Conformal higher spins
and partition functions

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“Partition function of free conformal higher spin theory,”
with M. Beccaria and X. Bekaert, arXiv:1406.3542

“Partition Functions and Casimir Energies in Higher Spin AdS$_{d+1}$/CFT$_d$,”
with S. Giombi and I. Klebanov, arXiv:1402.5396

“On partition function and Weyl anomaly of conformal higher spins,”
arXiv:1309.0785

“Weyl anomaly of conformal higher spins on six-sphere,” arXiv:1310.1795

Why conformal higher spin theory?

$s = 1$: Maxwell vector, $s = 2$: Weyl graviton, etc.

• fundamental role of local conformal invariance?
very constraining at quantum level: anomalies and unitarity issues
• existence of consistent (UV finite, anomaly free)
conformal higher spin theories?
• cancel anomalies: supersymmetry or summation over all spins?
• summation over spins may resolve unitarity issue?
• a limit of some string theory or alternative to string theory ?

recent interest:

formal relations between "triple" of theories:

- $\star$ free scalar CFT in M^d (e.g. R^d, S^d, $S^1 \times S^{d-1}$, ...)
- $\star$ conformal higher spins in M^d
- $\star$ massless higher spins in AdS_{d+1} with boundary M^d

Tree-level: CHS as induced theory from $\int \Phi\partial^2\Phi + \phi_s \cdot J_s(\Phi)$;
log singular part of action of massless HS in AdS: $\varphi_s\big|_{M^d} = \phi_s$

One-loop level: CHS partition function as ratio of CFT or massless AdS higher spin partition functions

$$Z_s\Big|_{M^d} = \frac{Z_{-\,s}}{Z_{+\,s}}\Big|_{M^d} = \frac{Z_s^{(-)}}{Z_s^{(+)}}\Big|_{AdS_{d+1}}$$

Conformal higher spin (CHS) theory

• maximal gauge invariance and irreducibility
consistent with locality:
pure spin states off shell [Fradkin, AT 85]

$$d = 4: \qquad L_s = \phi_s \, P_s \, \partial^{2s} \phi_s \ , \qquad s = 1, 2, ...$$

$\phi_s = (\phi_{m_1 ... m_s})$ totally symmetric, $\Delta = 2 - s$
$(P_s)^{m_1..m_s}_{n_1...n_s}$ totally symmetric traceless transverse projector
e.g. $(P_1)^m_n = \delta^m_n - \frac{\partial^m \partial_n}{\partial^2}$
• Gauge invariances: $\delta\phi_s = \partial\xi_{s-1} + \eta_2 \lambda_{s-2}$
differential (like reparam.) + algebraic (like Weyl)
• cf. two-derivative massless higher spin fields:
$L_s = \varphi_s \bar{P}_s \partial^2 \varphi_s$ where $\bar{P}_s$ chosen to have locality
$\bar{P}_1 = P_1, \qquad \bar{P}_2 = P_2 - 2P_0$ (Einstein)
mixture of spins off-shell

Free CHS action in flat $d = 4$

$$S_s = \int d^4x \, \phi_s P_s \, \partial^{2s} \, \phi_s = \int d^4x \, (-1)^s \, C_s C_s$$

$\phi_s = (\phi_{\mu_1 \ldots \mu_s}) \equiv \phi_{\mu(s)}$ totally symmetric

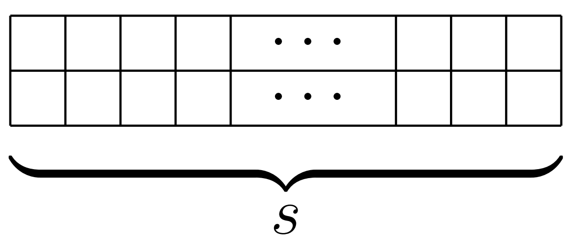
$P_s = (P^{\mu_1 \ldots \mu_s}_{\nu_1 \ldots \nu_s}) \equiv P^{\mu(s)}_{\nu(s)}$ transverse traceless symm. in μ and ν

$C_s \equiv C_{\mu(s),\nu(s)} = (C_{\mu_1 \ldots \mu_s, \, \nu_1 \ldots \nu_s})$ generalized Weyl tensor

$C_{\mu(s),\nu(s)} = \mathcal{P}^{\lambda(s),\rho(s)}_{\mu(s),\nu(s)} \partial^s_{\lambda(s)} \phi_{\rho(s)}$

$\mathcal{P}_s$ makes $C_{\mu(s),\nu(s)}$ symmetric and traceless in $\mu(s)$ and $\nu(s)$
and antisymmetric between:

$C_{\mu(s),\nu(s)}$ in (s, s) representation of $SO(4)$



Alternative: $C_{\mu_1\nu_1\,\mu_2\nu_2\,...\mu_s\nu_s}$ antisymm. in each μ_i and ν_i

$C_1 = (F_{\mu\nu})$ Maxwell, $C_2 = (C_{\mu_1\nu_1\mu_2\nu_2})$ linearized Weyl tensor

any even dimension d:

$$S_s = \int d^d x \, \phi_s P_s \, \partial^{2s+d-4} \, \phi_s = (-1)^s \int d^d x \; C_s \, \partial^{d-4} \, C_s$$

ϕ_s and C_s have d-independent $SO(d,2)$ scaling dimensions

$$\Delta(\phi_s) = 2 - s \; , \qquad\qquad \Delta(C_s) = 2$$

• Free (non-unitary) higher spin conformal theory in flat space

• Generalization to curved background?

Weyl-invariant quadratic action known for $s = 1$ and $s = 2$;

kinetic operator $K = D^{2s+d-4} + ...$ – complicated for $s \geqslant 3$

reparametrization and Weyl invariant and consistent with

CHS gauge symm. for any $g_{\mu\nu}$ solving Bach eqs of Weyl gravity

K simplifies / factorizes on conformally-flat background:
found for S^4 [AT 13; Metsaev 14; Nutma,Taronna 14]
and $S^1 \times S^3$ [Bekaert, Beccaria, AT 14]

• full interacting theory? need to include all higher spins
• cf. standard 2-derivative massless HS theory:
introducing consistent interactions difficult – no-go theorems;
incompatibility between higher-spin gauge symmetries
and minimal coupling with gravity around flat background;
resolved on constant curvature (A)dS background;
[Fradkin, Vasiliev 87; Vasiliev 90]
led to eqs for tower of interacting massless higher spins
• CHS theory is different:
interactions consistent with coupling to gravity even around
flat background and admits an action principle

non-linear CHS theory can be defined as induced theory

[AT 02; Segal 02; Bekaert, Joung, Mourad 10;

Giombi, Klebanov, Pufu, Safdi, Tarnopolsky 13]

• $\ln \varepsilon_{\rm UV}$ term in eff. action of free scalar CFT + $\phi_s \cdot J_s$
with source ("shadow") fields ϕ_s
for all conserved symmetric higher spin currents J_s
$\to$ local functional of ϕ_s starting with CHS kinetic term

• Interactions: $\sum \partial^{n_m} \phi_{s_1} ... \phi_{s_m}, \ n_m = d + \sum_{i=1}^m (s_i - 2)$

[Bekaert, Joung, Mourad 10]

Weyl graviton couples minimally to higher spins:
no increase of number of derivatives

• quantum consistency? anomalies?

Interactions with graviton – curved space:

conformal $\to$ Weyl symmetry: $g'_{mn} = \lambda^2(x) g_{mn}$

conformal anomaly free HS quantum theories?

$$T^m_m = -\mathrm{a}\, R^* R^* + \mathrm{c}\, C^2$$

Weyl gravity ($s = 2$) is anomalous: $\mathrm{a}_2 = \frac{87}{20}\,, \quad \mathrm{c}_2 = \frac{199}{30}$

• one possible resolution – supersymmetry

$\mathcal{N} = 4$ conformal supergravity + 4 $\mathcal{N} = 4$ Maxwell multiplets

is anomaly free: $\mathrm{a} = \mathrm{c} = 0$ [Fradkin, AT 82]

• Another option: CHS theory with sum over all spins

$\star$ $\sum_{s=0}^{\infty} \mathrm{a}_s = 0$ in a regularization [Giombi, Klebanov et al 13; AT 13]

$\star$ same for Casimir energy on S^3 (a priori unrelated)

$\sum_{s=0}^{\infty} E_s = 0$ [Bekaert, Beccaria, AT 14]

$\star$ conjecture: $\sum_{s=0}^{\infty} \mathrm{c}_s = 0$

Free complex scalar CFT$_d$

$$S_0 = \int d^d x \; \Phi^*_r \partial^2 \Phi_r , \qquad \Delta(\Phi) = \tfrac{1}{2}(d-2), \qquad r = 1, ..., N$$

tower of conserved higher spin currents

$$J_{\mu_1 ... \mu_s} = \Phi^*_r \partial_{\mu_1} ... \partial_{\mu_s} \Phi_r + ... \ , \qquad \partial \cdot J_s \Big|_{\text{on-shell}} = 0$$

"single-trace" CFT primaries: "singlet sector"

- introduce sources ϕ_s

$$\int d^d x \; J_s(x)\, \phi_s(x) \ , \qquad J_s \sim \Phi^*_r \partial^s \Phi_r$$

$$\Delta(J_s) = \Delta_+ \equiv s + d - 2 \ , \qquad \Delta(\phi_s) = d - \Delta_+ \equiv \Delta_- = 2 - s$$

⋆ ϕ_s: same representation as spin s "shadow" conformal field

⋆ ϕ_s: same gauge symmetries and dimension as CHS field

⋆ $\phi_s = \varphi_s \big|_{M^d}$ – bndry value of massless higher spin s in AdS$_{d+1}$

• Induced action or generating functional for CFT correlators:

$$\Gamma = -\ln Z(\phi_s) = \tfrac{1}{2} N \ln \det(\partial^2 + \phi_s P_s \partial^s)$$
$$= \tfrac{1}{2} N \int d^d x\, d^d x'\, \phi_s(x)\, \mathrm{K}(x,x')\, \phi_s(x') \;+\; O(\phi_s^3)$$
$$\mathrm{K} = \langle J_s(x) J_s(x')\rangle = \frac{P_s(x-x')}{(|x-x'|^2+\varepsilon_{\rm UV}^2)^{s+d-2}} \;\to\; P_s(p)\, p^{2s+d-4} \ln \frac{p^2}{\varepsilon_{\rm UV}^2}$$

$$\Gamma = N \ln \varepsilon_{\rm UV}\; S_{\rm CHS} + \ldots\,, \qquad S_{\rm CHS} = \int d^d x\; \phi_s P_s \partial^{2s+d-4} \phi_s + \ldots$$

• AdS$_{d+1}$ dual: massless higher spin (MHS) theory

$$S_{\rm MHS} = N \int d^{d+1}x\, \sqrt{g}\, \phi_s(-\nabla^2 + \mathrm{m}_s^2)\, \phi_s + \ldots$$
$$\mathrm{m}_s^2 = s^2 + (d-5)s - 2d + 4$$

$$S_{\rm MHS}(\varphi_s\big|_{M^d} \sim \phi_s) \;=\; N \ln \varepsilon_{\rm IR}\; S_{\rm CHS}(\phi_s) + \ldots$$

“tree-level” relation between CHS in M^d and MHS in AdS$_{d+1}$

• Original example of $\mathcal{N}=4$ SYM: background field sources for superconformal currents – $\mathcal{N}=4$ conformal SG multiplet integrate out SYM fields:

$$Z_{\rm SYM}(h,...) = \int [dA...] \exp - S_{\rm SYM}(A,..;h,...)$$

log UV term in $\ln Z_{\rm SYM}$ is 1-loop exact (related to trace anomaly) given by $\mathcal{N}=4$ conformal supergravity action [Liu, AT 98]

$$\ln Z_{\rm SYM} = {\rm c} \ln \varepsilon_{_{\rm UV}} \, S_{\rm CSG} + {\rm fin} \, , \qquad S_{\rm CSG} = \int d^4x \sqrt{g} \, C^2 + ...$$

• Trace anomaly (2-, 3-functions of s-c currents) protected – get same on AdS$_5$ side: $\mathcal{N}=8, d=5$ supergravity action on Dirichlet problem solution:

log IR term is $\mathcal{N}=4$ CSG action:

$$\int d^5x \sqrt{g} R \to \ln \varepsilon_{_{\rm IR}} \int d^4x \sqrt{g} \, C^2 + {\rm fin}$$

Simplest CFT data: spectrum of conformal operators

"one-particle" partition function $\mathcal{Z} = \sum_n \mathrm{d}_n\, q^{\Delta_n}$

radial quantization: operators in $R^d \to$ states in $R \times S^{d-1}$

spectrum of dimensions / energies $\omega_n = \Delta_n$ encoded in

in partition function in $S^1_\beta \times S^{d-1}$: $\quad -\ln Z = \frac{1}{2} \ln \det(-\nabla^2 + ...)$

"one-particle" or canonical partition function

$$\mathcal{Z} = \mathrm{tr}\, e^{-\beta H} = \sum_n \mathrm{d}_n\, e^{-\beta\omega_n} = \sum_n \mathrm{d}_n\, q^{\Delta_n} \ , \qquad q \equiv e^{-\beta}$$

"multi-particle" or grand canonical partition function

$$\ln Z = -\sum_n d_n \ln(1 - e^{-\beta\omega_n}) = \sum_{m=1}^{\infty} \tfrac{1}{m} \mathcal{Z}(q^m)$$

Two methods to compute $\mathcal{Z}(q) = \sum_n \mathrm{d}_n \, q^{\Delta_n}$:

I. Operator counting in flat R^d: [Cardy 91; Kutasov, Larsen 00]
enumerate all conformal primaries and their descendants
modulo eqs. of motion and identities

II. Partition function on $S^1_\beta \times S^{d-1}$:
define CFT on time $\times$ spatial sphere
and compute determinants

Counting method:

conformal scalar: $S_{\text{c.s.}} = \int d^dx\, (\partial\Phi)^2, \qquad \Delta(\Phi) = \frac{1}{2}(d-2)$

- lowest dim conformal operator Φ contributes $q^{\frac{1}{2}(d-2)}$
- its conformal descendants $\partial_{\mu_1}\ldots\partial_{\mu_k}\Phi$:

each power of derivative in given direction enters only once

get factor $\sum_{k=0}^{\infty} q^k = (1-q)^{-1}$ from each of d directions

- but some operators vanish due to e.o.m. $\partial^2\Phi = 0$

$\Delta(\partial^2\Phi) = \frac{1}{2}(d-2) + 2$ – need subtract $q^{\frac{1}{2}(d-2)+2}$

dressed again by derivative factor $(1-q)^{-d}$

Total partition function of conformal scalar

$$\mathcal{Z}_{\text{c.s.}}(q) = \frac{q^{\frac{d-2}{2}}(1-q^2)}{(1-q)^d}$$

$d = 4$ Maxwell vector: $\quad S_1 = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu}$

• lowest dimension gauge-invariant operator: $F_{\mu\nu}$:

$\Delta = 2,\;$ d $= 6$ components $\;\rightarrow 6q^2$

• its derivatives give $(1-q)^{-4}$ factor

• this overcounts ignoring e.o.m. $\partial^\mu F_{\mu\nu} = 0$ and gauge identities $\partial^\mu F^*_{\mu\nu} = 0$ (and their derivatives) implying subtraction of $-(4+4)q^3$ times $(1-q)^{-4}$

• but also overcounts as identities descending from $\partial_\mu\partial_\nu F^{\mu\nu} = 0$ and $\partial^\mu\partial^\nu F^*_{\mu\nu} = 0$ of $\Delta = 4$ are trivial requires adding back $2q^4(1-q)^{-4}$

Total $d = 4$ vector partition function

$$\mathcal{Z}_1(q) = \frac{6q^2 - (4+4)q^3 + (1+1)q^4}{(1-q)^4} = \frac{2q^2(3-q)}{(1-q)^3}$$

[generalization: conformal $s = 1$ in even d: $\;\int d^dx\, F_{\mu\nu}\,\partial^{d-4}F_{\mu\nu}$]

Similar count for operators of singlet sector of $U(N)$ scalar theory:

$J_0 = \Phi_r^* \Phi_r$ and conserved $J_s \sim \Phi_r^* \partial^s \Phi_r$ and their descendants

$\Delta(J_s) = \Delta_+ = s + d - 2$

analog of e.o.m.: $\partial^{\mu_1} J_{\mu_1 \dots \mu_s} = 0$ is rank $s-1$ with $\Delta = \Delta_+ + 1$

$$\mathcal{Z}_{+\,0} = \frac{q^{d-2}}{(1-q)^d}\,, \qquad \mathcal{Z}_{+\,s} = \frac{\mathrm{n}_s\, q^{\Delta_+} - \mathrm{n}_{s-1}\, q^{\Delta_+ +1}}{(1-q)^d}$$

$$\mathrm{n}_s = (2s+d-2)\, \tfrac{(s+d-3)!}{(d-2)!\, s!}$$

n_s = components of symmetric traceless rank s tensor in d dim

$$d = 4: \qquad \mathcal{Z}_{+\,s} = \frac{(s+1)^2\, q^{s+2} - s^2\, q^{s+3}}{(1-q)^4}$$

$\mathcal{Z}_{+\,s}$ has interpretation of character $\chi_{(\Delta_+, s, 0, \dots, 0)}(q, 1, \dots, 1)$ of short rep. of $SO(d,2)$ with dim Δ_+ and spin s [Dolan 05]

Full singlet sector partition function:

$$\mathcal{Z}_{+} = \sum_{s=0}^{\infty} \mathcal{Z}_{+\,s} = \frac{q^{d-2}(1+q)^2}{(1-q)^{2d-2}} = \left[\mathcal{Z}_{c.s.}(q)\right]^2$$

- N^0 term in singlet-sector $\ln Z$ of $U(N)$ scalar on $S^1 \times S^{d-1}$

[Shenker, Yin 11; Giombi, Klebanov, AT 14]

- relation between characters of $SO(2,d)$ (cf. [Flato, Fronsdal])

AdS$_{d+1}$/CFT$_d$ interpretation:

$J_s \leftrightarrow \varphi_s$ – massless higher spin gauge field in AdS$_{d+1}$

$\mathcal{Z}_{+\,s}(q) = \mathcal{Z}_s^{(+)}(q)$ – massless spin s partition function in thermal AdS$_{d+1}$ with $S^1 \times S^{d-1}$ boundary

[Gopakumar, Gupta, Lal 11, 12; Giombi, Klebanov, AT 14]

Massless higher spin partition function in AdS$_{d+1}$

quadratic action of massless symmetric HS fields in AdS_{d+1}

gauge fixing / ghosts $\rightarrow$ 1-loop massless HS partition function:

$$Z_s(\mathrm{AdS_{d+1}}) = \Big[\frac{\det\big(-\nabla^2 + \mathrm{m}'^2_{s-1}\big)_{s-1\,\perp}}{\det\big(-\nabla^2 + \mathrm{m}^2_s\big)_{s\perp}}\Big]^{1/2}$$

$$\mathrm{m}^2_s = (s-2)(s+d-2) - s\,, \qquad \mathrm{m}'^2_{s-1} = (s-1)(s+d-2)$$

$d=2$: [Gaberdiel, Gopakumar, Saha 10]; $d \geqslant 3$: [Gupta, Lal 12]

• mass m spin s field in AdS$_{d+1}$: $(-\nabla^2 + \mathrm{m}^2_s + m^2)\,\varphi_s = 0$,

solutions near $z \to 0$ bndry of $ds^2 = z^{-2}(dz^2 + dx_n dx_n)$

$$\varphi_s \sim z^{\Delta_\pm - s}\,, \qquad \Delta_\pm = \tfrac{1}{2}d \pm \sqrt{(s+\tfrac{1}{2}d-2)^2 + m^2}$$

• standard choice of b.c.: $\Delta = \Delta_+$

φ_s and ghost terms ($m^2 = 0$): $\mathrm{m}^2_s = \Delta_+(\Delta_+ - d) - s$

$$\Delta_+ = s + d - 2\,, \qquad \Delta'_+ = \Delta_+ + 1$$

same as dimensions of J_s and $\partial \cdot J_s$

massless HS partition function in thermal AdS$_{d+1}$

$$\ln Z_s^{(+)} = \sum_{m=1}^{\infty} \tfrac{1}{m} \mathcal{Z}_s^{(+)}(q^m)$$

$$\mathcal{Z}_s^{(+)}(q) = \frac{\mathrm{n}_s\, q^{s+d-2} - \mathrm{n}_{s-1}\, q^{s+d-1}}{(1-q)^d}$$

$$\mathcal{Z}_s^{(+)}(q) = Z_{+\,s}(q)$$

massless HS contribution $\leftrightarrow$ current contribution

ghost contribution $\leftrightarrow$ current conservation contribution

Back to CHS: $\mathcal{Z}_s(q)$ for $s > 1$ $(d = 4)$

counting method straightforward (modulo group theory)

$s = 2$: Weyl graviton

$$S_2 = \int d^4x \sqrt{g}\, C_{\mu\nu\lambda\rho}\, C^{\mu\nu\lambda\rho}$$

linearized theory in R^4:

$$\mathcal{Z}_2 = \sum_n \mathrm{d}_n\, q^{\Delta_n} = \frac{10q^2 - 18q^4 + 8q^5}{(1-q)^4}$$

count gauge-invariant conformal operators built out of linearized Weyl tensor $C \sim \partial\partial h$ modulo identities and e.o.m.

derivatives $\rightarrow$ universal denominator $(1-q)^4$; find numerator

• off-shell components of $C_{\mu_1\nu_1\mu_2\nu_2}$:

$\Delta(C) = 2$, 10 independent components $\rightarrow 10q^2$

• non-trivial gauge identities on $C \sim \partial\partial h$

$$\mathcal{B}^{\mu_1\mu_2} \equiv \varepsilon^{\mu_1\nu_1\gamma_1\delta_1}\,\varepsilon^{\mu_2\nu_2\gamma_2\delta_2}\,\partial_{\nu_1}\partial_{\nu_2}\,C_{\gamma_1\delta_1\gamma_2\delta_2} = 0$$

$\Delta(\mathcal{B}^{\mu\nu}) = 4,$ 9 components, subtracting $9q^4$

• subtracting all derivatives of $\mathcal{B}^{\mu\nu}$ overcounts:

$\partial_\mu \mathcal{B}^{\mu\lambda} = 0$ with dimension 5 and 4 components: add back $4q^5$

⋆ off-shell count thus gives

$$\mathcal{Z}_2^{\text{off}-\text{sh.}} = \frac{10\,q^2 - 9\,q^4 + 4\,q^5}{(1-q)^4}$$

• next subtract descendant operators $\partial ... \partial C$

that vanish due to e.o.m. for dynamical field $\phi_2 = (h_{\mu\nu})$

$$B_{\mu_1\mu_2} \equiv \partial^{\nu_1}\partial^{\nu_2} C_{\mu_1\nu_1\mu_2\nu_2} = 0$$

count of symmetric traceless $B_{\mu_1\mu_2}$ same as for $\mathcal{B}^{\mu_1\mu_2}$:

subtract $9q^4$, add back $4q^5$ to account for identity $\partial^\mu B_{\mu\lambda} = 0$

⋆ contribution of equations of motion to be subtracted

$$\mathcal{Z}_2^{\text{e.o.m.}} = \frac{9\,q^4 - 4\,q^5}{(1-q)^4}$$

total:

$$\mathcal{Z}_2 = \mathcal{Z}_2^{\text{off-sh.}} - \mathcal{Z}_2^{\text{e.o.m.}} = \frac{10q^2 - 2(9q^4 - 4q^5)}{(1-q)^4}$$

common features of $s = 1, 2$ cases, generalize to $s > 2$ in $d = 4$:

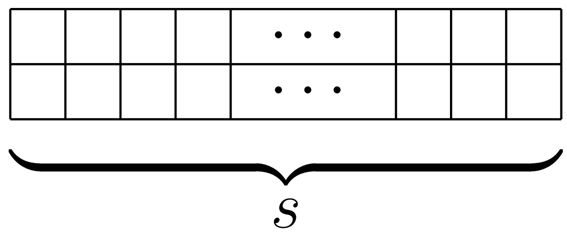
⋆ contributions of e.o.m. and identities are same – double

⋆ count of e.o.m. is identical to count of conserved traceless rank s current operator of dimension $\Delta_+ = s + d - 2$

$s > 2$: $\quad S_s \sim \int d^4x \, C_s C_s, \quad d = 4$

$$\mathcal{Z}_s = \frac{2(2s+1)q^2 - 2(s+1)^2 q^{s+2} + 2s^2 q^{s+3}}{(1-q)^4}$$

Weyl tensor $C_{\mu_1\nu_1,\dots,\mu_s\nu_s} \sim \partial^s \phi_s$: $(s, s, 0, .., 0)$ representation of $SO(d)$



$$\dim(s,s) = \mathrm{n}_{(s,s)} = \frac{(2s+d-4)(2s+d-3)(2s+d-2)(s+d-5)!(s+d-4)!}{s!\,(s+1)!\,(d-2)!(d-4)!}$$

$$\mathrm{n}_{(s,s)}\Big|_{d=4} = 2\,(2\,s+1)$$

- off shell count: $\quad \Delta(C_s) = 2 \;\; \rightarrow \;\; 2(2\,s+1)q^2$
- gauge identities: $\quad \mathcal{B}^{\mu_1\cdots\mu_s} = 0\;, \quad \partial_{\mu_1}\mathcal{B}^{\mu_1\cdots\mu_s} = 0$

$$\mathcal{B}^{\mu_1\cdots\mu_s} \equiv \varepsilon^{\mu_1\nu_1\gamma_1\delta_1} \;\dots\; \varepsilon^{\mu_s\nu_s\gamma_s\delta_s}\,\partial_{\nu_1}\cdots\partial_{\nu_s}\,C_{\gamma_1\delta_1\cdots\gamma_s\delta_s}$$

$\mathcal{B}^{\mu_1\cdots\mu_s}$: $\Delta = s+2$, symmetric traceless in $(s,0,...,0)$ of $SO(d)$

$$\mathrm{n}_s = (2s+d-2)\tfrac{(s+d-3)!}{(d-2)!\,s!}\ , \qquad \mathrm{n}_s\Big|_{d=4} = (s+1)^2$$

• subtract $(s+1)^2 q^{s+2}$; add back $s^2 q^{s+3}$ ("conservation" identity)

$$\mathcal{Z}_s^{\mathrm{off-sh.}} = \frac{2(2s+1)q^2 - (s+1)^2\, q^{s+2} + s^2\, q^{s+3}}{(1-q)^4}$$

• e.o.m. for conformal spin s: generalized linearized Bach eqs

$$B_{\mu_1...\mu_s} \equiv \partial^{\nu_1}...\partial^{\nu_s} C_{\mu_1\nu_1...\mu_s\nu_s} = 0\ , \qquad \partial^{\mu_1} B_{\mu_1...\mu_s} = 0$$

$B_{\mu_1...\mu_s}$ – same count as for $\mathcal{B}^{\mu_1\cdots\mu_s}$

$$\mathcal{Z}_s^{\mathrm{e.o.m.}} = \frac{(s+1)^2\, q^{s+2} - s^2\, q^{s+3}}{(1-q)^4}$$

$$\mathcal{Z}_s = \mathcal{Z}_s^{\mathrm{off-sh.}} - \mathcal{Z}_s^{\mathrm{e.o.m.}} = \frac{2(2s+1)q^2 - 2(s+1)^2 q^{s+2} + 2s^2 q^{s+3}}{(1-q)^4}$$

Generalization to arbitrary even d

CFT$_d$ interpretation / analogy:

$$\mathcal{Z}_s = \mathcal{Z}_s^{\text{off-sh.}} - \mathcal{Z}_s^{\text{e.o.m.}} = \mathcal{Z}_{-\,s} - \mathcal{Z}_{+\,s}$$

$\mathcal{Z}_s^{\text{e.o.m.}} = \mathcal{Z}_{+\,s}$ = counts conformal spin s current operators J_s

$\Delta(J_s) = \Delta_+ = s + d - 2$ (analog of $B_s \sim \partial^s C_s$)

$\mathcal{Z}_s^{\text{off-sh.}} = \mathcal{Z}_{-\,s}$ = counts spin s shadow operators $\widetilde{J}_s$ (analogs of ϕ_s)

$\Delta(\widetilde{J}_s) = \Delta_- = d - \Delta_+ = 2 - s$

$$\mathcal{Z}_{+\,s} = \frac{\mathrm{n}_s\, q^{\Delta_+} - \mathrm{n}_{s-1}\, q^{\Delta_+ + 1}}{(1-q)^d}\,, \qquad \Delta_+ = s + d - 2$$

guess for $\mathcal{Z}_{-\,s}$: replace Δ_+ by $\Delta_- = d - \Delta_+ = 2 - s$

but there is "correction" σ_s = character of conformal Killing tensor rep.

$$\mathcal{Z}_{-\,s} = \frac{\mathrm{n}_s\, q^{2-s} - \mathrm{n}_{s-1}\, q^{1-s}}{(1-q)^d} + \sigma_s(q)$$

$$\sigma_s(q) = \chi_{(s-1,s-1,0,...,0)}(q, 1, ..., 1)$$

total $\mathcal{Z}_{-\,s}$ contains only positive powers of q

$$\mathcal{Z}_{-\,s} = \widehat{\mathcal{Z}}_s(q) - \mathcal{Z}_{+\,s}(q)\ , \qquad \widehat{\mathcal{Z}}_s \equiv \frac{1}{(1-q)^d} \sum_{m=2}^{d-2} (-1)^m \, \mathrm{c}_{s,m} \, q^m$$

$\mathrm{c}_{s,m}$ dim of $\mathfrak{so}(d)$ reps with 2 rows of s boxes and $m-2$ of 1 box:

$$\begin{aligned} \mathrm{c}_{s,m} &= \dim(s, s, 1^{m-2}) \\ &= \frac{(2s+d-2)!(s+d-3)!(s+d-4)!(s+d-3-m)!(s+m-3)!}{(2s+d-5)!(s+m-1)!(s+d-1-m)!s!(s-1)!(d-2)!(d-2-m)!(m-2)!} \end{aligned}$$

$d = 6$:

$$\begin{aligned} \mathrm{n}_s &= \tfrac{1}{12}(s+1)(s+2)^2(s+3) \\ \mathrm{c}_{s,2} = \mathrm{c}_{s,4} &= \tfrac{1}{12}(s+1)^2(s+2)^2(2s+3) \\ \mathrm{c}_{s,3} &= \tfrac{1}{6}s(s+1)(s+2)(s+3)(2s+3) \end{aligned}$$

CHS partition function in even $d \geqslant 4$

$$\mathcal{Z}_s = \mathcal{Z}_{-\,s}(q) - \mathcal{Z}_{+\,s}(q) = \widehat{\mathcal{Z}}_s(q) - 2\mathcal{Z}_{+\,s}(q)$$

$$= \frac{1}{(1-q)^d}\Big[\sum_{m=2}^{d-2}(-1)^m\,\mathrm{c}_{s,m}\,q^m - 2\mathrm{n}_s\,q^{s+d-2} + 2\mathrm{n}_{s-1}\,q^{s+d-1}\Big]$$

• $\mathcal{Z}_s$ can be expressed in terms of characters of $\mathfrak{so}(d,2)$ Verma modules

• group-theoretic perspective: $\mathcal{Z}_{+\,s}$ and $\mathcal{Z}_{-\,s}$
associated with conformal current J_s and shadow field $\widetilde{J}_s$

• J_s generates unitary irrep of $\mathfrak{so}(d,2)$
$\widetilde{J}_s$ generates reducible indecomposable $\mathfrak{so}(d,2)$ rep.
non-unitarizable: $\Delta_- = 2 - s$ is below unitarity bound

• "Weyl-tensor" $\partial^s\widetilde{J}_s \leftrightarrow C_s \sim \partial^s\phi_s$ with $\Delta = 2$ is conf. primary

• analysis of relevant $\mathfrak{so}(d,2)$ modules [Shaynkman, Tipunin, Vasiliev 04]

Method II: Partition function on $S^1 \times S^{d-1}$

conformal scalar:

$$\ln Z_{\rm c.s.} = -\tfrac{1}{2}\, \ln\, \det \mathcal{O}_0 \ , \qquad \mathcal{O}_0 = -D^2 + \tfrac{d-2}{4\,(d-1)}\, R$$

$$D^2 = \partial_0^2 + \nabla^2 \ , \qquad \nabla^2 = \nabla^i \nabla_i = D^2_{S^{d-1}}$$

$$\partial_0 = \partial_t, \quad t \in (0, \beta); \qquad R = R(S^{d-1}) = (d-1)(d-2)$$

$$\mathcal{O}_0 = -\partial_0^2 - \nabla^2 + \tfrac{1}{4}\,(d-1)^2$$

eigenvalues and multiplicities of Laplacian $-\nabla^2$ on S^{d-1}

$$\lambda_n = n\,(n+d-2)\ , \qquad {\rm d}_n = (2n+d-2)\, \tfrac{(n+d-3)!}{n!\,(d-2)!}$$

eigenvalues of $\mathcal{O}_0$

$$\lambda_{k,n} = w^2 + \omega_n^2, \qquad w = \tfrac{2\pi k}{\beta}\ , \quad \omega_n = n + \tfrac{1}{2}(d-2)$$

$$-\ln Z_{\rm c.s.} = \tfrac{1}{2}\ln \det\mathcal{O}_0 = \tfrac{1}{2}\sum_{k,n} \mathrm{d}_n \ln\lambda_{k,n} = -\sum_{m=1}^{\infty} \mathcal{Z}_{\rm c.s.}(m\beta)$$

$$\mathcal{Z}_{\rm c.s.}(\beta) = \sum_{n=0}^{\infty} d_n\, e^{-\beta\,[n+\frac{1}{2}(d-2)]} = \frac{q^{\frac{d-2}{2}}(1-q^2)}{(1-q)^d}$$

same as in operator counting method

$d = 4$ Maxwell vector: $\quad S_1 = -\frac{1}{4}\int d^4x\sqrt{g}\, F_{\mu\nu}\, F^{\mu\nu}$

$Z(S^1 \times S^3)$ in Lorentz gauge ($R_{00} = 0$, $R_{ij} = 2\, g_{ij}$)

$$Z_1 = \frac{\det(-D^2)}{\big[\det(-g_{\mu\nu}D^2 + R_{\mu\nu})\big]^{1/2}} = \Big[\frac{\det(-D^2)}{\det(-g_{ij}D^2 + R_{ij})}\Big]^{1/2}$$

$$= \frac{1}{\big[\det(-g_{ij}D^2 + R_{ij})_\perp\big]^{1/2}} = \frac{1}{\big[\det\mathcal{O}_{1\,\perp}\big]^{1/2}}$$

$$\mathcal{O}_{1\,ij} = (-\partial_0^2 - \nabla^2 + 2)_{ij}$$

same found directly in $A_0 = 0$ gauge

from spectrum of 3-vector Laplacian $(-\nabla^2)_{1\perp}$ on S^3

get spectrum of $\mathcal{O}_{1\perp}$ $(\Delta_0 \to iw,\ w = \frac{2\pi k}{\beta})$

$$\lambda_{k,n} = w^2 + \omega_n^2 \,, \qquad \omega_n = n + 2 \,, \qquad \mathrm{d}_n = 2(n+1)(n+3)$$

$$-\ln Z_1 = \tfrac{1}{2} \sum_{k,n} \mathrm{d}_n \ln \lambda_{k,n} = -\sum_{m=1}^{\infty} \mathcal{Z}_1(m\beta)$$

$$\mathcal{Z}_1(\beta) = \sum_{n=0}^{\infty} \mathrm{d}_n \, e^{-\beta(n+2)} = \frac{2q^2(3-q)}{(1-q)^3}$$

same as in operator counting method

$s = 2$: Weyl graviton

$$S_2 = \tfrac{1}{2} \int d^4x \sqrt{g}\, C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho} = \int d^4x \sqrt{g} \left(R_{\mu\nu} R_{\mu\nu} - \tfrac{1}{3} R^2 \right)$$

expand Weyl action near curved background to 2nd order in $\phi_2 = (h_{\mu\nu})$

$$L^{(2)} = \tfrac{1}{4} D^2 h_{\mu\nu}\, D^2 h^{\mu\nu} - R^{\mu}{}_{\rho} h_{\mu\nu}\, D^2 h^{\nu\rho} + \tfrac{1}{2} R^{\mu\nu}\, h_{\alpha\beta}\, D_\mu D_\nu\, h^{\alpha\beta}$$
$$- \tfrac{3}{2} R_{\rho\sigma}\, R^{\sigma\mu} h_{\mu\nu}\, h^{\nu\rho} + \tfrac{1}{2} R^{\nu\rho} R^{\sigma\mu} h_{\mu\nu} h_{\rho\sigma} + \tfrac{1}{6} (h_{\mu\nu}\, R^{\mu\nu})^2 + \tfrac{1}{4} R_{\mu\nu} R^{\mu\nu} h_{\alpha\beta} h^{\alpha\beta}$$
$$+ \tfrac{1}{2}\, R\, R_{\rho}{}^{\mu}\, h_{\mu\nu}\, h^{\nu\rho} - \tfrac{1}{9}\, R^2\, h_{\mu\nu}\, h^{\mu\nu}$$

4-order operator factorizes on conformally-flat background

S^4: $R_{\mu\nu} = \frac{1}{4} R\, g_{\mu\nu}, R = 12$; on TT tensors $h_{\mu\nu}$

$$L^{(2)} = \tfrac{1}{4}\, h^{\mu\nu}\, \widetilde{\mathcal{O}}_2\, h_{\mu\nu}\ , \quad \widetilde{\mathcal{O}}_2 = \big(- D^2 + \tfrac{1}{6} R\big)\big(- D^2 + \tfrac{1}{3} R\big)$$

$S^1 \times S^3$: $R_{ij} = \frac{1}{3} R\, g_{ij}$, $R = 6$; gauge $h_{0i} = h_{00} = 0$

$$L^{(2)} = \tfrac{1}{4}\, h^{ij}\, \mathcal{O}_2\, h_{ij}\ , \qquad \mathcal{O}_2 = (\partial_0^2 + \nabla^2)^2 - \tfrac{2}{3} R\, (2\partial_0^2 + \nabla^2) + \tfrac{1}{9} R^2$$
$$\mathcal{O}_2 = \left[(\partial_0 - 1)^2 + \nabla^2 - 3\right] \left[(\partial_0 + 1)^2 + \nabla^2 - 3\right]$$

$$Z_2 = \frac{1}{\left[\det \mathcal{O}_{2\,\perp}\, \det{}' \mathcal{O}_{1\,\perp}\right]^{1/2}}$$

$\mathcal{O}_{1\perp} = -\partial_0^2 - \nabla^2 + 2$ on $V_i^\perp$ from $h_{ij} \to h_{ij}^\perp + D_i V_j + D_j V_i$

$$(-\nabla^2)_{2\perp} : \quad \lambda_n = (n+2)(n+4) - 2\,, \qquad \mathrm{d}_n = 2(n+1)(n+5)$$

spectrum ($\partial_0 \to iw, \;\; w = 2\pi k \beta^{-1}$)

$$\mathcal{O}_{2\perp} : \qquad \lambda_{k,n} = \left[w^2 + (n+2)^2\right]\left[w^2 + (n+4)^2\right]$$

factorization related to conformal invariance of spin 2 theory

$$\det \mathcal{O}_{2\perp} : \qquad \mathcal{Z}_{2,0} = \sum_{n=0}^{\infty} 2\,(n+1)(n+5)\,(q^{n+2} + q^{n+4})$$

$$\det{}' \mathcal{O}_{1\perp} : \qquad \mathcal{Z}_{1,1} = \sum_{n=1}^{\infty} 2\,(n+1)(n+3)\,q^{n+2}$$

$$\mathcal{Z}_2 = \mathcal{Z}_{2,0}(q) + \mathcal{Z}_{1,1}(q) = \frac{10q^2 - 18q^4 + 8q^5}{(1-q)^4}$$

same as in operator counting method

Conformal higher spin partition function on $S^1 \times S^3$

$$S_s = \int d^4x \sqrt{g}\, \phi_s (D^{2s} + ...) \phi_s$$

$2s$-order kinetic operator on TT 3d tensors $\phi_{i_1 \dots i_s}$ factorizes

s=even :
$$\mathcal{O}_s = \prod_{r=1}^{s} \left[(\partial_0 + 2r - s - 1)^2 + \nabla^2 - s - 1 \right]$$

s=odd :
$$\mathcal{O}_s = - \prod_{r=-\frac{1}{2}(s-1)}^{\frac{1}{2}(s-1)} \left[(\partial_0 + 2r)^2 + \nabla^2 - s - 1 \right]$$

e.g. $\mathcal{O}_3 = \left(\partial_0^2 + \nabla^2 - 4\right) \left[(\partial_0 + 2)^2 + \nabla^2 - 4 \right] \left[(\partial_0 - 2)^2 + \nabla^2 - 4 \right]$

$$Z_s = \frac{1}{\left[\prod_{r=1}^{s} \det{}' \mathcal{O}_{r\,\perp} \right]^{1/2}}$$

det$'$: first $s-r$ modes are to be omitted

spectrum of spin s Laplacian $-\nabla^2$ on S^3

$$(-\nabla^2)_{s\perp}: \quad \lambda_n = (n+s)(n+s+2)-s\ , \quad \mathrm{d}_n = 2(n+1)(n+2s+1)$$

for $\mathcal{O}_{r\perp}$ $(w = 2\pi k\beta^{-1})$

$$\lambda_{k,n} = \prod_{r=1}^{s} \left(w^2 + \omega_{n,r}^2\right)\ , \qquad\qquad \omega_{n,r} = n + 2r$$

$$\det{}'\mathcal{O}_{r\perp}: \quad \mathcal{Z}_{r,s-r} = \textstyle\sum_{n=s-r}^{\infty} 2\,(n+1)(n+2r+1)\ \sum_{p=1}^{r} q^{n+2p}$$

$$\mathcal{Z}_s = \sum_{r=1}^{s} \mathcal{Z}_{r,s-r} = \frac{2\,q^2}{(1-q)^4}\ \left[(s+1)^2\,(1-q^s) - s^2\,(1-q^{s+1})\right]$$

$$= \frac{2(2s+1)q^2 - 2(s+1)^2 q^{s+2} + 2s^2 q^{s+3}}{(1-q)^4}$$

same as in operator counting method

Conformal higher spin partition function on S^4

• Weyl-invariant operator on curved background

$\partial^{2s} \to D^{2s} + R\, D^{2s-2} + ... + R^s$ not known explicitly for $s > 2$

consistent on any Weyl gravity solution (conformally-flat, Einstein, ...)

• can be found in factorized form on S^4 (or dS_4 or AdS_4) [AAT 13]

also derived in [Metsaev 14; Nutma, Taronna 14]

• examples: Maxwell theory on S^4 $(R = 12, \ \ r = 1)$

$$Z_1 = \Big[\frac{\det\widehat{\Delta}_0(0)}{\det\widehat{\Delta}_{1\perp}(3)}\Big]^{1/2}, \qquad \widehat{\Delta}_s(M^2) \equiv -\nabla_s^2 + M^2$$

• Weyl graviton: $C^2 \to \frac{1}{2} h\, \widehat{\Delta}_{2\perp}(2)\, \widehat{\Delta}_{2\perp}(4)\, h$

cf. Einstein graviton: $-\nabla^2 h_{mn} - 2R_{mknl}h^{kl} \ \to \ \widehat{\Delta}_2(2)\, h_{mn}$

$$Z_2 = Z_{2,1} Z_{2,0} = \Big[\frac{\det\widehat{\Delta}_{1\perp}(-3)}{\det\widehat{\Delta}_{2\perp}(2)}\Big]^{1/2} \Big[\frac{\det\widehat{\Delta}_0(-4)}{\det\widehat{\Delta}_{2\perp}(4)}\Big]^{1/2}$$

Einstein graviton $Z_{2,1}$ and "partially-massless" $Z_{2,0}$ factors

- General CHS: factorization into all "partially-massless" operators

$$D^{2s} + ... = \prod_{k=0}^{s-1} \widehat{\Delta}_{s\perp}(M^2_{s,k}) \ , \qquad M^2_{s,k} = 2 + s - k - k^2$$

- add ghost factors $\to$ remarkably simple generalization of flat-space Z

$$Z_s = \prod_{k=0}^{s-1} Z_{s,k} \ , \qquad Z_{s,k} = \Big[\frac{\det \widehat{\Delta}_{k\perp}(M^2_{k,s})}{\det \widehat{\Delta}_{s\perp}(M^2_{s,k})}\Big]^{1/2}$$

$$Z_{s,k} = \Big(\frac{\det[-\nabla^2 + (2 + k - s - s^2)\ ]_{k\perp}}{\det[-\nabla^2 + (2 + s - k - k^2)\ ]_{s\perp}}\Big)^{1/2}$$

- $k = s - 1$ term: massless spin s partition function in (A)dS$_4$

$$Z_{s,s-1} = \Big(\frac{\det[-\nabla^2 + (1 - s^2)\epsilon\ ]_{s-1\,\perp}}{\det[-\nabla^2 + (2 + 2s - s^2)\epsilon\ ]_{s\perp}}\Big)^{1/2} , \quad \epsilon = \pm 1$$

partition function on S^4: extract conformal anomaly coefficient a_s

$$\ln Z = -B_4 \ln \varepsilon_{\rm UV} + \text{finite}$$

$$B_4 = \frac{1}{(4\pi)^2} \int d^4x \sqrt{g}\, b_4 \Big|_{S^4} = -\mathrm{a}_s \ , \qquad b_4 = \tfrac{1}{4}(-\mathrm{a}\, R^* R^* + \mathrm{c}\, C^2)$$

$$\text{Maxwell}: \quad \mathrm{a}_1 = \frac{31}{45} \ , \quad \mathrm{c}_1 = \frac{2}{5} \ , \qquad \text{Weyl}: \quad \mathrm{a}_2 = \frac{87}{5} \ , \quad \mathrm{c}_2 = \frac{398}{15}$$

- apply standard b_4-algorithm to each 2-nd order operator [AT 13]

$$\mathrm{a}_s = \sum_{k=0}^{s-1} \Big(\mathrm{a}[\widehat{\Delta}_{s\perp}(2+s-k-k^2)] - \mathrm{a}[\widehat{\Delta}_{k\perp}(2+k-s-s^2)] \Big)$$

$$\mathrm{a}_s = \frac{1}{180} \nu^2 (14\nu + 3) \ , \qquad \nu = s(s+1)$$

- same coefficient found via massless HS AdS_5 relation

[Giombi, Klebanov et al 13]

$$\ln Z_s^{(-)} - \ln Z_s^{(+)} = \mathrm{a}_s \ln \varepsilon_{\rm IR} + \text{finite}, \qquad \mathrm{Vol}_{\mathrm{AdS}_5} \sim \ln \varepsilon_{\rm IR}$$

relation between 1-loop partition functions:

- conformal spin s in conformally-flat M^d (e.g. S^d)
- spin s part of singlet sector CFT$_d$: current J_s and shadow $\widetilde{J}_s$ in M^d
- massless spin s field with $\pm$ b.c. in AdS$_{d+1}$ with bndry M^d

$$Z_s\Big|_{M^d} = \frac{Z_{-\,s}}{Z_{+\,s}}\Big|_{M^d} = \frac{Z_s^{(-)}}{Z_s^{(+)}}\Big|_{AdS_{d+1}}$$

second equality implied by vectorial AdS$_{d+1}$/CFT$_d$

- an argument for via "double-trace" deformation of CFT$_d$

  [Giombi, Klebanov, Pufu, Safdi, Tarnapolsky 13]

- check by direct computation on Z_s for CHS on S^d, $d = 4, 6$ [AT 13,14]

- direct proof in case of $M^d = S^1 \times S^{d-1}$ [Beccaria, Bekaert, AT 14]

Summing over spins

total CHS partion function: sum over all spins $s = 0, 1, 2, ..., \infty$

$$\mathcal{Z} = \sum_{s=0}^{\infty} (\mathcal{Z}_{-\,s} - \mathcal{Z}_{+\,s}) = \mathcal{Z}_{-} - \mathcal{Z}_{+}$$

$\mathcal{Z}_{+}$ is finite

$$\mathcal{Z}_{+} = \sum_{s=0}^{\infty} \mathcal{Z}_{+\,s} = \frac{q^{d-2}(1-q^2)^2}{(1-q)^{2d}}$$

• $\mathcal{Z}_{+}$ = partition function of massless HS Vasiliev theory in AdS_{d+1}
= partition function of singlet sector of $U(N)$ scalar CFT$_d$ on $S^1 \times S^{d-1}$
(counts spin s conserved current operators and their descendants)
• $\mathcal{Z}_{-} = \sum_s \mathcal{Z}_{-\,s} = \sum_s \mathcal{Z}_s^{\rm off-sh.}$ formally divergent:
$c_{s,m}$ are polynomials in s not suppressed by s-independent powers of q

Natural regularization of sum over spins

- Physical meaning – preservation of symmetries of theory

(cf. string theory: sums of fields of growing spins and masses in 2d description that should be consistent with target space symmetries)

[Brink, Nielsen 73; Brink, Fairlie, 74; Nahm 77]

- special regularization of infinite sums over spins necessary in AdS/CFT: 1-loop log ∞ in massless HS theory in AdS_4 then vanishes as required by $O(N^0)$ check of AdS_4/CFT_3 [Giombi, Klebanov 13]

$$\sum_{s=1}^{\infty} e^{-\epsilon s}\, \mathrm{a}_s = \tfrac{1}{180} \sum_{s=1}^{\infty} e^{-\epsilon s} s^2 (s+1)^2 (14s^2 + 14s + 3)$$

$$= \tfrac{14}{\epsilon^7} + \tfrac{7}{\epsilon^6} + \tfrac{3}{2\epsilon^5} + \tfrac{1}{6\epsilon^4} + 0 + \tfrac{\epsilon}{7560} + O(\epsilon^2)$$

finite part =0: $\zeta(-2n) = 0$ and $\frac{1}{3}\zeta(-3) + \frac{7}{10}\zeta(-5) = 0$

- conformal higher spin theory has vanishing a-anomaly coefficient in proper regularization [Giombi, Klebanov et al 13,14; AT 13,14]

• generalized ζ-function or cutoff regularization in any even d

$$\sum_{s=0}^{\infty} e^{-(s+\frac{d-3}{2})\,\epsilon}\, \mathrm{a}_s\Big|_{\epsilon\to 0,\ \mathrm{fin}} = 0$$

$$d=4: \quad \mathrm{a}_s = \tfrac{1}{180}\nu^2(14\nu+3)\ , \qquad\qquad \nu = s(s+1)$$

$$d=6: \quad \mathrm{a}_s = \frac{1}{151200}\nu^2(22\nu^3-55\nu^2-2\nu+2)\ , \qquad \nu=(s+1)(s+2)$$

• regularization consistent with underlying symmetries of CHS

– use it also to define partition function

$$\mathcal{Z} = \sum_{s=0}^{\infty} e^{-(s+\frac{d-3}{2})\,\epsilon}\, \mathcal{Z}_s\Big|_{\epsilon\to 0,\ \mathrm{fin}} = \widehat{\mathcal{Z}}(q) - 2\mathcal{Z}_+(q)$$

$$\widehat{\mathcal{Z}}(q) = \frac{1}{(1-q)^d}\sum_{m=2}^{d-2}(-1)^m\,\widehat{\mathrm{c}}_m\, q^m\ , \qquad \widehat{\mathrm{c}}_m \equiv \sum_{s=0}^{\infty} e^{-(s+\frac{d-3}{2})\,\epsilon}\, \mathrm{c}_{s,m}\Big|_{\epsilon\to 0,\ \mathrm{fin}}$$

explicitly:

$$d = 4: \qquad \mathcal{Z} = -\frac{q^2(11+26q+11q^2)}{6(1-q)^6}$$

$$d = 6: \qquad \mathcal{Z} = \frac{q^2(407-5298q-466311q^2-992956q^3-466311q^4-5298q^5+407q^6)}{241920(1-q)^{10}}$$

summed over spins $\mathcal{Z}_+$ and $\mathcal{Z}$ have $\beta \to -\beta$ symmetry:

$$\mathcal{Z}(q) = \mathcal{Z}(1/q) , \qquad\qquad q = e^{-\beta}$$

implies vanishing of associated Casimir (vacuum) energy on $R \times S^{d-1}$:

$$\mathcal{Z}(\beta) = \mathrm{tr}\, e^{-\beta H} = \sum_n \mathrm{d}_n\, e^{-\beta\omega_n}$$

$$E_c = \frac{1}{2}\sum_n \mathrm{d}_n\, \omega_n = \frac{1}{2}\zeta_E(-1)$$

$$\zeta_E(z) = \sum_n \mathrm{d}_n\, \omega_n^{-z} = \frac{1}{\Gamma(z)}\int_0^\infty d\beta\, \beta^{z-1}\mathcal{Z}(\beta)$$

$E_c = 0$: suggests CHS theory is special – vac energy vanishes also in

- $\mathcal{N} > 4$ gauged SG in AdS$_4$ [Allen, Davis 83; Gibbons, Nicolai 84]
- massless HS in AdS$_{d+1}$ ($Z_+ = Z^{(+)}$) [Giombi, Klebanov, AT 14]
- $\mathcal{N} = 4$ conformal SG + four $\mathcal{N} = 4$ SYM multiplets [Beccaria, AT]

conformal spin s Casimir energies:

$$d = 4: \; E_{c,s} = \tfrac{1}{720}\nu(18\nu^2 - 14\nu - 11)\,, \qquad \nu = s(s+1)$$

$$d = 6: \; E_{c,s} = \frac{1}{241920}\,\nu^2\,(12\nu^3 - 58\nu^2 - 6\nu + 117), \quad \nu = (s+1)(s+2)$$

$$\sum_{s=0}^{\infty} e^{-\,(s+\frac{d-3}{2})\,\epsilon}\, E_{c,s}\Big|_{\epsilon\to 0,\ \mathrm{fin}} = 0$$

$\star$ $E_{c,s}$ of CHS on $R \times S^{d-1}$ is - 2 of Casimir energy from $Z_{+\,s} = Z_s^{(+)}$ or -2 vac. energy of massless spin s in AdS$_{d+1}$

$\star$ $E_{c,s}$ similar but different from a_s

(T_{00} in general depends on derivative terms in T_m^m [Herzog, Huang 13])

CHS theory in $d = 2$

$d = 2$ CHS action is trivial for $s > 1$: $C_s = 0$ (no Weyl tensor in $d = 2$)

$S^1 \times S^1$ partition function from gauge fixing and ghosts in path integral

$$\mathcal{Z}_s = \mathcal{Z}_{-\,s} - \mathcal{Z}_{+\,s} = -2\mathcal{Z}_{+\,s} = -\frac{4\,q^s}{1-q}\,, \qquad \mathcal{Z}_{-\,s} = -\mathcal{Z}_{+\,s}\,, \quad s > 1$$

$$s = 1\text{: } \int d^2x\, F^{\mu\nu}\partial^{-2}F_{\mu\nu}; \quad s = 0\text{: } \int d^2x\, \phi\, \partial^{-2}\phi, \qquad \Delta(F) = \Delta(\phi) = 2$$

$$\mathcal{Z}_1 = -\frac{2q}{1-q}\,, \qquad \mathcal{Z}_0 = -\frac{1+q}{1-q}$$

$$\mathcal{Z} = \mathcal{Z}_0 + \mathcal{Z}_1 + \sum_{s=2}^{\infty} \mathcal{Z}_s = -\frac{(1+q)^2}{(1-q)^2}\,, \qquad \mathcal{Z}(q) = \mathcal{Z}(1/q)$$

$$E_{c,0} = E_{c,1} = \tfrac{1}{12}\,, \qquad E_{c,s} = \tfrac{1}{6}\left[1 + 6\,s\,(s-1)\right]\,, \qquad s > 1$$

$$E_{c,0} + E_{c,1} + \sum_{s=2}^{\infty} e^{-\,(s-\frac{1}{2})\,\epsilon}\, E_{c,s}\Big|_{\epsilon\to 0,\ \mathrm{fin}} = 0$$

$d=2$: Casimir energy on S^1 related to conformal anomaly $c \equiv \mathrm{a}$

$$E_{c,s} = -\frac{1}{12} c_s$$

CHS partition function on S^2 $\quad (\widehat{\Delta}_k(M^2) \equiv -\nabla^2 + M^2)$

$$Z_s(S^2) = \frac{\prod_{k=0}^{s-1} \left[\det \widehat{\Delta}_{k\perp}(k - s(s-1))\right]^{1/2}}{\prod_{k'=1}^{s-1} \left[\det \widehat{\Delta}_{s\perp}(s - k'(k'-1))\right]^{1/2}}$$

$$\ln Z = -B_2 \ln \varepsilon_{\text{UV}} + \text{finite}$$
$$B_2[\widehat{\Delta}_k(M^2)] = N_k(\tfrac{1}{6}R - M^2), \quad R = 2, r = 1$$

$$B_2^{(s)} = \sum_{k'=1}^{s-1} B_2[\widehat{\Delta}_{s\perp}(s - k'(k'-1))] - \sum_{k=0}^{s-1} B_2[\widehat{\Delta}_{k\perp}(k - s(s-1))]$$
$$= -\tfrac{2}{3}\big[1 + 6s(s-1)\big]$$

$B_2 = \frac{c}{24\pi} \int d^2x \sqrt{g} R = \frac{1}{3} c$

$s > 1: \quad B_2^{(s)} = \frac{1}{3} c_s = -\frac{2}{3} \left[1 + 6\, s\, (s-1) \right]$

$s = 0, 1: \quad B_2 = \frac{1}{3} c = -\frac{1}{3}$

computation of c_s via massless spin s in AdS$_3$ [Giombi, Klebanov et al 13]

Total central charge thus also vanishes:

$$c_0 + c_1 + \sum_{s=2}^{\infty} e^{-\,(s-\frac{1}{2})\,\epsilon}\, c_s \Big|_{\epsilon \to 0,\ \mathrm{fin}} = 0$$

• $d = 2$ CHS theory closely related to spin s W-gravity model [Hull 91]

same linearized symmetries – generalized diffs and Weyl transfs

same anomaly: W-gravity anomaly given by bc ghost contribution

$c_{gh} = -2(1 + 6s^2 - 6s)$ [Hull; Yamagishi; Pope et al 91]

• $d = 2$ case, while degenerate still limit of d-dimensional CHS theory which itself may be viewed as $d > 2$ generalization of W-gravity

Summary

relations between partition functions

$$\frac{Z_{-\,s}}{Z_{+\,s}}\Big|_{\rm M^d} = Z_s\Big|_{\rm M^d}$$

Z_s – 1-loop CHS partition function on conformally flat M^d

$Z_{+\,s}$ – free CFT partition function in spin s singlet sector ($Z = \prod_s Z_{+\,s}$)

$Z_{-\,s}$ is spin s shadow operator counterpart

AdS/CFT: $\quad Z_{\pm\,s}\big|_{\rm M^d} = Z_s^{(\pm)}\big|_{\rm AdS_{d+1}}$

partition function massless spin s field φ_s in AdS$_{d+1}$ with bndry M^d

computed with standard $\varphi_s \sim z^{\Delta_+ - s}$ or alternative $\varphi_s \sim z^{\Delta_- - s}$ b.c.

$$\frac{Z_s^{(-)}}{Z_s^{(+)}}\Big|_{AdS_{d+1}} = Z_s\Big|_{M^d}$$

verified explicitly for $M^d = S^d$ (matching of $\rm a_s$ coefficients)

Same relations derived for Z_s on $M^d = S^1 \times S^{d-1}$

$$\ln Z_s = \sum_{m=1}^{\infty} \tfrac{1}{m} \mathcal{Z}_s(q^m)$$

$$\mathcal{Z}_s(q) = \mathcal{Z}_{-\,s}(q) - \mathcal{Z}_{+\,s}(q) \,, \qquad \mathcal{Z}_{\pm\,s}(q) = \mathcal{Z}_s^{(\pm)}(q)$$

• $\mathcal{Z}_{+\,s}$ counts components of traceless symmetric current operator J_s of dim Δ_+ and its conformal descendants modulo $\partial \cdot J_s = 0$

• $\mathcal{Z}_{-\,s}$ counts shadow spin s operators (modulo gauge degeneracy) is given by $\mathcal{Z}_{+\,s}$ with $\Delta_+ \to \Delta_- = d - \Delta_+$ plus character of conformal Killing tensor rep. of $SO(d,2)$

• in $d = 4$:

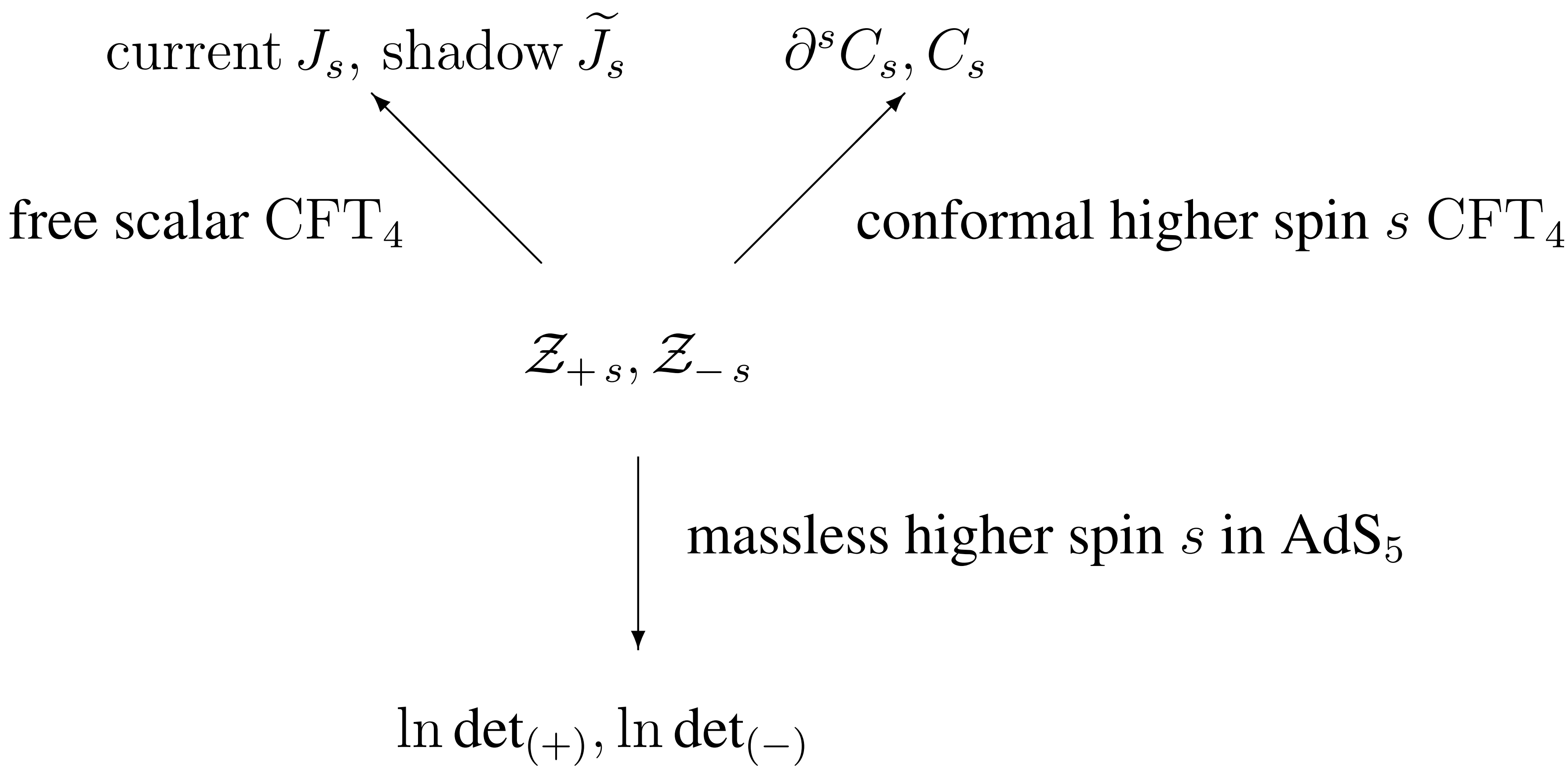
$$\mathcal{Z}_{-\,s} = \widehat{\mathcal{Z}}_s - \mathcal{Z}_{+\,s}, \quad \mathcal{Z}_s = \widehat{\mathcal{Z}}_s - 2\mathcal{Z}_{+\,s}$$

$$\widehat{\mathcal{Z}}_s = \frac{2(2\,s+1)\,q^2}{(1-q)^4} \,, \qquad \mathcal{Z}_{+\,s} = \frac{(s+1)^2\,q^{s+2} - s^2\,q^{s+3}}{(1-q)^4}$$

• interpretation: $\widehat{\mathcal{Z}}_s$ counts components of CHS field strength C_s

$$\mathcal{Z}_{-\,s} = \mathcal{Z}_s^{\rm off-sh.} \,, \qquad \mathcal{Z}_{+\,s} = \mathcal{Z}_s^{\rm e.o.m.} \,, \qquad \mathcal{Z}_s = \mathcal{Z}_s^{\rm off-sh.} - \mathcal{Z}_s^{\rm e.o.m.}$$

off-shell CHS fields have same symmetries and dimensions
as shadow operators: $\mathcal{Z}_{-\,s}$ counts off-shell shadow fields
$\mathcal{Z}_s$ counts physical CHS operators or "on-shell" shadow fields



current J_s, shadow $\widetilde{J}_s$
$\partial^s C_s, C_s$
free scalar CFT_4
conformal higher spin s CFT_4
$\mathcal{Z}_{+\,s}, \mathcal{Z}_{-\,s}$
massless higher spin s in AdS_5
$\ln\det_{(+)}, \ln\det_{(-)}$

Summing over all spins:

- CHS partition function on conformally-flat M^d is UV finite:

$$\sum_{s=0}^{\infty} \mathrm{a}_s\Big|_{\mathrm{reg.}} = 0$$

- CHS partition function on $S^1 \times S^{d-1}$ satisfies $\mathcal{Z}(q) = \mathcal{Z}(1/q)$, e.g.,

$$d = 4: \qquad \mathcal{Z}(q) = \sum_{s=0}^{\infty} \mathcal{Z}_s(q)\Big|_{\mathrm{reg.}} = -\frac{q^2\,(11 + 26q + 11q^2)}{6(1-q)^6}$$

- implies vanishing of associated Casimir or vacuum energy on S^{d-1}

$$\sum_{s=0}^{\infty} E_{c,s}\Big|_{\mathrm{reg.}} = 0$$

as in case of massless higher spin partition function in AdS_{d+1}

• conjecture:

all anomaly coefficients vanish in same regularization, e.g. in $d = 4$

$$\sum_{s=0}^{\infty} \mathrm{c}_s \Big|_{\mathrm{reg.}} = 0$$

requires understanding CHS partition function in Ricci-flat background

• may summation over spins help also with unitarity issue?

need to study CHS interactions and S-matrix