu \Psi_r^\mu$$

$\chi_r \rightsquigarrow$ constraints $P \cdot \Psi_r = 0 \rightsquigarrow$ worldline $N = 2$ susy

$$D_r = \Psi_r \cdot \frac{\partial}{\partial X} + P \cdot \frac{\partial}{\partial \Psi_r}\,, \qquad \{D_r, D_s\} = \delta_{rs} D_0\,.$$

Super ambitwistor space:
$\mathbb{A}_s =$ symplectic quotient of (X, P, Ψ_r)-space by $P^2, P \cdot \Psi_r$.

Symplectic potential: $\theta = P \cdot dX + \Psi_r \cdot d\Psi_r$

Super LeBrun correspondence holds with perturbations

$$\delta\theta = \mathrm{e}^{ik\cdot X}\,\bar{\delta}(k \cdot P) \prod_{r=1}^{2} \epsilon_{r\mu}(P^\mu + \Psi_r^\mu k \cdot \Psi_r)\,.$$

Note: polarization states $\epsilon_{1\mu}\epsilon_{2\nu} \rightsquigarrow$ NS sector of type II sugra.

Use chiral RNS-like action

$$S[X, P, \Psi] = \int_\Sigma P \cdot \bar{\partial} X - \frac{e}{2} P_\mu P^\mu + \sum_{r=1}^{2} \Psi_r \cdot \bar{\partial} \Psi_r + \chi_r P \cdot \Psi_r$$

with $N = 2$ susy.

- To quantize, gauge fix $\chi_r = 0 \rightsquigarrow$ bosonic ghosts (β_r, γ_r) in $(K^{3/2}, T^{1/2})$ for fermionic symmetry (and (b, c), $(\tilde{b}, \tilde{c})$).
- We obtain BRST operator

$$Q = \int cT + \tilde{c} P^2 + \gamma_r P \cdot \Psi_r \,.$$

- For $Q^2 = 0$ central charge C must vanish

$$C = 2d + \frac{d}{2} + \frac{d}{2} - 26 + 11 - 26 + 11 = 3(D - 10)$$

- So critical in $d = 10$ dimensions.

- Integrated vertex operator

$$\mathcal{V}_i = \int_\Sigma e^{ik\cdot X(\sigma_i)}\, \bar{\delta}(k\cdot P(\sigma_i)) \prod_{r=1}^{2} \epsilon_{r\mu}(P^\mu(\sigma_i) + \Psi_r^\mu(\sigma_i) k\cdot \Psi_r(\sigma_i))$$

- need two fixed operators for γ_r zero modes (fixing susy)

$$U_i = e^{k_i\cdot X(\sigma_i)} \prod_r \epsilon_r \cdot \Psi_r(\sigma_i)$$

- and an extra fixed one to fix 3rd c and $\tilde{c}$ zero modes

$$V_i = \prod_{r=1}^{2} \epsilon_{r\mu}(P^\mu(\sigma_i) + \Psi_r^\mu(\sigma_i) k\cdot \Psi_r(\sigma_i))$$

So amplitudes are given by

$$\mathcal{M}(1,\ldots,n) = \left\langle c_1 \tilde{c}_1 \prod_r \gamma_{r1} U_1 c_2 \tilde{c}_2 \prod_r \gamma_{r2} U_2 c_3 \tilde{c}_3 V_3 \mathcal{V}_4 \ldots \mathcal{V}_n \right\rangle .$$

Much works as before giving $\delta^d(\sum_i k_i) \prod_i{}' \bar{\delta}(k_i\cdot P)$ etc..

Amplitude formulae with Pfaffians

- New ingredient is the correlation function of the Ψ's.
- Ψ_1, Ψ_2 independent so contractions computed separately.
- Ψ's appear twice in $\mathcal{V}_i$, as $k_i \cdot \Psi_i$ or $\epsilon_i \cdot \Psi_i$.
- Contractions give for example

$$A_{ij} := \left\langle k_i \cdot \Psi_i k_j \cdot \Psi_j \right\rangle = \frac{k_i \cdot k_j}{\sigma_i - \sigma_j}\,, \quad B_{ij} := \left\langle \epsilon_i \cdot \Psi_i\, \epsilon_j \Psi_j \right\rangle = \frac{\epsilon_i \cdot \epsilon_j}{\sigma_i - \sigma_j}\,.$$

- For $C_{ij} = \langle k_i \cdot \psi_i\, \epsilon_j \cdot \psi_j \rangle$, $P(\sigma_i) \cdot \epsilon_i$ gives diagonal entry.
- Two $k \cdot \Psi$s are missing in $U_i +$ ghost contribution $\rightsquigarrow Pf'(M)$.

Theorem

We obtain CHY formula

$$\mathcal{M}(1,\ldots,n) = \delta^d\left(\sum_i k_i\right) \int_{\mathbb{CP}^{1\,n}} \frac{Pf'(M_1)Pf'(M_2)}{\text{Vol SL}(2,\mathbb{C})} \prod_i{}' \bar{\delta}(k_i \cdot P(\sigma_i))d\sigma_i$$

Other models

We can start with other formulations of null superparticles

- **Heterotic model:** as above but $r = 1$ and current algebra (SO(32) or $E_8 \times E_8$ for $Q^2 = 0$) $\rightsquigarrow$ CHY Yang-Mills formula.
- Bosonic case $+2$ current algebras $\rightsquigarrow$ CHY scalar formula.
- Pure spinor version (Berkovits) $S = \int P \cdot \bar\partial X + p_\alpha \bar\partial \theta^\alpha + \dots$.
- In 4d have twistor representation [Geyer, Lipstein, M. 1404.6219]

$$\mathbb{A} = \{(Z, W) \in \mathbb{T} \times \mathbb{T}^* | Z \cdot W = 0\} / \{Z \cdot \partial_Z - W \cdot \partial_W\}$$

$$S = \int_\Sigma Z \cdot \bar\partial W - W \cdot \bar\partial Z + aZ \cdot W$$

1. Not same as twistor string, (Z, W) spinors on world sheet.
2. Valid for any amount of supersymmetry.
3. New simpler 4d formulae with reduced moduli.

Relation to null infinity, BMS and soft gravitons

Geyer,Lipstein & M. 1406.1462, (cf Adamo, Casali & Skinner 1405.5122).

All real null geodesics intersect $\mathscr{I}^{\pm}$ so $\mathbb{A}_{\mathbb{R}} = T^*\mathscr{I}^{\pm}$; so

$$\mathbb{A} = T^*\mathscr{I}^{+}_{\mathbb{C}} \cup T^*\mathscr{I}^{-}_{\mathbb{C}} \qquad \text{glued over } \mathbb{A}_{\mathbb{R}}\,.$$

and

$$\text{Vertex op} = \delta\theta = \bar\partial(\text{infinitesimal diffeo } T * \mathscr{I}^{-} \to T^*\mathscr{I}^{+})\,.$$

The diffeos intertwine with BMS transformations.
Soft gravitons: $k \to 0$ then diffeo $\to$ supertranslation.
Gives Strominger/Weinberg soft graviton thm as Ward identity.

Theorem (Strominger/Weinberg)

Weinberg soft theorem: as $k_n \to 0$

$$\mathcal{M}(1,\ldots,n) \to \mathcal{M}(1,\ldots,n-1)\sum_{i=1}^{n-1}\frac{(\epsilon_n\cdot k_i)^2}{k_n\cdot k_i}\,,$$

follows from supertranslation equivariance.

Summary

We have chiral $\alpha' = 0$ ambitwistor strings based on LeBrun's correspondence that gives theory underlying CHY formulae

- NS sector of type II sugra extends to Ramond as in RNS string via spin-operator from bosonizing Ψs.
- Incorporates colour/kinematics Yang-Mills/gravity ideas.
- criticality gives extension to loops. [Adamo, Casali, Skinner]
    - At genus g, P is a 1-form and acquires dg zero-modes.
    - These are the loop momenta for g-loops.
    - Conceivabbly gives correct answer for loop processes.
- Quantization ties scattering of null geodesics into that for gravitational waves.

Thank You