JOHANNES GUTENBERG UNIVERSITÄT MAINZ

JG|U

On five-particle scattering amplitudes

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amplitudes 2015

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Applying N=4 sYM insights to QCD

- easier to obtain multi-leg/loop results in N=4 sYM

  simplicity allows to `see` things analytically

  sometimes leads to techniques that are universally applicable

- important example: connection between leading singularities of integrals and weight properties (``transcendentality``) of integrated answer

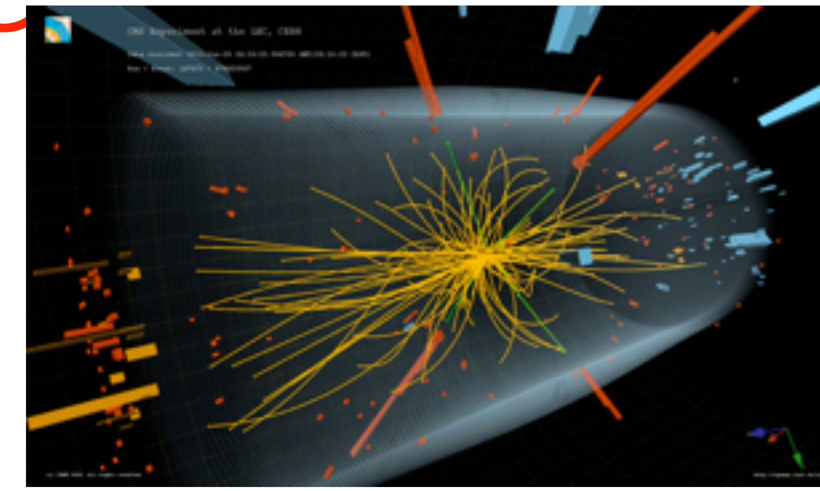
- these ideas have led to efficient methods for computing loop integrals in QCD

Outline of talk

- motivation for multi-leg computations
- technique
- analytic results for 2-loop 5-particle integrals
- applications to amplitudes

Experiment and theory

- The Higgs boson has been found at the LHC



Huge success both for theory and experiment

- What's next?

  - determine properties of the new particle

  - search for deviations from the standard model

- Increasing experimental precision puts new challenges to theory community

Les Houches wishlist

NNLO QCD and NLO EW Les Houches Wishlist

Wishlist part 1 - Higgs (V=W,Z)

Process	known	desired	motivation
H	d\sigma @ NNLO QCD d\sigma @ NLO EW finite quark mass effects @ NLO	d\sigma @ NNNLO QCD + NLO EW MC@NNLO finite quark mass effects @ NNLO	H branching ratios and couplings
H+j	d\sigma @ NNLO QCD (g only) d\sigma @ NLO EW	d\sigma @ NNLO QCD + NLO EW finite quark mass effects @ NLO	H p_T
H+2j	\sigma_tot(VBF) @ NNLO(DIS) QCD d\sigma(gg) @ NLO QCD d\sigma(VBF) @ NLO EW	d\sigma @ NNLO QCD + NLO EW	H couplings
H+V	d\sigma(V decays) @ NNLO QCD d\sigma @ NLO EW	with H→bb @ same accuracy	H couplings
t\bar tH	d\sigma(stable tops) @ NLO QCD	d\sigma(NWA top decays) @ NLO QCD + NLO EW	top Yukawa coupling
HH	d\sigma @ LO QCD finite quark mass effects d\sigma @ NLO QCD large m_t limit	d\sigma @ NLO QCD finite quark mass effects d\sigma @ NNLO QCD	Higgs self coupling

Wishlist part 2 - jets and heavy quarks

Process	known	desired	motivation
t\bar t	\sigma_tot @ NNLO QCD d\sigma(top decays) @ NLO QCD d\sigma(stable tops) @ NLO EW	d\sigma(top decays) @ NNLO QCD + NLO EW	precision top/QCD, gluon PDF effect of extra radiation at high rapidity top asymmetries
t\bar t+j	d\sigma(NWA top decays) @ NLO QCD	d\sigma(NWA top decays) @ NLO QCD + NLO EW	precision top/QCD, top asymmetries
single-top	d\sigma(NWA top decays) @ NLO QCD	d\sigma(NWA top decays) @ NNLO QCD (t channel)	precision top/QCD, V_tb
dijet	d\sigma @ NNLO	d\sigma @ NNLO QCD +	Obs.: incl. jets, dijet mass

Les Houches wishlist

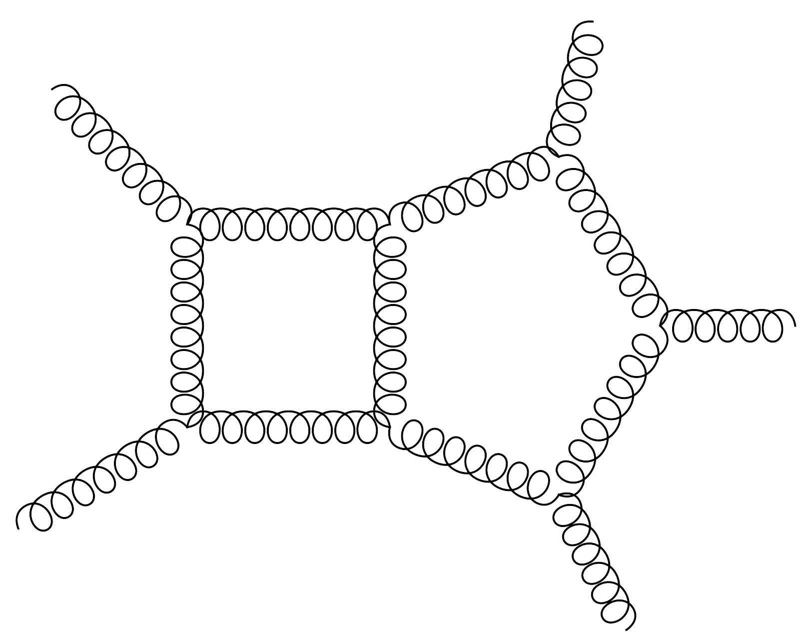
	QCD (g only) d\sigma @ NLO weak	NLO EW	–> PDF fits (gluon at high x) –> alpha_s CMS x sections: http://arxiv.org/abs/1212.6660 [http://arxiv.org/abs/1212.6660]
3j	d\sigma @ NLO QCD	d\sigma @ NNLO QCD + NLO EW	Obs.: R3/2 or similar –> alpha_s at high pT dom. uncertainty: scales see http://arxiv.org/abs/1304.7498 [http://arxiv.org/abs/1304.7498] (CMS)
\gamma+j	d\sigma @ NLO QCD d\sigma @ NLO EW	d\sigma @ NNLO QCD + NLO EW	gluon PDF, \gamma+b for bottom PDF

Wishlist part 3 - EW gauge bosons (V=W,Z)

Process	known	desired	motivation
V	d\sigma(lept. V decay) @ NNLO QCD + EW	d\sigma(lept. V decay) @ NNNLO QCD + NLO EW MC@NNLO	precision EW, PDFs
V+j	d\sigma(lept. V decay) @ NLO QCD + EW	d\sigma(lept. V decay) @ NNLO QCD + NLO EW	Z+j for gluon PDF W+c for strange PDF
V+jj	d\sigma(lept. V decay) @ NLO QCD	d\sigma(lept. V decay) @ NNLO QCD + NLO EW	study of systematics of H+jj final state
VV'	d\sigma(V decays) @ NLO QCD d\sigma(stable V) @ NLO EW	d\sigma(V decays) @ NNLO QCD + NLO EW	bkg H → VV TGCs
gg → VV	d\sigma(V decays) @ LO	d\sigma(V decays) @ NLO QCD	bkg to H→VV
V\gamma	d\sigma(V decay) @ NLO QCD d\sigma(PA, V decay) @ NLO EW	d\sigma(V decay) @ NNLO QCD + NLO EW	TGCs
Vb\bar b	d\sigma(lept. V decay) @ NLO QCD massive b	d\sigma(lept. V decay) @ NNLO QCD massless b	bgk to VH(→bb)
VV'\gamma	d\sigma(V decays) @ NLO QCD	d\sigma(V decays) @ NLO QCD + NLO EW	QGCs

Challenges for calculations in QFT

- Many processes involve several variables (masses, scattering angles), e.g. 2->3 processes
- One of the main obstacles: often, no analytic expressions for the Feynman integrals are available
- In this talk, I will focus on virtual contributions and present tools for the evaluation of the Feynman integrals



Connection between integrands and integrated functions

$$\int d^D k_1 \dots d^D k_L \frac{1}{\prod_i P_i} \to \sum \text{(special functions)}$$

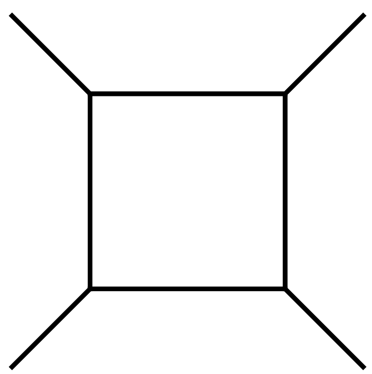
- I will review how ``looking at`` the LHS can be used to learn a lot about the RHS

- this talk: RHS evaluates to multiple polylogarithms

Analyzing loop integrands: maximal cuts, leading singularities

- maximal cuts

$$D_1 = k^2 \quad D_2 = (k+p_1)^2 \quad D_3 = (k+p_1+p_2)^2 \quad D_4 = (k+p_1+p_2+p_3)^2$$



$$= \int d^4k\,\delta(D_1)\delta(D_2)\delta(D_3)\delta(D_4) \sim \frac{1}{st}$$

note: there are two solutions that localize the loop momentum (related by complex conjugation); these correspond to the leading singularities [Cachazo; Cachazo, Skinner]

- at higher loops, maximal cuts do not completely localize the loop momenta; leading singularities `cut` also Jacobian factors

Pentagon example

- one-loop pentagon integrals

$$D_1 = k^2 \quad D_2 = (k+p_1)^2 \quad D_3 = (k+p_1+p_2)^2 \quad D_4 = (k+p_1+p_2+p_3)^2 \quad D_5 = (k-p_5)^2$$

- now there are five different maximal cuts we can take
- leading singularities of the scalar pentagon integral cannot all be normalized to one

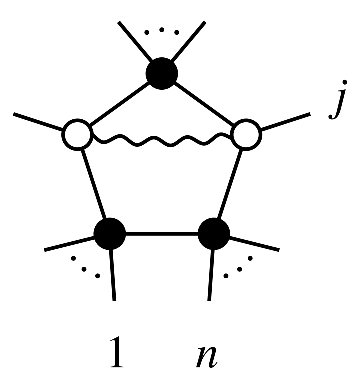
- consider a pentagon integral with numerator:

$$\int d^4k \frac{N(k)}{D_1 D_2 D_3 D_4 D_5}$$

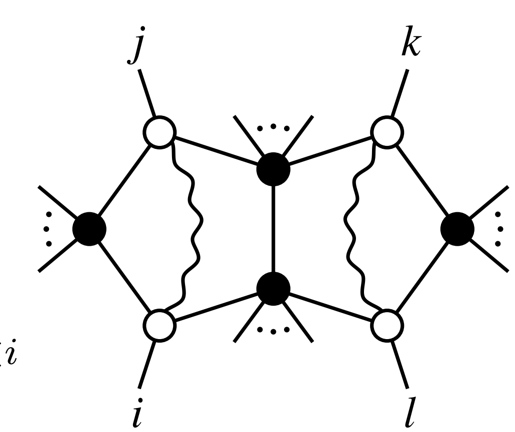
- can choose numerator such that integral has constant leading singularities

- Such integrals `naturally` appear in N=4 SYM [Arkani-Hamed et al, 2010]

$$\mathcal{A}_{\text{MHV}}^{1-\text{loop}} = \sum_{i<j}$$

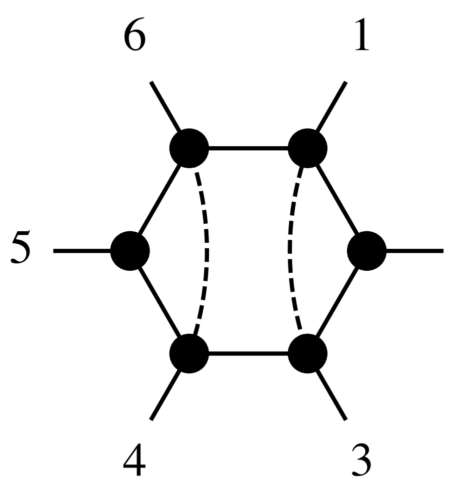


$$\mathcal{A}_{\text{MHV}}^{2-\text{loop}} = \frac{1}{2} \sum_{i<j<k<l<i}$$



Leading singularities, weight conjecture

• observation: these integrals have homogeneous logarithmic weight (`transcendentality`); e.g.,



$$= \mathrm{Li}_2(1-u_1) + \mathrm{Li}_2(1-u_2) + \mathrm{Li}_2(1-u_3) + \log(u_3)\log(u_1) - \frac{\pi^2}{3}$$

assign log weight: $w(\log) = 1 \quad w(\mathrm{Li}_n) = n \qquad w(\pi) = 1$

$$w(ab) = w(a) + w(b)$$

function has uniform weight 2 and kinematic-independent prefactors

• weight conjecture

[Arkani-Hamed, Bourjaily, Cachzao, Trnka, 2010]
[Arkani-Hamed et al, 2012]

integrals with constant leading singularities should have uniform weight

• as we will see, differential equations can shed more light on the weight properties

[JMH, 2013]

Example: choice of integral basis

three-loop N=4 SYM form factor

$$
\begin{aligned}
F_S^{(3)} = R_\epsilon^3 \Bigg[&+\frac{(3D-14)^2}{(D-4)(5D-22)} A_{9,1} - \frac{2(3D-14)}{5D-22} A_{9,2} - \frac{4(2D-9)(3D-14)}{(D-4)(5D-22)} A_{8,1} \\
&- \frac{20(3D-13)(D-3)}{(D-4)(2D-9)} A_{7,1} - \frac{40(D-3)}{D-4} A_{7,2} + \frac{8(D-4)}{(2D-9)(5D-22)} A_{7,3} \\
&- \frac{16(3D-13)(3D-11)}{(2D-9)(5D-22)} A_{7,4} - \frac{16(3D-13)(3D-11)}{(2D-9)(5D-22)} A_{7,5} \\
&- \frac{128(2D-7)(D-3)^2}{3(D-4)(3D-14)(5D-22)} A_{6,1} \\
&- \frac{16(2D-7)(5D-18)\left(52D^2-485D+1128\right)}{9(D-4)^2(2D-9)(5D-22)} A_{6,2} \\
&- \frac{16(2D-7)(3D-14)(3D-10)(D-3)}{(D-4)^3(5D-22)} A_{6,3} \\
&- \frac{128(2D-7)(3D-8)\left(91D^2-821D+1851\right)(D-3)^2}{3(D-4)^4(2D-9)(5D-22)} A_{5,1} \\
&- \frac{128(2D-7)\left(1497D^3-20423D^2+92824D-140556\right)(D-3)^3}{9(D-4)^4(2D-9)(3D-14)(5D-22)} A_{5,2} \\
&+ \frac{4(D-3)}{D-4} B_{8,1} + \frac{64(D-3)^3}{(D-4)^3} B_{6,1} + \frac{48(3D-10)(D-3)^2}{(D-4)^3} B_{6,2} \\
&- \frac{16(3D-10)(3D-8)\left(144D^2-1285D+2866\right)(D-3)^2}{(D-4)^4(2D-9)(5D-22)} B_{5,1} \\
&+ \frac{128(2D-7)\left(177D^2-1584D+3542\right)(D-3)^3}{3(D-4)^4(2D-9)(5D-22)} B_{5,2} \\
&+ \frac{64(2D-5)(3D-8)(D-3)}{9(D-4)^5(2D-9)(3D-14)(5D-22)} \\
&\quad \times \left(2502D^5-51273D^4+419539D^3-1713688D^2+3495112D-2848104\right) B_{4,1} \\
&+ \frac{4(D-3)}{D-4} C_{8,1} + \frac{48(3D-10)(D-3)^2}{(D-4)^3} C_{6,1} \Bigg] \, . \qquad \text{(B.1)}
\end{aligned}
$$

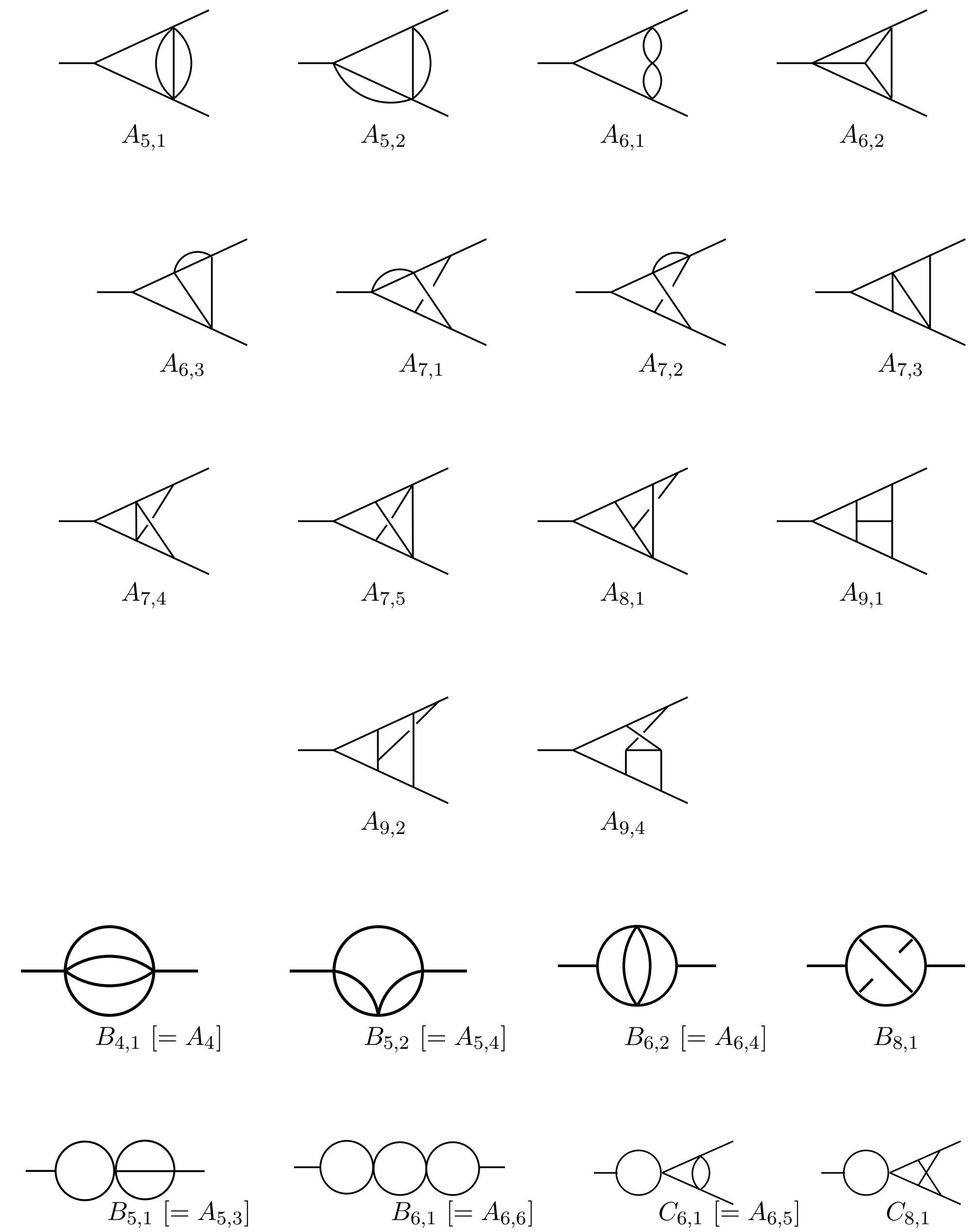


Figure 1: Master integrals for the three-loop form factors. Labels in brackets indicate the naming convention of Ref. [25].

Gehrmann, J.M.H., Huber (2011)

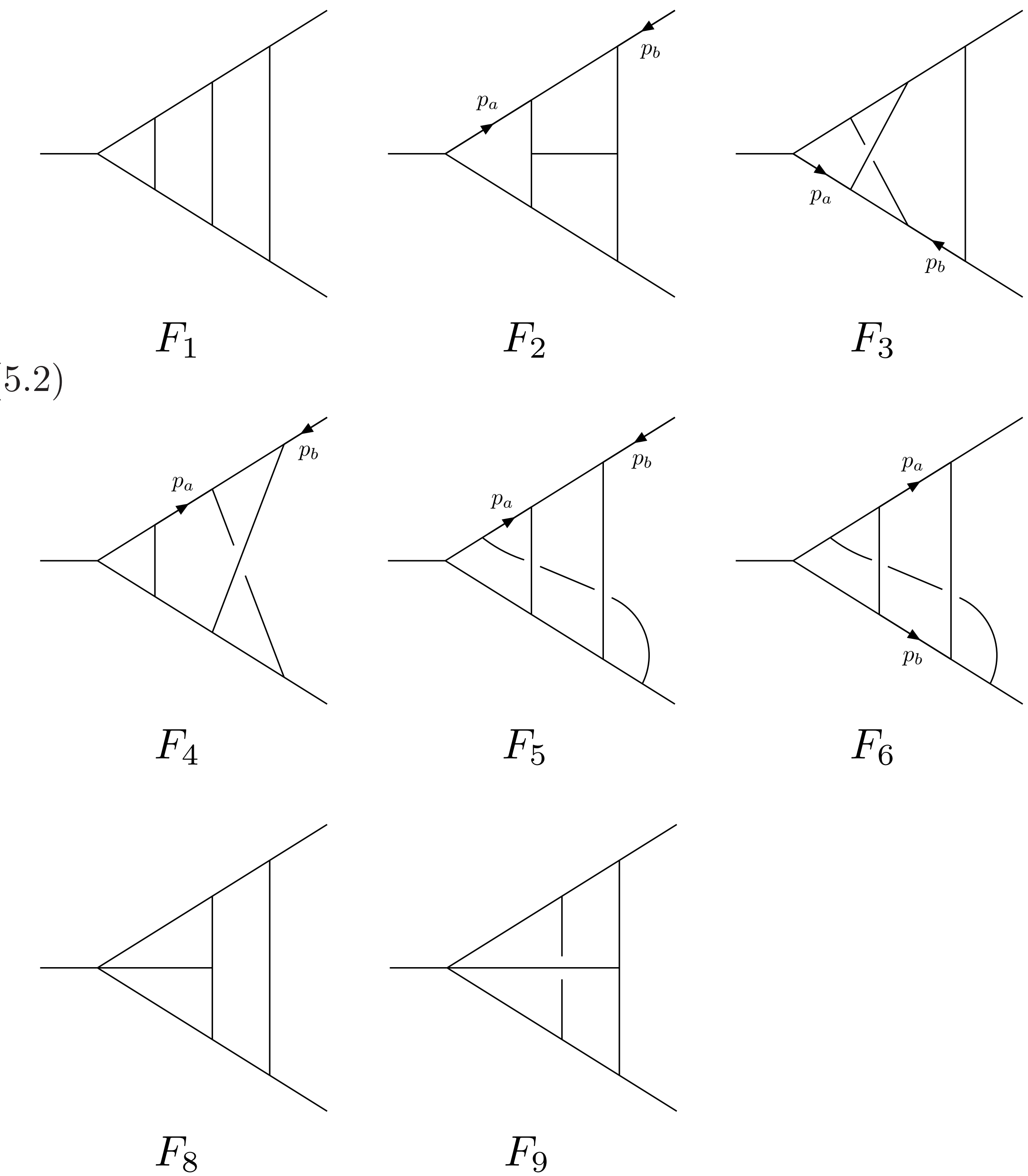
Gehrmann, Glover, Huber, Ikizlerli, Studerus;
Lee, Smirnov & Smirnov

Example: choice of integral basis

three-loop N=4 SYM form factor

$$F_S^{(3)} = R_\epsilon^3 \cdot [8\,F_1^{\rm exp} - 2\,F_2^{\rm exp} + 4\,F_3^{\rm exp} + 4\,F_4^{\rm exp} - 4\,F_5^{\rm exp} - 4\,F_6^{\rm exp} - 4\,F_8^{\rm exp} + 2\,F_9^{\rm exp}]$$

$$\begin{aligned} F_S^{(3)} &= R_\epsilon^3 \cdot [8\,F_1^{\rm exp} - 2\,F_2^{\rm exp} + 4\,F_3^{\rm exp} + 4\,F_4^{\rm exp} - 4\,F_5^{\rm exp} - 4\,F_6^{\rm exp} - 4\,F_8^{\rm exp} + 2\,F_9^{\rm exp}] \\ &= -\frac{1}{6\epsilon^6} + \frac{11\zeta_3}{12\epsilon^3} + \frac{247\pi^4}{25920\epsilon^2} + \frac{1}{\epsilon}\left(-\frac{85\pi^2\zeta_3}{432} - \frac{439\zeta_5}{60}\right) \\ &\quad - \frac{883\zeta_3^2}{36} - \frac{22523\pi^6}{466560} + \epsilon\left(-\frac{47803\pi^4\zeta_3}{51840} + \frac{2449\pi^2\zeta_5}{432} - \frac{385579\zeta_7}{1008}\right) \\ &\quad + \epsilon^2\left(\frac{1549}{45}\zeta_{5,3} - \frac{22499\zeta_3\zeta_5}{30} + \frac{496\pi^2\zeta_3^2}{27} - \frac{1183759981\pi^8}{7838208000}\right) + \mathcal{O}(\epsilon^3)\,. \end{aligned} \qquad (5.2)$$



- each integral has uniform (and maximal) ``transcendentality''
  T[Zeta[n]] = n
  T[eps^-n] = n
  T[A B] = T[A] + T[B]

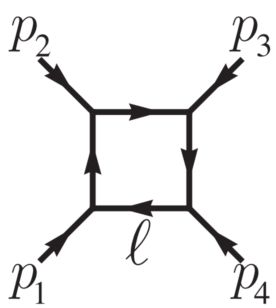
- for theories with less susy, other integrals also needed

Gehrmann, J.M.H., Huber (2011)

`d-log forms`

- observation: sometimes, loop integrand can be rewritten in suggestive form

$$\mathcal{A}_4^{\ell=0} \times$$



$$= \mathcal{A}_4^{\ell=0} \times \int \frac{d^4\ell \;\; (p_1+p_2)^2 (p_1+p_3)^2}{\ell^2 (\ell+p_1)^2 (\ell+p_1+p_2)^2 (\ell-p_4)^2}$$

[Arkani-Hamed et al, 2012]
[Caron-Huot, talk at Trento, 2012]
[Lipstein and Mason, 2013-2014]

[also see recent work, on non-planar cases: Arkani-Hamed et al, 2014; Bern et al., 2015]

$$\frac{d^4\ell \;\; (p_1+p_2)^2 (p_1+p_3)^2}{\ell^2 (\ell+p_1)^2 (\ell+p_1+p_2)^2 (\ell-p_4)^2}$$
$$= d\log\left(\frac{\ell^2}{(\ell-\ell^*)^2}\right) d\log\left(\frac{(\ell+p_1)^2}{(\ell-\ell^*)^2}\right) d\log\left(\frac{(\ell+p_1+p_2)^2}{(\ell-\ell^*)^2}\right) d\log\left(\frac{(\ell-p_4)^2}{(\ell-\ell^*)^2}\right)$$

- `d-log forms`: make leading singularities obvious

Summary integrand investigations

- leading singularities - maximal weight conjecture
- allows to systematically construct uniform weight integrals
- works both in planar/non-planar case
- assign weight -1 to 1/eps to extend to dimensionally regulated integrals

Next step:

[JMH, 2013]

- prove uniform weight properties using differential equations
- extend to uniform but non-maximal weights

Differential equations (DE) technique

• idea: differentiate Feynman integral w.r.t. external variables, e.g. s, t, masses

Some general facts:

• a given Feynman integral f satisfies an n-th order DE

• equivalently described by a system of n first-order equations for $\vec{f}$

$$\partial_x \vec{f}(x,\epsilon) = A(x,\epsilon)\vec{f}(x,\epsilon)$$

since they come from Feynman integrals, they can only have regular singularities. Constrains matrix $A(x,\epsilon)$

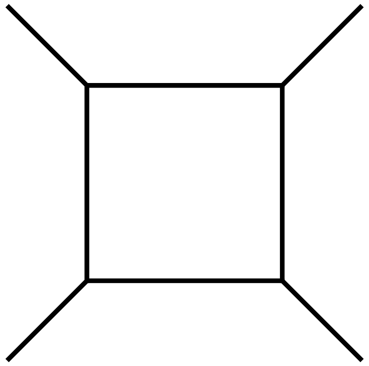
Long and successful history:

[Kotikov, 1991] [Remiddi, 1997] [Gehrmann, Remiddi, 2000] [...]

New idea: use integrals with constants leading singularities as basis for DE system [JMH, 2013]

Example: one-loop four-point integral

- choose basis according to [JMH, 2013]
- differential equations $x = t/s$ $D = 4 - 2\epsilon$



$$\partial_x \vec{f}(x,\epsilon) = \epsilon \left[\frac{a}{x} + \frac{b}{1+x} \right] \vec{f}(x,\epsilon)$$

$$a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & -1 \end{pmatrix} \quad b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 1 \end{pmatrix}$$

  - make singularities manifest
  - asymptotic behavior governed by matrices a, b

- Solution: expand to any order in ϵ

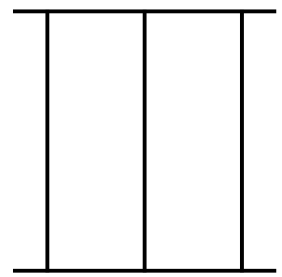
$$\vec{f} = \epsilon^{-p} \sum_{k \geq 0} \epsilon^k \vec{f}^{(k)}$$

$\vec{f}^{(k)}$ is k-fold iterated integral (uniform weight k)

Technique applies to QCD integrals

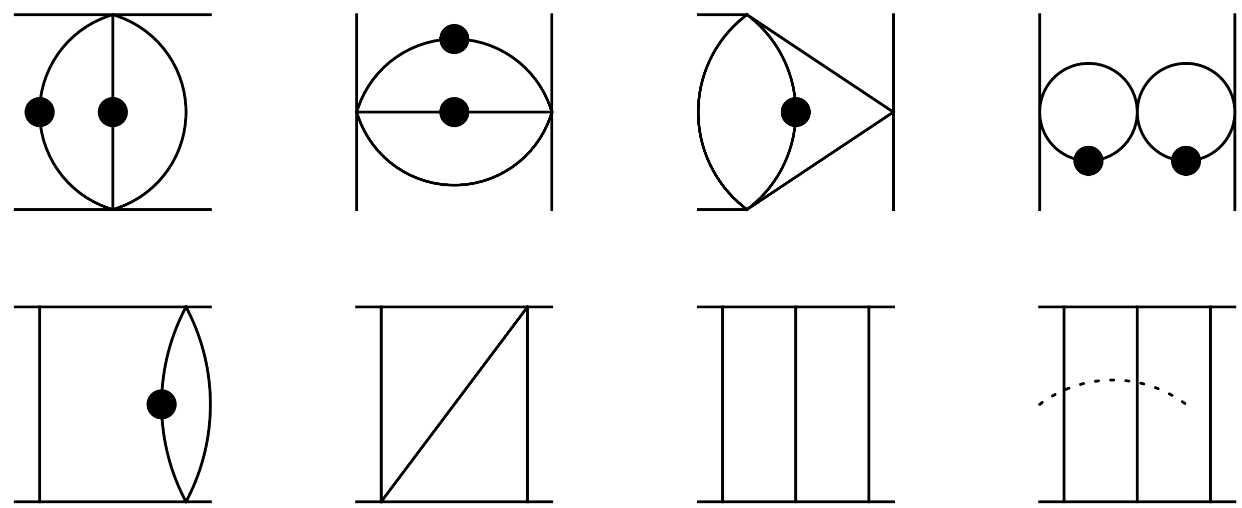
- system of DE for `N=4` integral contains QCD integrals

$$f(x,\epsilon) =$$



$$x = t/s$$
$$D = 4 - 2\epsilon$$

$$\vec{f}(x,\epsilon) =$$



$$\partial_x \vec{f}(x,\epsilon) = \epsilon \left[\frac{a}{x} + \frac{b}{1+x} \right] \vec{f}(x,\epsilon)$$

Multi-variable case and the alphabet

- Natural generalization to multi-variable case

$$d\vec{f}(\vec{x};\epsilon) = \epsilon\, d\left[\sum_k A_k \log \alpha_k(\vec{x})\right] \vec{f}(\vec{x};\epsilon)$$

A_k → constant matrices; $\alpha_k(\vec{x})$ → letters (alphabet)

- Examples of alphabets:

4-point on-shell $\quad \alpha = \{x, 1+x\}$

two-variable example (from 1-loop Bhabha scattering): $\quad \alpha = \{x\,, 1 \pm x\,, y\,, 1 \pm y\,, x+y\,, 1+xy\}$

[J.M.H., Smirnov]

``hexagon functions`` in N=4 SYM

$$\alpha = \{x, y, z, 1-x, 1-y, 1-z, \\ 1-xy, 1-xz, 1-yz, 1-xyz\}$$

[Goncharov, Spradlin, Vergu, Volovich] [Caron-Huot, He]

[Dixon, Drummond, J.M.H.] [Dixon et al.]

- Matrices and letters determine solution
- Immediate to solve in terms of iterated integrals

Physics applications of new ideas for DE

[JMH, 2013]

- vector boson production

  VV' planar and non-planar NNLO integrals

  [Caola, JMH, Melnikov, Smirnov, Smirnov, 2014]

  equal mass case: [Gehrmann, von Manteuffel, Tancredi, Weihs, 2014]

essential ingredient for ZZ and W+W- production at NNLO

[Cascioli et al, 2014] [Gehrmann et al, 2014]

- 3-loop QCD cusp anomalous dimension (determines IR structure of planar QCD scattering amplitudes)

  [Grozin, JMH, Korchemsky, Marquard, 2014]

- B physics [Bell, Huber, 2014] [Huber, Kraenkl, 2015]

- integrals for H production in gluon fusion at N3LO

  [Dulat, Mistlberger, 2014] [Hoeschele, Hoff, Ueda, 2014]

  physics result: [Anastasiou et al, 2014]

  …

Beyond iterated integrals

- Note: functions beyond iterated integrals can appear in Feynman integrals

- One such class are elliptic functions, needed e.g. in top quark physics [Czakon and Mitov, 2010]

- A generalization of the above methods is required here

For more information, cf. recent lecture notes:
JMH, arXiv:1412.2296 [hep-ph]

New results for penta-box integrals and five-particle amplitudes at NNLO

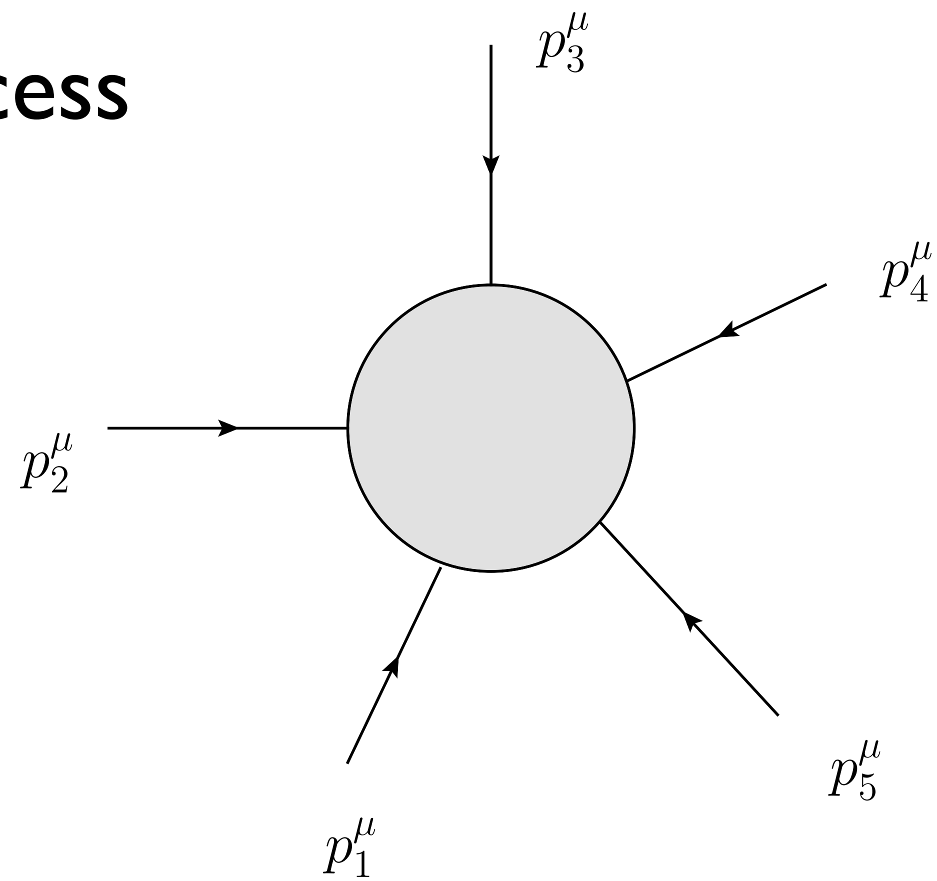
[Gehrmann, JMH, Lo Presti]

[related work with Frellesvig on one-loop pentagon integrals]

five-point kinematics

- massless 5->0 process

$$s_{ij} = (p_i + p_j)^2$$

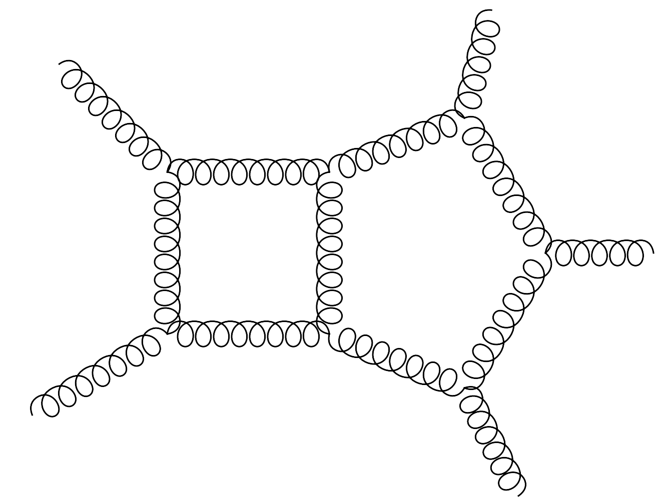


- independent variables $\vec{x} = \{s_{12}\,, s_{23}\,, s_{34}\,, s_{45}\,, s_{51}\}$

- convenient to start with non-physical region where all planar integrals are real-valued $s_{i,i+1} < 0$

- other kinematic regions can be reached by analytic continuation

differential equations for penta-box integrals



- 61 planar master integrals

$$d\vec{f}(\vec{x};\epsilon) = \epsilon\, d\left[\sum_k A_k \log \alpha_k(\vec{x})\right] \vec{f}(\vec{x};\epsilon)$$

- integral basis chosen following [JMH, 2013]

$$\vec{x} = \{s_{12}\,, s_{23}\,, s_{34}\,, s_{45}\,, s_{51}\}$$

- alphabet of 24 letters $\alpha_k(\vec{x})$ e.g.

$$s_{12} \qquad s_{12} - s_{34}$$

$$s_{12} + s_{23} \qquad s_{12} - s_{34} + s_{51}$$

$$(s_{23} - s_{51})\sqrt{\Delta} + s_{12}s_{23}^2 - s_{34}s_{23}^2 + s_{34}s_{45}s_{23} - 2s_{12}s_{51}s_{23}$$
$$+ s_{34}s_{51}s_{23} + s_{45}s_{51}s_{23} + s_{12}s_{51}^2 - s_{45}s_{51}^2 + s_{34}s_{45}s_{51}$$

Gram determinant Δ

boundary conditions (1)

- the boundary conditions can be obtained from physical conditions
- no singularities in non-physical region $s_{i,i+1} < 0$
- this means that certain singularities are spurious (on the first sheet of the multivalued functions), e.g. at

$$s_{12} = s_{34}$$
$$s_{12} + s_{51} = s_{34}$$

- similarly, no branch cuts should start at $\Delta = 0$
- these conditions fix everything except trivial single-scale integrals that are evaluated in terms of gamma functions

boundary conditions (2)

- boundary values at symmetric point

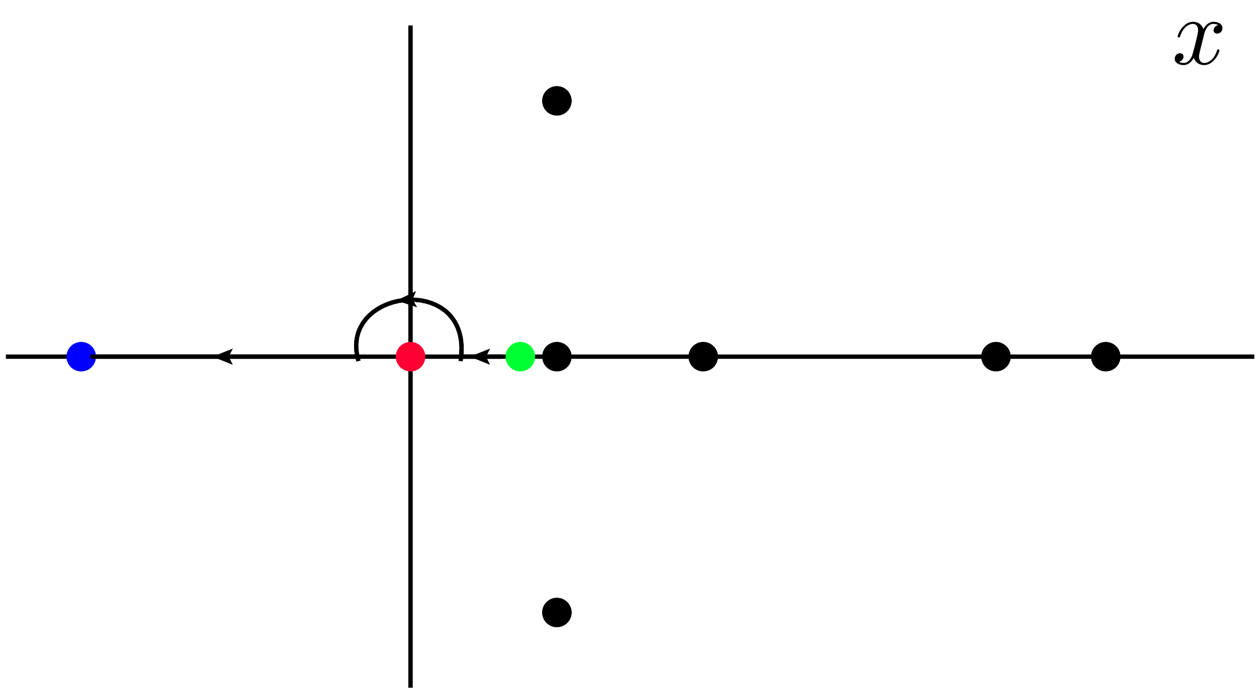
$$s_{12} = -\frac{x}{(1-x)^2}\,, \quad s_{23} = -1\,, \quad s_{34} = -1\,, \quad s_{45} = -1\,, \quad s_{51} = -1$$

- reduced alphabet (no square root)

$$\{x+1, x, x-\frac{1}{2}, x-1, x-2, 1-3x+x^2, 1-x+x^2\}$$

$$s_{12} = -1 \quad \longleftrightarrow \quad 1-3x+x^2 = 0$$

$$\Delta = 0 \quad \longleftrightarrow \quad x = -1$$



analytic solution

- we have

$$d\vec{f}(\vec{x},\epsilon) = \epsilon\, d\tilde{A}\, \vec{f}(\vec{x},\epsilon) \qquad \tilde{A} = \sum_k A_k \alpha_k(\vec{x})$$

- solution in terms of iterated integrals

[cf. Panzer's overview lecture]

$$\vec{f}(\vec{x},\epsilon) = \mathbb{P}\exp\left[\epsilon \int_\gamma d\,\tilde{A}\right] \vec{f}(\vec{x}_0,\epsilon)$$

$$\gamma : [0,1] \longrightarrow \mathcal{M}$$

$$\gamma(0) = \vec{x}_0 \qquad \gamma(1) = \vec{x}$$

- can be written in terms of Goncharov polylogarithms (for a convenient choice of γ)

- Note: knows about all ``symbol`` simplifications, but has exact information about boundary values

application to five-particle amplitudes

- five-particle scattering amplitudes were conjectured to have the following form (in modern language) [Bern, Dixon, Smirnov, 2003]

$$\log M_5 = \sum_{L\geq 1} a^L \left[-\frac{\gamma^{(L)}}{8(L\epsilon)^2} - \frac{\mathcal{G}_0^{(L)}}{4L\epsilon} + f^{(L)} \right] \sum_{i=1}^{5} \left(\frac{\mu^2}{s_{i,i+1}} \right)^{L\epsilon}$$

$$+ \frac{\gamma(a)}{4} F_n^{(1)}(s_{ij}) + C(a) + \mathcal{O}(\epsilon)$$

- This is in part due to the infrared structure of amplitudes
- The BDS conjecture fixes the finite part; it is now understood to follow from dual conformal symmetry

[Drummond, JMH, Korchemsky, Sokatchev, 2008]

- previously, this formula had been tested numerically

[Cachazo, Spradlin, Volovich, 2006] (parity-even part)
[Bern, Czakon, Kosower, Roiban, Smirnov, 2006]

application to five-particle amplitudes

$$\log M_5 = \sum_{L\geq 1} a^L \left[-\frac{\gamma^{(L)}}{8(L\epsilon)^2} - \frac{\mathcal{G}_0^{(L)}}{4L\epsilon} + f^{(L)} \right] \sum_{i=1}^{5} \left(\frac{\mu^2}{s_{i,i+1}} \right)^{L\epsilon}$$

$$+ \frac{\gamma(a)}{4} F_n^{(1)}(s_{ij}) + C(a) + \mathcal{O}(\epsilon)$$

- all ingredients now known analytically
- we verified the parity-even part of it using our analytic results

$$M_5^{(2)} = \sum_{\text{cyclic}} \left[\epsilon^2 \left(f_{60} + f_{54} + f_{52} \right) \right]$$

$$f_{60}^{(0)} = -3C[]$$

$$f_{60}^{(1)} = 2C[5] + C[7] + C[8] + C[10] + C[11] + 3C[12] - 4C[14] - 3C[16] + 2C[21]$$

$$\ldots$$

- reproduces everything, including constants

Summary

- unitarity-based methods for determining integrands complemented with a new method for evaluating the integrals
- both rely on analyzing the integrand's singularity structure
- DE method particularly useful for problems with many scales
- new result: all planar on-shell five-particle two-loop integrals

Outlook

- opens the door for applications to 2->3 amplitudes
- can be used to compute QCD +++++ amplitude

[Badger, Frellesvig, Zhang, 2013]

- non-planar integrals
- extension to Higgs plus jet integrals

Thank you!

Extra slides

Algebraic approach to differential equations

- The leading singularities approach allows to find a canonical form of the differential equations in an efficient way

- The exist also approaches (mostly) ignoring the Feynman integral origin, and working directly at the level of the DE

- Differential equations for Feynman integrals only have regular singularities (Fuchsian differential equations)

- Algorithms exist to make this manifest [Moser, 1960; Barkatou]

- Recently proposed to apply this to obtain canonical form [JMH, 2014]

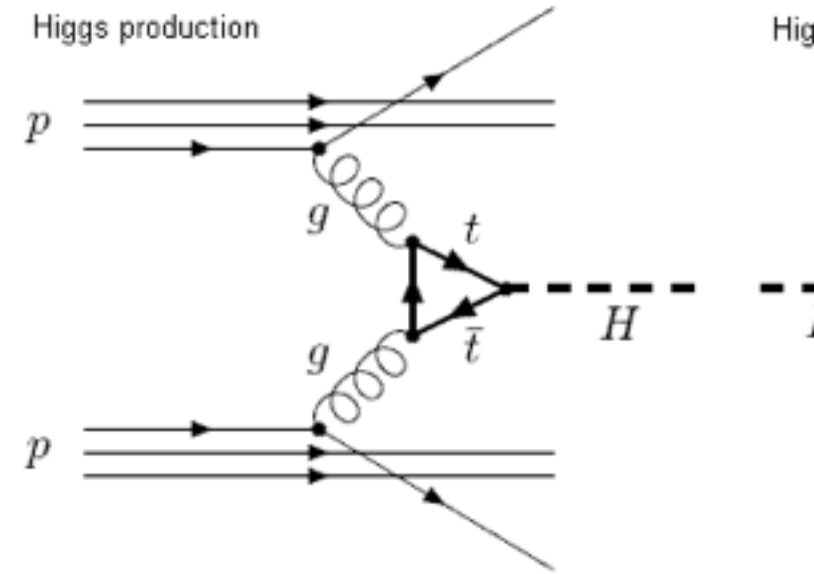
- Implementation (with improvements) [Lee, 2014]

The alphabet and perfect bricks (1)

Can we parametrize variables such that alphabet is rational?

Not essential, but nice feature.

- Example: Higgs production



encounter $\sqrt{1 - 4m^2/s}$

choose $-m^2/s = x/(1-x)^2$

$\alpha = \{x, 1-x, 1+x\}$ (to two loops)

Note: this is a purely kinematical question. Independent of basis choice.

- Related to diophantine equations

e.g. find rational solutions to equations such as

$$1 + 4\,a = b^2$$

here we found a 1-parameter solution

$$a = \frac{x}{(1-x)^2} \qquad b = \frac{1+x}{1-x}$$

The alphabet and perfect bricks (2)

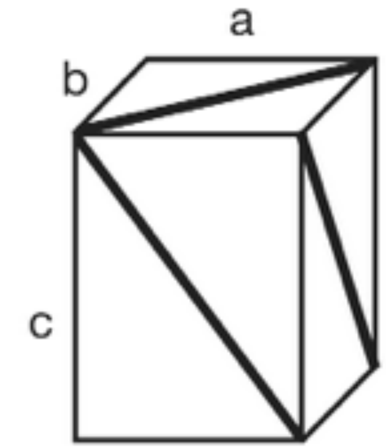
- Classic example: Euler brick problem

Find a brick with sides a, b, c and diagonals d, e, f integers

smallest solution (P. Halcke):

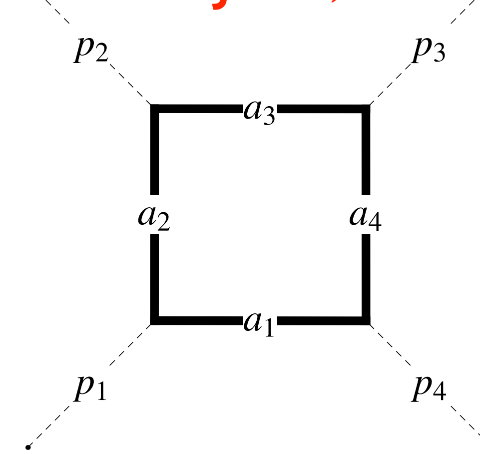
(a,b,c)=(44,117,240)

$$a^2 + b^2 = d^2\,,\quad a^2 + c^2 = e^2\,,\quad b^2 + c^2 = f^2\,.$$



Perfect cuboid (add eq. $a^2 + b^2 + c^2 = g^2$): open problem in mathematics!

- Similar equations for particle kinematics [Caron-Huot JMH, 2014]



e.g encountered in 4-d light-by-light scattering

$$u = -4m^2/s \qquad v = -4m^2/t$$

$$\beta_u = \sqrt{1+u},\ \beta_v = \sqrt{1+v},\ \beta_{uv} = \sqrt{1+u+v}$$

Need two-parameter solution to

$$\beta_u^2 + \beta_v^2 = \beta_{uv}^2 + 1$$

e.g. $$\beta_u = \frac{1-wz}{w-z}\,,\quad \beta_v = \frac{w+z}{w-z}\,,\quad \beta_{uv} = \frac{1+wz}{w-z}\,.$$

more roots in D-dim and at 3 loops! - in general alphabet changes with the loop order!

Find such solutions systematically? Minimal polynomial order?

Feynman integrals as iterated integrals (1)

- Logarithm and dilogarithm are first examples of iterated integrals with special ``d-log`` integration kernels

$$\frac{dt}{t} = d\log t \qquad \frac{-dt}{1-t} = d\log(1-t) \qquad \frac{dt}{1+t} = d\log(1+t)$$

- these are called harmonic polylogarithms (HPL) [Remiddi, Vermaseren]

e.g. $H_{1,-1}(x) = \int_0^x \frac{dx_1}{1-x_1} \int_0^{x_1} \frac{dx_2}{1+x_2}$

  - shuffle product algebra
  - coproduct structure
  - Mathematica implementation [Maitre]

- weight: number of integrations

- special values related to multiple zeta values (MZV)

$$\zeta_{i_1,i_2,\ldots,i_k} = \sum_{a_1>a_2>\ldots a_k\geq 1} \frac{1}{a_1^{i_1} a_2^{i_2} \ldots a_k^{i_k}}$$

cf. e.g. [Bluemlein, Broadhurst, Vermaseren]

e.g. $H_{0,1}(1) = \mathrm{Li}_2(1) = \zeta_2$

Feynman integrals as iterated integrals (2)

- Natural generalization: multiple polylogarithms [also called hyperlogarithms; Goncharov polylogarithms]

  allow kernels $w = d\log(t-a)$

$$G_{a_1,\ldots a_n}(z) = \int_0^z \frac{dt}{t-a_1} G_{a_2,\ldots,a_n}(t)$$

  numerical evaluation: GINAC [Vollinga, Weinzierl]

- Chen iterated integrals

$$\int_C \omega_1 \omega_2 \ldots \omega_n \qquad C : [0,1] \longrightarrow M$$

(space of kinematical variables)

Alphabet: set of differential forms $\omega_i = d\log \alpha_i$

integrals we discuss will be monodromy invariant on $M \setminus S$

S (set of singularities)

more flexible than multiple polylogarithms!

- Uniform weight functions (pure functions):

  $\mathbb{Q}$-linear combinations of functions of the same weight