THE UNIVERSITY
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CENTRE FOR THEORETICAL PHYSICS · Higgs

Diagrammatic representation of the coproduct of one-loop Feynman diagrams

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Motivation – Feynman diagrams evaluating to polylogarithms

Discontinuities of Feynman diagrams have a diagrammatic representation as cuts.

[Landau ('59), Cutkosky ('60), t'Hooft & Veltman ('73), ...]

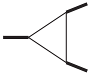
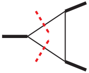
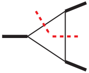
Discontinuities are naturally found within the coproduct of the Hopf algebra of multiple polylogarithms (MPLs).

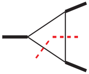
For Feynman integrals, coproduct entries corresponding to discontinuities have a diagrammatic representation as cuts

[SA, Britto, Duhr, Gardi, JHEP 1410 (2014) 125; SA, Britto, Grönqvist, arXiv:1504.00206 (to appear JHEP)]

Ex: 'first entry condition'

[Gaiotto, Maldacena, Sever, Vieira, JHEP 1112 (2011) 011]

$\Delta_{1,n-1}\Bigg($  $\Bigg) = \log\left(-p_1^2\right) \otimes$  $+ \log\left(-p_2^2\right) \otimes$ 

$+ \log\left(-p_3^2\right) \otimes$ 

The coproduct of the Hopf algebra of polylogarithms encodes a lot of the analytic information of these functions:

- discontinuities ;
- derivatives ;
- ...

Is there a completely diagrammatic representation of the coproduct of one-loop Feynman integrals?

i.e., is there an operator Δ that maps a graph F to two other graphs and is consistent with the coproduct of MPLs?

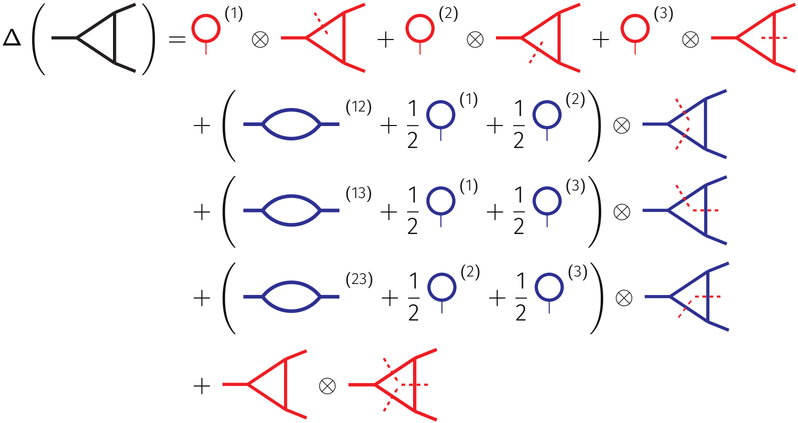
Outline

Diagrammatic operations on polygons

The diagrammatic coproduct and the coproduct of MPLs

Conclusion and outlook

Diagrammatic operations on polygons



- Odd number of cut edges, one graph on the left.
- Even number of cut edges, two types of graphs on the left.

Rules for diagrammatic coproduct

$$\Delta(F) = \sum_i L_i \otimes R_i$$

F graph with n edges, out of which c are cut.

$\Rightarrow R_i$ graph with n edges out of which m are cut, such that $c \leq m \neq 0$.

Case 1: m odd.

L_i is a graph with m edges obtained by pinching the uncut edges in R_i.

Case 2: m even.

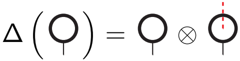
L_i is a sum of graphs:

+ the diagram with m edges obtained by pinching the uncut edges in R_i ;
+ $\frac{1}{2}$ times the graphs with $m-1$ edges obtained by pinching an extra edge.

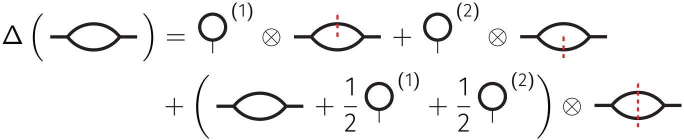
If F has cut edges they are never pinched.

DIAGRAMMATIC COPRODUCT OF UNCUT GRAPHS ($c = 0$)

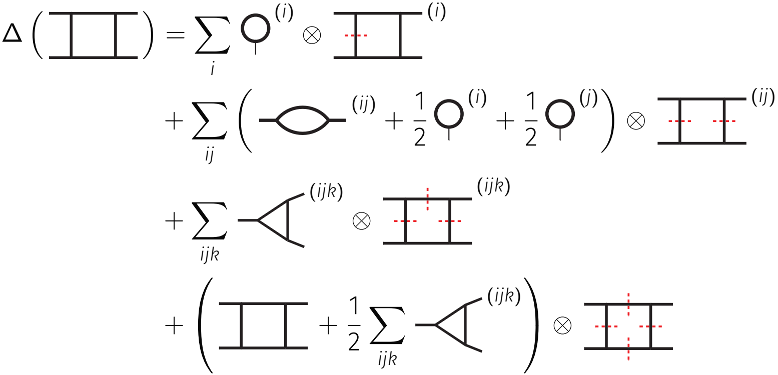
One edge ($n = 1$, $c = 0$) — **tadpole**:



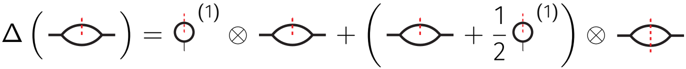
Two edges ($n = 2$, $c = 0$) — **bubble**:



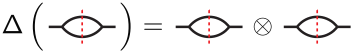
Four edges ($n = 4$, $c = 0$) — **box**:



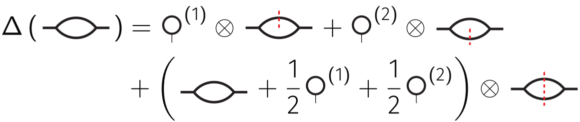
Two edges, one cut ($n = 2$, $c = 1$) — **single cut bubble**:



Two edges, two cuts ($n = 2$, $c = 2$) — **double cut bubble**:



Compare with uncut bubble:

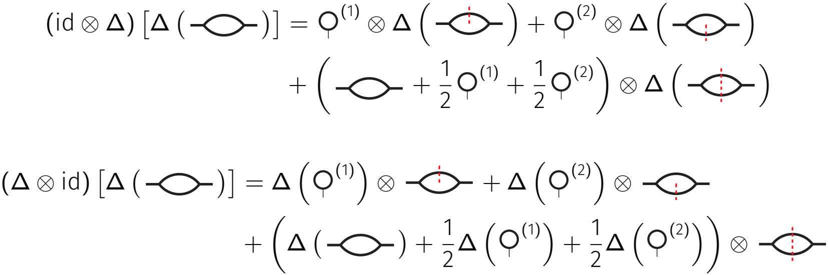


Δ is coassociative:

$$(\mathrm{id} \otimes \Delta)\, \Delta F = (\Delta \otimes \mathrm{id})\, \Delta F$$

[No proof, but checked up to 20 edges]

Example:



The diagrammatic coproduct and the coproduct of MPLs

Multiple Polylogarithms:

$$G\left(a_1,\ldots,a_n;z\right)=\int_0^z\frac{dt}{t-a_1}G\left(a_2,\ldots,a_n;t\right)\qquad a_i,z\in\mathbb{C}$$

A large class of Feynman diagrams can be written in terms of MPL.

$\mathbb{Q}$-vector space of MPL forms Hopf algebra (graded by weight) — $\mathcal{H}$

Equipped with a **coproduct** $\Delta:\mathcal{H}\longrightarrow\mathcal{H}\otimes\mathcal{H}$

Coassociativity

$$(\mathrm{id}\otimes\Delta)\,\Delta=(\Delta\otimes\mathrm{id})\,\Delta$$

Coproduct and discontinuities

$$\Delta \mathrm{Disc} = (\mathrm{Disc} \otimes \mathrm{id})\, \Delta$$

Discontinuities act on the **first entry** of the coproduct

Coproduct and differential operators

$$\Delta \frac{\partial}{\partial z} = \left(\mathrm{id} \otimes \frac{\partial}{\partial z}\right) \Delta$$

Differential operators act on the **last entry** of the coproduct

$$F = \frac{e^{\gamma_E \epsilon}}{\pi^{\frac{D}{2}}} \int d^D k \prod_{j=1}^{n} \frac{1}{q_j^2 - m_j^2 + i0}$$

$$q_j = \alpha_j k + \sum_{l=1}^{n} \beta_{jk} q_l, \quad \alpha_j, \beta_{jk} \in \{-1, 0, 1\}$$

We choose $D = d - 2\epsilon$ with $d \in \mathbb{N}$, even, such that $d - 2 < n \leq d$. E.g.:

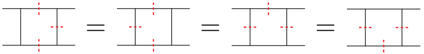
- tadpoles and bubbles: $D = 2 - 2\epsilon$;
- triangles and boxes: $D = 4 - 2\epsilon$;
- pentagons and hexagons: $D = 6 - 2\epsilon$;
- $\ldots$;

F evaluates to MPLs and is a pure function of weight $\frac{d}{2}$

(N.B.: we assume $w(\epsilon) = -1$)

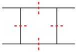
One, two and three propagator cuts as in **‘real kinematics’**:

- replace propagator by delta function, keep real integration contour ;
- triple cuts isolating a three-point vertex with massless particles vanish.

ex:  $= 0$

Four, five, . . . propagator cuts computed in **‘complex kinematics’**:

- compute residues, change integration contour ;
- cuts isolating a three-point vertex with massless particles don’t vanish.

ex:  $\neq 0$

Cuts of F evaluate to MPLs and are pure functions of weight $\frac{d}{2} - \left\lceil \frac{m}{2} \right\rceil$

Coproduct of MPLs and diagrammatic coproduct

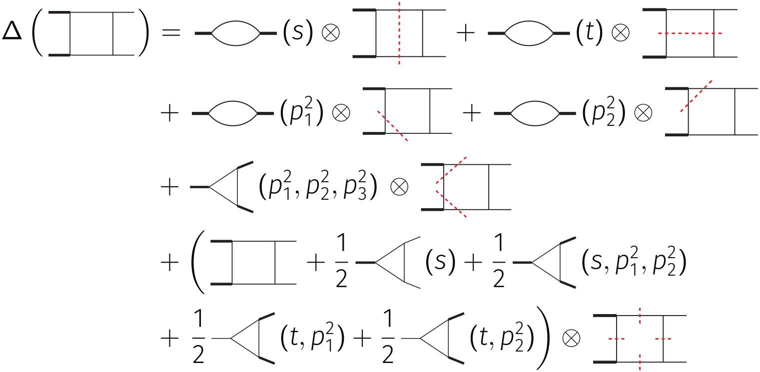
Use the coproduct of MPLs to check diagrammatic coproduct conjecture

Make the following identifications:

$$\text{Feynman diagram} \longleftrightarrow \text{MPLs it evaluates to}$$
$$\text{Cut Feynman diagram} \longleftrightarrow \text{MPLs it evaluates to}$$
$$\text{diagrammatic coproduct} \longleftrightarrow \text{coproduct of MPLs}$$

- Diagrams in dimensional regularisation $\Rightarrow$ relations between different weight MPLs, order by order in ϵ ;
- Diagrammatic rules $\Rightarrow$ relations between a priori unrelated diagrams ;
- Intricate interplay between ϵ-expansions required to cancel singularities of finite diagrams.

Example: two-mass-hard box $B(s, t; p_1^2, p_2^2)$



Checked all coproduct components up to weight 4 (i.e., ϵ^2).

Checks

Explicitly checked for several orders in ϵ for:

- **tadpole:** trivial ;
- **bubbles:** $\mathrm{Bub}(p^2)$, $\mathrm{Bub}(p^2; m^2)$ and $\mathrm{Bub}(p^2; m_1^2, m_2^2)$;
- **triangles:** several combinations of internal and external masse ;
- **box:** $B(s,t)$, $B(s,t,p_1^2)$, $B(s,t,p_1^2,p_3^2)$, $B(s,t,p_1^2,p_2^2)$, $B(s,t;m_{12}^2)$ and $B(s,t;m_{12}^2,m_{23}^2)$.

Consistency checks for:

- **box:** $B(s,t,p_1^2,p_2^2,p_3^2)$ and $B(s,t,p_1^2,p_2^2,p_3^2,p_4^2)$;
- **pentagon:** zero mass pentagon ;
- **hexagon:** zero mass hexagon.

Discontinuity operators act on first entry of the coproduct

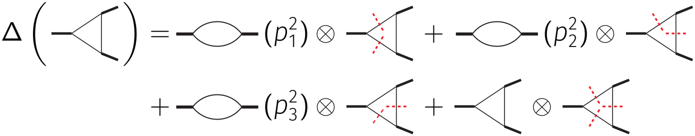
$$\Delta \mathrm{Disc} = (\mathrm{Disc} \otimes \mathrm{id})\, \Delta$$

First entries of coproduct of graph have same number of cut edges as graph

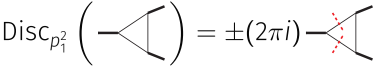
$\Rightarrow$They have the same discontinuity structure (Landau equations).

The graphical coproduct is consistent with the action of discontinuity operators

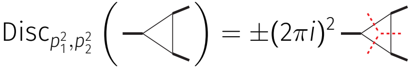
First entry condition — satisfied by construction by the diagrammatic conjecture of a Feynman diagram: first entry is always a Feynman diagram.



First entry condition:



Iterated discontinuities:



Differential operators act on last entry of the coproduct

$$\Delta \frac{\partial}{\partial z} = \left(\mathrm{id} \otimes \frac{\partial}{\partial z}\right) \Delta$$

Last entries of coproduct of graph have same number of edges as graph

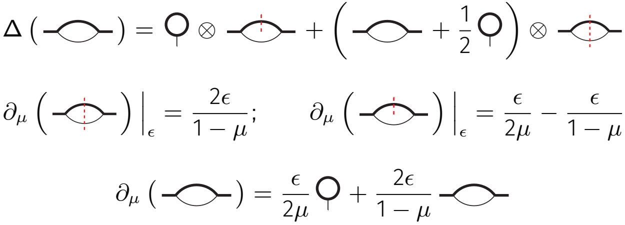
$\Rightarrow$They obey the same differential equations.

The graphical coproduct is consistent with the action of the differential operators

Reverse unitarity — cut diagrams obey same differential equation as uncut diagrams.

Diagrammatic coproduct predicts differential equations

Example: Differential equations without IBPs ($\mu = m^2/p^2$ and $\partial_\mu = \partial/\partial\mu$)



Same strategy can be used for cut graphs $\Rightarrow$ reverse unitarity

Coefficient of differential equations are derivatives of the weight one term in the ϵ-expansion **of cuts**

Conclusion and outlook

Conclusion

We conjecture and give evidence that:

The coproduct of all one-loop Feynman diagrams has a diagrammatic representation

We give explicit rules to construct the diagrammatic representation for any one-loop diagram.

Explicitly checked for several non-trivial examples.

Diagrammatic representation consistent with **differential equations** and **discontinuitities**.

Can our construction be generalised to two and more loops?

- What is a good basis of pure Feynman integral beyond one-loop?
- Which combinations of diagrams appear in the first entry?

Elliptic functions appear beyond one loop. Can our construction be generalised to diagrams that do not evaluate to MPLs?

- Can a coproduct be defined for elliptic functions?

Thank you!