

Energy-energy correlations in $\mathcal{N} = 4$ SYM

Emery Sokatchev
LAPTH-Annecy & CERN

In collaboration with

Andrei Belitsky, Stefan Hohenegger, Gregory Korchemsky, Alexander Zhiboedov

arXiv:1311.6800, 1309.1424, 1309.0769

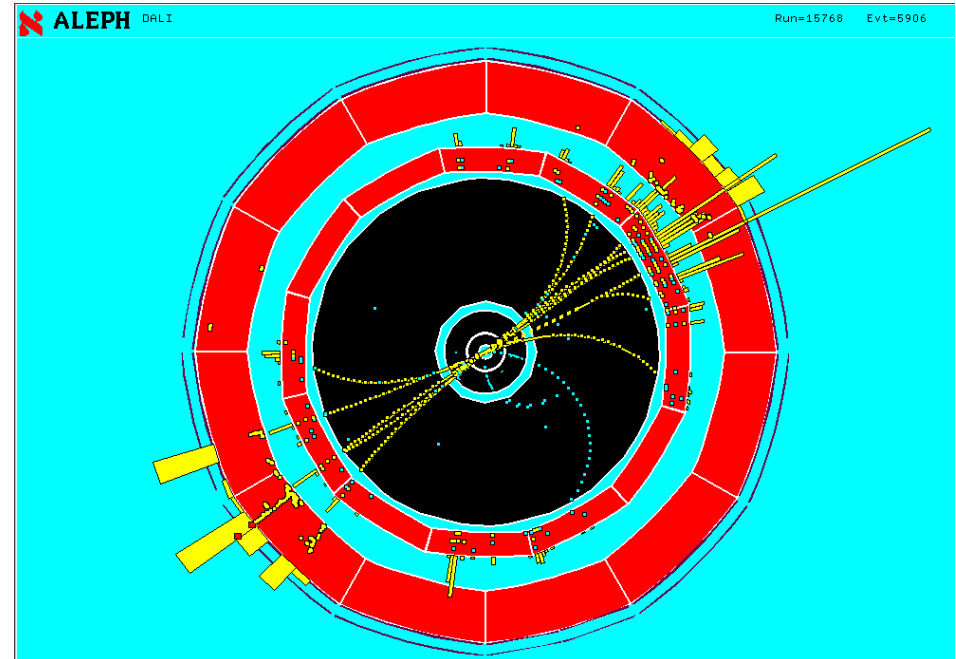
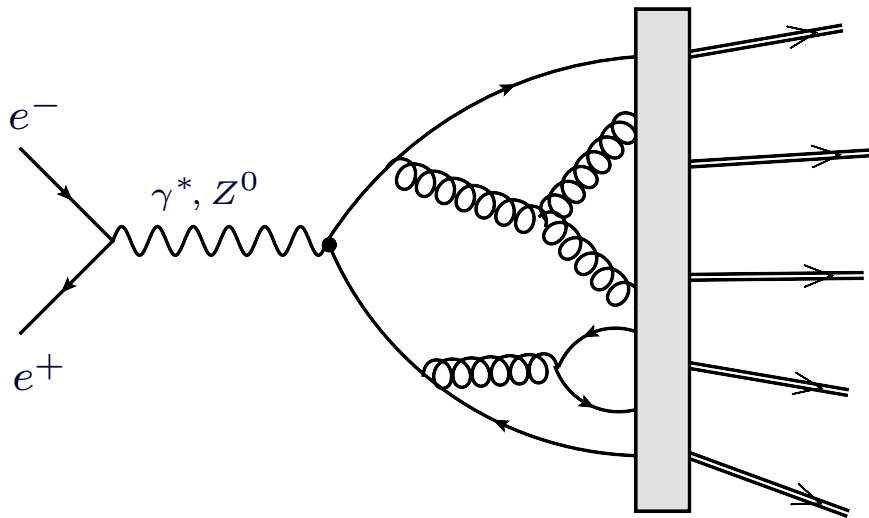
Can we define and compute

“real” physical observables

related to scattering amplitudes in $\mathcal{N} = 4$ SYM?

e^+e^- annihilation in QCD

- ✓ PETRA (1978-1986) and LEP (1989-2010)



- ✓ Virtual photon or Z^0 –boson decays into quarks and gluons that undergo hadronization
- ✓ Final states can be described using a class of *infrared finite* observables (event shapes): energy-energy correlations (EEC), thrust, heavy mass, ...

Energy-energy correlation

✓ Function of the angle $0 \leq \chi \leq \pi$ between the detected particles [Basham,Brown,Ellis,Love]

✓ Conventional ('amplitude') approach

$$EEC(\chi) = \frac{1}{\sigma_{\text{tot}}} \sum_{a,b,X} \int d\sigma_{a+b+X} \frac{E_a E_b}{Q^2} \delta(\cos \theta_{ab} - \cos \chi)$$

σ_{tot} total cross section $e^+e^- \rightarrow \text{hadrons}$; total energy
 $\sum_a E_a = Q$

✓ Weak coupling expansion in QCD

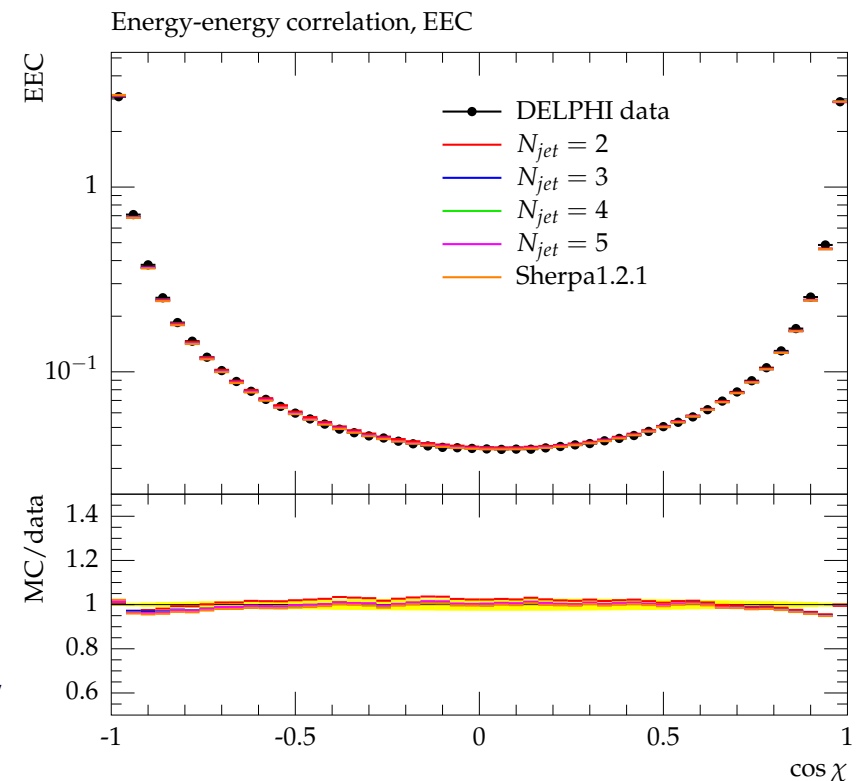
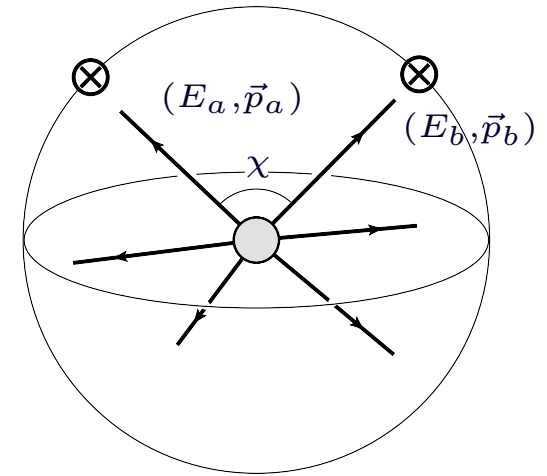
$$EEC(\chi) = a EEC_{LO}(\chi) + a^2 EEC_{NLO}(\chi) + O(a^3)$$

✓ Current status (1978 – today):

✗ Very precise experimental data

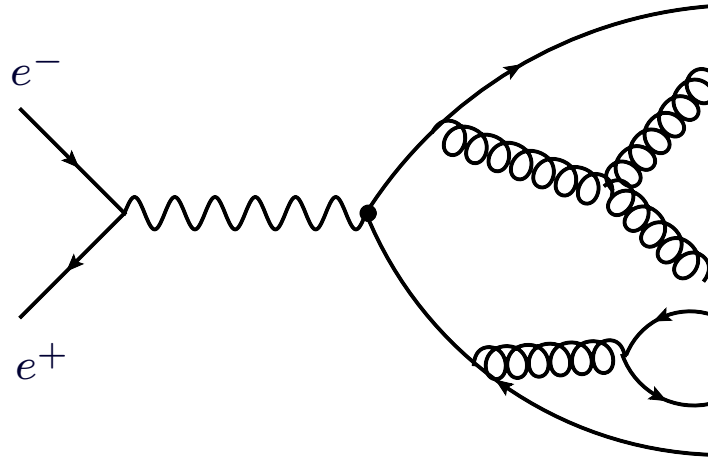
✗ Poor analytical control, EEC_{NLO} is evaluated numerically

✓ Our goal: develop more efficient method for computing EEC



“ e^+e^- annihilation” in $\mathcal{N} = 4$ SYM

- ✓ Define EEC in $\mathcal{N} = 4$ SYM and evaluate it at weak/strong coupling



- ✓ Pair of $e^+e^- \Rightarrow$ electromagnetic current \Rightarrow pair of quarks \Rightarrow fragmentation into hadrons.
- ✓ From QCD to $\mathcal{N} = 4$ SYM: introduce an analog of the electromagnetic current
 - ✗ (protected) half-BPS operator built from the six real scalars Φ^I (with $I = 1, \dots, 6$)

$$O_{20'}^{IJ}(x) = \text{tr} \left[\Phi^I \Phi^J - \frac{1}{6} \delta^{IJ} \Phi^K \Phi^K \right]$$

- ✗ To lowest order in the coupling, $O_{20'}(x)$ produces a pair of scalars out of the vacuum
- ✗ At higher orders, the state $O_{20'}(x)|0\rangle$ is a sum of on-shell states with any number of $\mathcal{N} = 4$ “particles”: scalars (s), gluinos (λ) and gluons (g)

$$\int d^4x e^{iqx} O_{20'}(x)|0\rangle = |ss\rangle + |ssg\rangle + |s\lambda\lambda\rangle + \dots$$

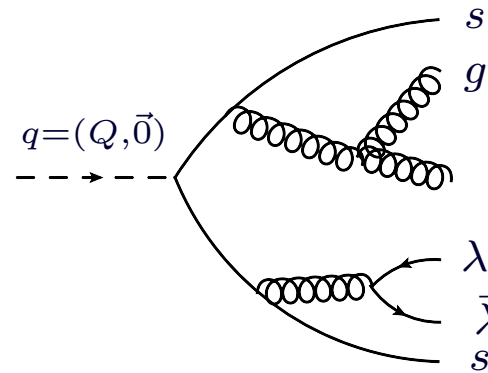
EEC in $\mathcal{N} = 4$ SYM

✓ Conventional approach

$$\text{EEC}(\chi) = \frac{1}{\sigma_{\text{tot}}} \sum_{a,b,X} \int d\text{LIPS} |\mathcal{A}_{a+b+X}|^2 \frac{E_a E_b}{Q^2} \delta(\cos \chi - \cos \theta_{ab})$$

Amplitude of creation of the final state $|a, b, X = \text{everything}\rangle$

$$\mathcal{A}_{a+b+X} = \int d^4x e^{iqx} \langle a, b, X | O_{\mathbf{20}'}(x) | 0 \rangle$$



✓ Problems:

- ✗ presence of **infrared divergences** in the transition amplitudes \mathcal{A}_{a+b+X}
- ✗ integration over the Lorentz invariant phase space of the final states **dLIPS**
- ✗ need to sum over all final states \sum_X
- ✗ no analytical results beyond one loop

✓ New approach: EEC can be computed from *correlation functions of energy flow operators*

EEC from amplitudes I

✓ Transition amplitude

$$\mathcal{A}_{a+b+X} = \text{---} \textcircled{1} \begin{array}{l} \nearrow s \\ \searrow s \end{array} + \text{---} \textcircled{0} \begin{array}{l} \nearrow s \\ \searrow s \\ \text{---} g \end{array} + \text{---} \textcircled{0} \begin{array}{l} \nearrow \lambda \\ \searrow s \\ \text{---} \lambda \end{array} + \dots$$

✓ Matrix elements ($s_{ij} = (p_i + p_j)^2$ with $p_i^2 = 0$; 't Hooft coupling $a = g_{\text{YM}}^2 N / (4\pi^2)$)

$$|\mathcal{A}_{ss}|^2 = |\langle s(p_1)s(p_2) | O_{\mathbf{20}'} | 0 \rangle|^2 = \frac{2}{s_{12}} [1 + a F_{\text{virt}}(q^2)]$$

$$|\mathcal{A}_{ssg}|^2 = |\langle s(p_1)s(p_2)g(p_3) | O_{\mathbf{20}'} | 0 \rangle|^2 = a \frac{s_{12}}{s_{13}s_{23}}$$

$$|\mathcal{A}_{s\lambda\lambda}|^2 = |\langle \lambda(p_1)\lambda(p_2)s(p_3) | O_{\mathbf{20}'} | 0 \rangle|^2 = a \frac{2}{s_{12}}$$

✓ Total transition amplitude

$$\sigma_{\text{tot}}(q) = \int \text{dLIPS}_2 |\mathcal{A}_{ss}|^2 + \int \text{dLIPS}_3 (|\mathcal{A}_{ssg}|^2 + |\mathcal{A}_{s\lambda\lambda}|^2) + O(a^2)$$

$$= \frac{1}{8\pi} [1 + a F_{\text{virt}}(q^2)] + a \int \text{dLIPS}_3 \frac{s_{12}^2 + 2s_{13}s_{23}}{s_{12}s_{13}s_{23}} + O(a^2) = \frac{1}{8\pi} + \mathbf{0} \cdot a + O(a^2)$$

EEC from amplitudes II

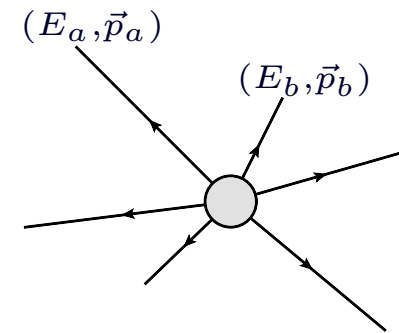
- ✓ Weighted cross-section: insert weight factor $w(p)$

$$\text{EEC} = \left[\int \text{dLIPS}_2 w(p_1, p_2) |\mathcal{A}_{ss}|^2 + \int \text{dLIPS}_3 w(p_1, p_2, p_3) (|\mathcal{A}_{ssg}|^2 + |\mathcal{A}_{s\lambda\lambda}|^2) + O(a^2) \right] / \sigma_{\text{tot}}$$

- ✓ Weight factors for energy-energy correlations

$$w(p_1, p_2, \dots) = \sum_{a,b} \frac{E_a E_b}{q^2} \delta(\cos \theta_{ab} - \cos \chi)$$

$$\text{dLIPS}_\ell = \prod_i \frac{d^4 p_i}{(2\pi)^3} \delta_+(p_i^2) \delta^{(4)} \left(\sum p_i - q \right)$$



- ✓ Conformal symmetry: EEC depends only on the scaling variable

$$z = \frac{1}{2}(1 - \cos \chi), \quad 0 < z < 1$$

- ✓ Non-protected quantity: one-loop contribution

[Zhiboedov],[Engelund,Roiban]

$$\text{EEC}_{\mathcal{N}=4} = \frac{a}{4z^2(1-z)} \ln \frac{1}{1-z} + O(a^2)$$

IR finite function of $0 < \chi < \pi$

- ✓ Two-loop correction is hard to compute ($\sim 10^2$ diagrams)

EEC from correlation functions I

- ✓ Use the completeness condition $\sum_X |X\rangle\langle X| = 1$

$$\begin{aligned}\sigma_{\text{tot}}(q) &= \sum_X (2\pi)^4 \delta^{(4)}(q - p_X) |\mathcal{A}_{O_{20'} \rightarrow X}|^2 \\ &= \sum_X \int d^4x e^{iqx} \langle 0|O^\dagger(0)|X\rangle e^{-ixp_X} \langle X|O(0)|0\rangle = \int d^4x e^{iqx} \langle 0|O^\dagger(x) \sum_X |X\rangle\langle X|O(0)|0\rangle \\ &= \int d^4x e^{iqx} \langle 0|O^\dagger(x)O(0)|0\rangle = \int d^4x e^{iqx} \frac{C_N}{(x^2 - i\epsilon x^0)^2} = \frac{1}{16\pi} (N^2 - 1) \theta(q^0) \theta(q^2)\end{aligned}$$

Wightman correlation function (no time ordering!), **protected** for 1/2-BPS operators

- ✓ Generalization to EEC: insert weight factor

$$\text{EEC} \sim \sum_X \langle 0|O^\dagger(x)|X\rangle w(X) \langle X|O(0)|0\rangle = \langle 0|O^\dagger(x) \mathcal{E}(\vec{n}_1) \mathcal{E}(\vec{n}_2) O(0)|0\rangle$$

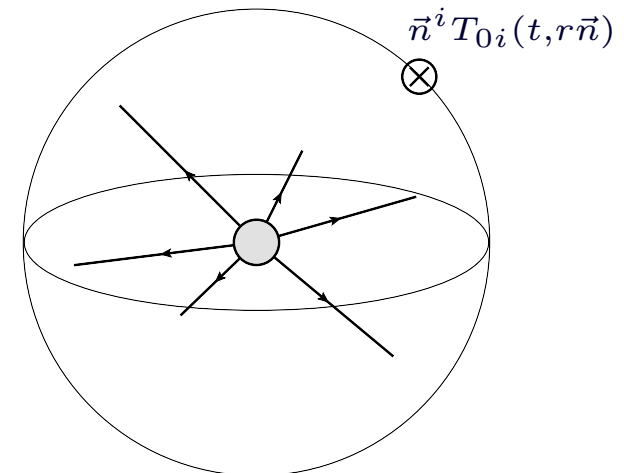
Energy flow operator

$$\mathcal{E}(\vec{n})|X\rangle = \sum_a E_a \delta^{(2)}(\Omega_{\vec{p}_a} - \Omega_{\vec{n}})|X\rangle$$

- ✓ Relation to the energy-momentum tensor

[Sveshnikov, Tkachov], [Korchemsky, Oderda, Sterman]

$$\mathcal{E}(\vec{n}) = \int_0^\infty dt \lim_{r \rightarrow \infty} r^2 \vec{n}^i T_{0i}(t, r\vec{n})$$



EEC from correlation functions II

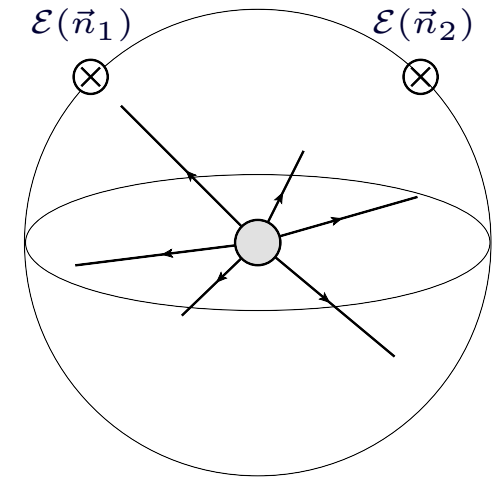
- ✓ Energy flow correlations [Korchemsky,Sterman],[Belitsky,Korchemsky,Sterman],[Hofman,Maldacena]

$$\langle \mathcal{E}(\vec{n}_1) \mathcal{E}(\vec{n}_2) \rangle_q = \sigma_{\text{tot}}^{-1} \int d^4x e^{iqx} \langle 0 | O^\dagger(x) \mathcal{E}(\vec{n}_1) \mathcal{E}(\vec{n}_2) O(0) | 0 \rangle$$

Energy flow in the direction of \vec{n}_1 and \vec{n}_2

- ✓ Average over the orientations \vec{n}_1 and \vec{n}_2 with the relative angle χ kept fixed

$$\text{EEC} = \int d\Omega_1 d\Omega_2 \delta(\vec{n}_1 \cdot \vec{n}_2 - \cos \chi) \langle \mathcal{E}(\vec{n}_1) \mathcal{E}(\vec{n}_2) \rangle_q / q^2$$



- ✓ Final form of EEC: Multi-fold integral of the 4-point *Wightman* function

$$\text{EEC} \sim \underbrace{\int d^4x e^{iqx}}_{\text{Fourier}} \underbrace{\int_0^\infty dt_1 dt_2 \lim_{r_i \rightarrow \infty} r_1^2 r_2^2}_{\text{Detector limit}} \underbrace{\langle 0 | O^\dagger(x) T_{0\vec{n}_1}(x_1) T_{0\vec{n}_2}(x_2) O(0) | 0 \rangle_W}_{\text{Wightman correlation function}} \Big|_{x_i = (t, r\vec{n}_i)}$$

$\mathcal{N} = 4$ SUSY and analytic continuation

✓ How to obtain $\langle OT_{\mu\nu}T_{\rho\sigma}O \rangle$?

✗ Starting point: Euclidean 4-point function of scalar half-BPS operators

$$\langle O(x_1)O(x_2)O(x_3)O(x_4) \rangle_{\mathbb{E}} = \frac{1}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2} \Phi(u, v; a), \quad u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{23}^2 x_{41}^2}{x_{13}^2 x_{24}^2}$$

✗ O and $T_{\mu\nu}$ are members of the $\mathcal{N} = 4$ energy-momentum tensor multiplet

$$\mathcal{T}(x, \theta, \bar{\theta}) = O(x) + \theta\sigma^\mu\bar{\theta} R_\mu(x) + \dots + \theta\sigma^\mu\bar{\theta} \theta\sigma^\nu\bar{\theta} T_{\mu\nu}(x) + \dots + (\theta)^4 \mathcal{L}_{\mathcal{N}=4 \text{ SYM}}(x)$$

✗ $\langle OT_{\mu\nu}T_{\rho\sigma}O \rangle$ is obtained from $\langle OOOO \rangle$ by $\mathcal{N} = 4$ conformal SUSY

$$\xrightarrow{\mathcal{N}=4 \text{ SUSY}} \langle O(x_1)T_{\mu\nu}(x_2)T_{\rho\sigma}(x_3)O(x_4) \rangle_{\mathbb{E}} = D_{\mu\nu\rho\sigma}(\partial_2, \partial_3) \langle O(x_1)O(x_2)O(x_3)O(x_4) \rangle_{\mathbb{E}}$$

with $D_{\mu\nu\rho\sigma}$ a 4th-order differential operator.

✓ How to analytically continue $\Phi(u, v; a)$ from Euclid to Minkowski&Wightman?

[Lüscher, Mack]

✗ Most easily done using the Mellin transform

$$\Phi(u, v; a) = \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{dj_1 dj_2}{(2\pi i)^2} M(j_1, j_2; a) u^{j_1} v^{j_2}$$

by the simple replacement $x_{ij}^2 \rightarrow x_{ij}^2 - i\epsilon x_{ij}^0$ for $i < j$ in u, v .

All-order formula for EEC

Master formula

$$\text{EEC}(\chi) = \frac{1}{4z^2(1-z)} \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{dj_1 dj_2}{(2\pi i)^2} \underbrace{M(j_1, j_2; a)}_{\text{corr. function}} \underbrace{K(j_1, j_2)}_{\text{detector}} \underbrace{\left(\frac{1-z}{z}\right)^{j_1+j_2}}_{\text{angular dependence}}$$

The dependence on the angle χ enters through

$$z = (1 - \cos \chi)/2$$

Detector function depending on the type of observable (energy, charge, ...) but not on the coupling

$$K_{EE}(j_1, j_2) = \frac{2\Gamma(1-j_1-j_2)}{\Gamma(j_1+j_2)[\Gamma(1-j_1)\Gamma(1-j_2)]^2}$$

The dependence on the coupling constant comes from the Mellin amplitude

$$\Phi(u, v; a) = \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{dj_1 dj_2}{(2\pi i)^2} M(j_1, j_2; a) u^{j_1} v^{j_2}$$

$$M(j_1, j_2; a) = \underbrace{aM^{(1)}(j_1, j_2) + a^2M^{(2)}(j_1, j_2) + \dots}_{\text{are known}}$$

Mellin amplitude also known at **strong coupling** via AdS/CFT

[Arutyunov, Frolov]

EEC up to two loops

✓ Four-point correlator at weak coupling

[Eden,Schubert,ES],[Bianchi et al]

$$\begin{aligned} \Phi(u, v; a) = & a \Phi^{(1)}(u, v) + a^2 \left\{ \frac{1}{2} (1 + u + v) \left[\Phi^{(1)}(u, v) \right]^2 \right. \\ & \left. + 2 \left[\Phi^{(2)}(u, v) + \frac{1}{u} \Phi^{(2)}(v/u, 1/u) + \frac{1}{v} \Phi^{(2)}(1/v, u/v) \right] \right\} + O(a^3) \end{aligned}$$

✓ Euclidean ladder (or 'scalar box') integrals $\Phi^{(1)}$ and $\Phi^{(2)}$

[Usyukina,Davydychev]

✓ Mellin amplitude to two loops:

$$\begin{aligned} M(j_1, j_2) = & a M^{(1)}(j_1, j_2) + a^2 \left[\frac{1}{2} \widetilde{M}^{(2)}(j_1, j_2) + \widetilde{M}^{(2)}(j_1, j_2 - 1) \right. \\ & \left. + 2M^{(2)}(j_1, j_2) + 4M^{(2)}(j_1, -1 - j_1 - j_2) \right] \end{aligned}$$

$$M^{(1)}(j_1, j_2) = -\frac{1}{4} [\Gamma(-j_1)\Gamma(-j_2)\Gamma(1 + j_1 + j_2)]^2$$

$$M^{(2)}(j_1, j_2) = -\frac{1}{4} \Gamma(-j_1)\Gamma(-j_2)\Gamma(1 + j_1 + j_2)$$

$$\times \int \frac{dj'_1 dj'_2}{(2\pi i)^2} M^{(1)}(j'_1, j'_2) \frac{\Gamma(j'_1 - j_1)\Gamma(j'_2 - j_2)\Gamma(1 + j_1 + j_2 - j'_1 - j'_2)}{\Gamma(1 - j'_1)\Gamma(1 - j'_2)\Gamma(1 + j'_1 + j'_2)}$$

$$\widetilde{M}^{(2)}(j_1, j_2) = \int \frac{dj'_1 dj'_2}{(2\pi i)^2} M^{(1)}(j_1 - j'_1, j_2 - j'_2) M^{(1)}(j'_1, j'_2)$$

Warm-up exercise: One loop

- ✓ Master formula at one loop

$$\text{EEC}^{(1 \text{ loop})} = \frac{a}{4z^2(1-z)} \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{dj_1 dj_2}{(2\pi i)^2} M^{(1)}(j_1, j_2; a) K(j_1, j_2) \left(\frac{1-z}{z}\right)^{j_1+j_2}$$

Mellin amplitude

$$M^{(1)}(j_1, j_2) = -\frac{1}{4} [\Gamma(-j_1)\Gamma(-j_2)\Gamma(1+j_1+j_2)]^2$$

$$K(j_1, j_2) = \frac{2\Gamma(1-j_1-j_2)}{\Gamma(j_1+j_2)[\Gamma(1-j_1)\Gamma(1-j_2)]^2}$$

- ✓ Change integration variable $j_1 + j_2 \rightarrow j_1$

$$\begin{aligned} \text{EEC}^{(1 \text{ loop})} &= -\frac{a}{4z^2(1-z)} \int \frac{dj_1 dj_2}{(2\pi i)^2} \frac{j_1^2}{2(j_1-j_2)^2 j_2^2} \frac{\pi}{\sin(\pi j_1)} \left(\frac{1-z}{z}\right)^{j_1} \\ &= \frac{a}{4z^2(1-z)} \ln \frac{1}{1-z} \end{aligned}$$

- ✓ Not so simple at two loops! $M^{(2)}(j_1, j_2)$ is given by a two-fold Mellin integral.

EEC at two loops I

Final result for EEC

$$\text{EEC}_{\mathcal{N}=4} = \frac{1}{4z^2(1-z)} \left\{ aF_1(z) + a^2 \left[(1-z)F_2(z) + \frac{1}{4}F_3(z) \right] \right\}, \quad z = \frac{1}{2}(1 - \cos \chi)$$

$F_w(z)$ are linear combinations of functions of homogenous weight $w = 1, 2, 3$

$$F_1(z) = -\ln(1-z)$$

$$F_2(z) = 4\sqrt{z} \left[\text{Li}_2(-\sqrt{z}) - \text{Li}_2(\sqrt{z}) + \frac{1}{2} \ln z \ln \left(\frac{1+\sqrt{z}}{1-\sqrt{z}} \right) \right] \\ + (1+z) \left[2\text{Li}_2(z) + \ln^2(1-z) \right] + 2 \ln(1-z) \ln \left(\frac{z}{1-z} \right) + z \frac{\pi^2}{3},$$

$$F_3(z) = (1-z)(1+2z) \left[\ln^2 \left(\frac{1+\sqrt{z}}{1-\sqrt{z}} \right) \ln \left(\frac{1-z}{z} \right) - 8\text{Li}_3 \left(\frac{\sqrt{z}}{\sqrt{z}-1} \right) - 8\text{Li}_3 \left(\frac{\sqrt{z}}{\sqrt{z}+1} \right) \right] \\ - 4(z-4)\text{Li}_3(z) + 6(3+3z-4z^2)\text{Li}_3 \left(\frac{z}{z-1} \right) - 2z(1+4z)\zeta_3 + 2[(3-4z)z \ln z \\ + 2(2z^2 - z - 2) \ln(1-z)] \text{Li}_2(z) + \frac{1}{3} \ln^2(1-z) [4(3z^2 - 2z - 1) \ln(1-z) \\ + 3(3-4z)z \ln z] + \frac{\pi^2}{3} [2z^2 \ln z - (2z^2 + z - 2) \ln(1-z)]$$

EEC at two loops II

- ✓ Sum of basis functions with nontrivial arguments and rational prefactors R_2 and R_3

$$\text{EEC}^{(2 \text{ loops})} = \sum R_2(z, \sqrt{z}) W_2(z, \sqrt{z}) + \sum R_3(z, \sqrt{z}) W_3(z, \sqrt{z})$$

Weight two $W_2 = \{\text{Li}_2, \ln \ln, \pi^2\}$; weight three $W_3 = \{\text{Li}_3, \text{Li}_2 \ln, \ln \ln \ln, \pi^2 \ln, \zeta_3\}$

- ✓ W_2 and W_3 depend on $\sqrt{z} = |\sin(\chi/2)|$, but EEC is manifestly invariant under $\sqrt{z} \rightarrow -\sqrt{z}$
- ✓ Scattering amplitudes have homogenous weight in planar $\mathcal{N} = 4$ SYM at weak coupling

$$A_{a+b+X} \sim \exp \left(-\text{Div}(1/\epsilon) + \sum_{\ell} a^{\ell} W_{2\ell} + O(\epsilon) \right)$$

This property is '*minimally*' violated for EEC after the phase space integration

$$\text{EEC}(\chi) = \frac{1}{\sigma_{\text{tot}}} \sum_{a,b,X} \int \text{dLIPS} |A_{a+b+X}|^2 \frac{E_a E_b}{Q^2} \delta(\cos \chi - \cos \theta_{ab})$$

- ✓ Strong coupling from AdS supergravity: uniform distribution at leading order
Agrees with the strong coupling prediction for EEC in planar $\mathcal{N} = 4$ SYM

[Hofman, Maldacena]

$$\text{EEC}_{\mathcal{N}=4} \stackrel{a \rightarrow \infty}{\sim} \frac{1}{2} \left[1 + a^{-1} (1 - 6z(1-z)) + O(a^{-3/2}) \right]$$

Conclusions and open questions

- ✓ Energy flow correlations are good/nontrivial physical observables in $\mathcal{N} = 4$ SYM
- ✓ Relation to energy flow correlations in QCD (most complicated part)?
- ✓ All symmetries of $\mathcal{N} = 4$ SYM are preserved, what is the manifestation of integrability?
- ✓ Interpolation between weak and strong coupling?
- ✓ Bootstrap?
- ✓ Other proposals for 'good' observables?