Form factors and the dilatation operator from on-shell methods

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[1410.6309]
[1410.8485] with D. Nandan, C. Sieg and G. Yang
[1504.06323] with F. Loebbert, D. Nandan, C. Sieg and G. Yang

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Motivation to study form factors (1)

$$\mathcal{A}(1,\dots,n) = \langle 1,\dots,n|0\rangle$$

$$\mathcal{C}_{\mathcal{O}_1\dots\mathcal{O}_n}(x_1,\dots,x_n) = \langle 0|\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)|0\rangle$$

$$\mathcal{F}_{\mathcal{O}}(1,\dots,n;x) = \langle 1,\dots,n|\mathcal{O}(x)|0\rangle$$

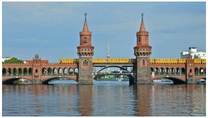
$\Rightarrow$ Form factors as bridge between purely on-shell amplitudes and purely off-shell correlation functions [van Neerven (1985)] [Boels, Bork, Brandhuber, Engelund, Gehrmann, Gurdogan, Henn, Huber, Kazakov, Kniehl, Moch, Mooney, Naculich, Penante, Roiban, Spence, Tarasov, Travaglini, Vartanov, Wen, Yang (2010–2014)]

Previous studies have focused on a special class of operators

$\rightarrow$ Study form factor of generic operator [MW(2014)]

Motivation to study form factors (2)

on-shell methods



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integrability

Connection between dilatation operator and amplitudes [Zwiebel (2011)]

Motivated study of amplitudes at weak coupling via integrability [Ferro, Lukowski, Meneghelli, Plefka, Staudacher, Chicherin, Derkachov, Kirschner, Frassek, Kanning, Ko, Beisert, Broedel, Rosso, de Leeuw, Bargheer, Huang, Loebbert, Yamazaki (2012–2014)]

Form factors as bridge between on-shell methods and integrability

→ Revisit spectral problem via on-shell methods, field-theoretic derivation and extension of connection between dilatation operator and amplitudes [MW (Oct.2014)], [Nandan, Sieg, MW, Yang (Oct.2014)] [Loebbert, Nandan, Sieg, MW, Yang (Apr.2015)]

Further on-shell approaches to the dilatation operator via correlation functions [Engelund, Roiban (2012)], [Koster, Mitev, Staudacher (Oct.2014)], [Brandhuber, Penante, Travaglini, Young (Dec.2014, Feb.2015)]

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Super spinor helicity variables for super form factors

Fourier transform to momentum space

$$\mathcal{F}_{\mathcal{O}}(1,\dots,n;q) = \int \mathrm{d}^4x\, \mathrm{e}^{-iqx} \langle 1,\dots,n|\mathcal{O}(x)|0\rangle$$
$$= \delta^4\left(q - \sum_{i=1}^{n} p_i\right) \langle 1,\dots,n|\mathcal{O}(0)|0\rangle$$

Spinor helicity variables: $p_i^{\alpha\dot{\alpha}} = p_i^{\mu}\sigma_{\mu}^{\alpha\dot{\alpha}} = \lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}} \qquad (p_i^2 = 0)$

Nair's $\mathcal{N} = 4$ on-shell super field

$$\Phi = g^{+} + \eta^{A}\,\bar{\psi}_{A} + \frac{1}{2!}\eta^{A}\eta^{B}\,\phi_{AB} + \eta^{A}\eta^{B}\eta^{C}\,\psi_{ABC} + \eta^{1}\eta^{2}\eta^{3}\eta^{4}\,g^{-}$$

Colour-ordered super form factors

$$\mathcal{F}_{\mathcal{O}}(1,\dots,n;q) = \sum_{\sigma\in\mathbb{S}_n/\mathbb{Z}_n} \mathrm{tr}[\mathrm{T}^{a_{\sigma(1)}}\cdots\mathrm{T}^{a_{\sigma(n)}}]\hat{\mathcal{F}}_{\mathcal{O}}(\sigma(1),\dots,\sigma(n);q)$$
$$+ \text{multi-trace terms}$$

Gauge-invariant local composite operators

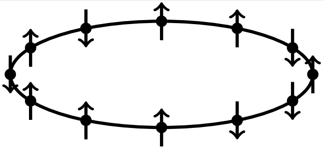
Single-trace operators

$$\mathcal{O}(x) = \text{tr}(W_1(x)W_2(x)\dots W_L(x))$$

with $W_i \in$

$$\begin{aligned}
\{\, \mathrm{D}_{(\alpha_1\dot\alpha_1}\cdots\mathrm{D}_{\alpha_k\dot\alpha_k}\,\bar F_{\dot\alpha_{k+1}\dot\alpha_{k+2})} &\;\widehat{=}\; (\mathbf{a}^\dagger)^{k+2}(\mathbf{b}^\dagger)^{k}\;\mathbf{d}^{\dagger1}\mathbf{d}^{\dagger2}\mathbf{d}^{\dagger3}\mathbf{d}^{\dagger4}\,|\,0\,\rangle\,,\\
\mathrm{D}_{(\alpha_1\dot\alpha_1}\cdots\mathrm{D}_{\alpha_k\dot\alpha_k}\,\bar\psi_{\dot\alpha_{k+1})A} &\;\widehat{=}\; (\mathbf{a}^\dagger)^{k+1}(\mathbf{b}^\dagger)^{k}\;\mathbf{d}^{\dagger A}\mathbf{d}^{\dagger B}\mathbf{d}^{\dagger C}\,|\,0\,\rangle\,,\\
\mathrm{D}_{(\alpha_1\dot\alpha_1}\cdots\mathrm{D}_{\alpha_k\dot\alpha_k)}\,\phi_{AB} &\;\widehat{=}\; (\mathbf{a}^\dagger)^{k}\;(\mathbf{b}^\dagger)^{k}\;\mathbf{d}^{\dagger A}\mathbf{d}^{\dagger B}\,|\,0\,\rangle\,,\\
\mathrm{D}_{(\alpha_1\dot\alpha_1}\cdots\mathrm{D}_{\alpha_k\dot\alpha_k}\,\psi_{\alpha_{k+1})ABC} &\;\widehat{=}\; (\mathbf{a}^\dagger)^{k}\;(\mathbf{b}^\dagger)^{k+1}\mathbf{d}^{\dagger A}\,|\,0\,\rangle\,,\\
\mathrm{D}_{(\alpha_1\dot\alpha_1}\cdots\mathrm{D}_{\alpha_k\dot\alpha_k}\,F_{\alpha_{k+1}\alpha_{k+2})} &\;\widehat{=}\; (\mathbf{a}^\dagger)^{k}\;(\mathbf{b}^\dagger)^{k+2}\,|\,0\,\rangle\,\}
\end{aligned}$$

Irreducible fields transforming in the singleton representation $\mathcal{V}_S$ of $\mathfrak{psu}(2,2|4)$ in oscillator picture using $(\mathbf{a}_i^{\dagger\alpha}, \mathbf{b}_i^{\dagger\dot\alpha}, \mathbf{d}_i^{\dagger A})$ [Günaydin, Saçlioglu (1982)], [Günaydin, Minic, Zagermann (1998)], [Beisert (2003)]



Form factors as spin chains

Colour-ordered minimal ($n = L$) tree-level super form factor for generic operator $\mathcal{O}$

$$\hat{\mathcal{F}}^{(0)}_{\mathcal{O}}(1,\dots,L;q) = L\delta^4\left(q - \sum_{i=1}^{L} \lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}}\right)\left(\begin{array}{c} \mathbf{a}_i^{\dagger\alpha} \to \lambda_i^{\alpha} \\ \mathbf{b}_i^{\dagger\dot{\alpha}} \to \tilde{\lambda}_i^{\dot{\alpha}} \\ \mathbf{d}_i^{\dagger A} \to \eta_i^{A} \\ \text{in oscillator picture} \end{array}\right)$$

$\Rightarrow$ Minimal form factors yield the spin chain of $\mathcal{N} = 4$ SYM theory in super spinor helicity variables, making it accessible to on-shell methods from the study of amplitudes

[MW(2014)]

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Spin chain, dilatation operator and integrability

Dilatation operator measures (anomalous) scaling dimensions

$\rightarrow$ Observables in a CFT

One-loop dilatation operator $\mathfrak{D}_2$
$=$ Hamiltonian of integrable spin chain

Nearest-neighbour interaction: $\mathfrak{D}_2 = \sum_{i=1}^{L} (\mathfrak{D}_2)_{i\,i+1}$

Spectral problem can be solved by Bethe ansatz techniques

[Minahan, Zarembo (2002)] [Beisert (2003)] [Beisert, Staudacher (2003)]

General structure of loop corrections to from factors

ℓ-loop minimal form factor

$$\mathcal{F}_{\mathcal{O}} = \left(1 + g^2\mathcal{I}^{(1)} + g^4\mathcal{I}^{(2)} + \dots\right)\mathcal{F}_{\mathcal{O}}^{(0)}$$

$\mathcal{I}^{(\ell)}$ are operators, as the $\mathcal{O}$ do not renormalise diagonally!

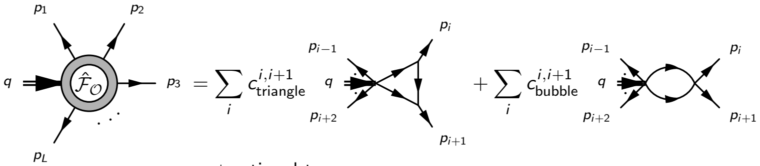
General structure of divergences in $4 - 2\varepsilon$ dimensions

$$\log(\mathcal{I}) = \sum_{\ell=1}^{\infty} g^{2\ell}\left[-\frac{\gamma_{\text{cusp}}^{(\ell)}}{8(\ell\varepsilon)^2} - \frac{\mathcal{G}_0^{(\ell)}}{4\ell\varepsilon}\right]\sum_{i=1}^{n}(-s_{i\,i+1})^{-\ell\varepsilon}$$
$$- \sum_{\ell=1}^{\infty} g^{2\ell}\frac{\mathfrak{D}_{2\ell}}{2\ell\varepsilon} + \text{Finite}(g^2) + \mathcal{O}(\varepsilon)$$

Universal IR divergences [Mueller (1979)], [Collins (1980)], [Sen (1981)], [Magnea, Sterman (1990)], [Bern, Dixon, Smirnov (2005)], ...
but operator-valued UV divergences

$\Rightarrow$ We can compute $\mathcal{I}$ via on-shell methods and extract the ℓ-loop dilatation operator $\mathfrak{D}_{2\ell}$

General ansatz from integral basis:



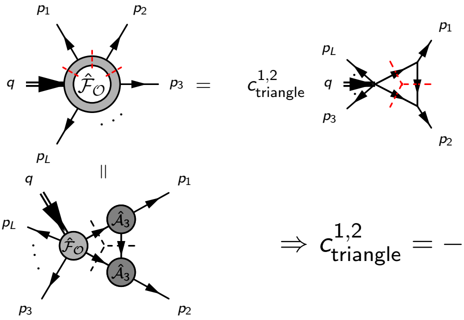
$+$ rational terms

$\Rightarrow$ Determine coefficients via cuts

Cut: $\dfrac{1}{l^2} \to \delta(l^2)\Theta(l_0)$

Triple cut and triangle coefficient

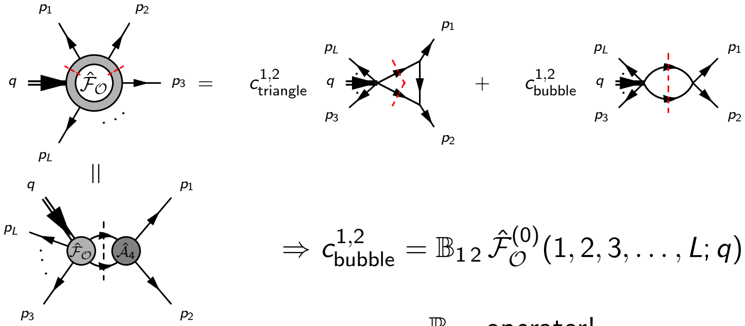
Triple cut between p_1, p_2 and the rest of the diagram:



$$\Rightarrow c^{1,2}_{\text{triangle}} = -s_{1\,2}\hat{\mathcal{F}}^{(0)}_{\mathcal{O}}(1, 2, 3, \ldots, L; q)$$

Double cut and bubble coefficient

Double cut between p_1, p_2 and the rest of the diagram:



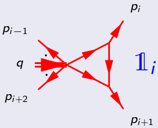
$$\Rightarrow c^{1,2}_{\text{bubble}} = \mathbb{B}_{1\,2}\,\hat{\mathcal{F}}^{(0)}_{\mathcal{O}}(1,2,3,\ldots,L;q)$$

$\mathbb{B}_{1\,2}$ operator!

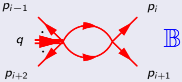
One-loop minimal form factor of a generic operator $\mathcal{O}$

$$\hat{\mathcal{F}}^{(1)}_{\mathcal{O}}(1,\dots,L;q) = g^2 \mathcal{I}^{(1)} \hat{\mathcal{F}}^{(0)}_{\mathcal{O}}(1,\dots,L;q)\,, \quad \mathcal{I}^{(1)} = \sum_{i=1}^{L} \mathcal{I}^{(1)}_{i\,i+1}$$

$\mathcal{I}^{(1)}_{i\,i+1} = -\,s_{i\,i+1}$



$\mathbb{1}_{i\,i+1} +$



$\mathbb{B}_{i\,i+1}$ + rational terms

IR divergence, Universal

UV divergence $\Rightarrow (\mathfrak{D}_2)_{i\,i+1} = -2\mathbb{B}_{i\,i+1}$

$\mathfrak{D}_2$ agrees with result of [Beisert (2003)] in formulation of [Zwiebel (2007)] after replacing $(\mathbf{a}_i^{\dagger\alpha}, \mathbf{b}_i^{\dagger\dot{\alpha}}, \mathbf{d}_i^{\dagger A})$ by $(\lambda_i^{\alpha}, \tilde{\lambda}_i^{\dot{\alpha}}, \eta_i^{A})$.
Field-theoretic derivation of a connection between amplitudes and dilatation operator which was observed in [Zwiebel (2011)].

[MW(2014)]

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Konishi operator

- Prime example of non-protected operators
- $\mathcal{K} = \text{tr}[\phi^I \phi^I]$
- No mixing
- Anomalous dimension $\gamma_{\mathcal{K}}$ known via field theory up to five loops [Eden, Heslop, Korchemsky, Smirnov, Sokatchev (2012)] and via integrability up to ten loops [Marboe, Volin (2014)]

Subtleties in on-shell method occur

Observed missmatch for $\gamma_{\mathcal{K}}^{(2)}$

[Boucher-Veronneau, Dixon, Pennington (private communication)]

Subtleties concerning the regularisation in $D = 4 - 2\varepsilon$

Four-dimensional-helicity scheme [Bern, Kosower (1991)],...	Dimensional reduction from $D = 10$ to $D = 4 - 2\varepsilon$ [Siegel (1979)],...
$N_\phi = 6$ $\mathcal{K}_6 = \frac{1}{2}\epsilon^{ABCD}\,\mathrm{tr}(\phi_{AB}\phi_{CD})$	$N_\phi = 6 + 2\varepsilon$ $\mathcal{K} = \mathrm{tr}(\phi^I\phi^I)$

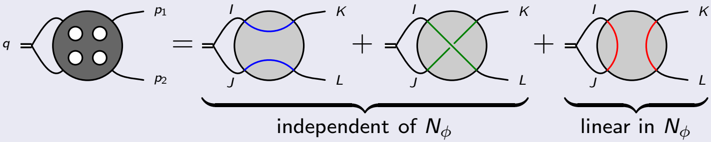
$\mathcal{K}$ and $\mathcal{K}_6$ differ by 2ε scalars,
similar to $\mu^2 = \ell^2_{D=4} - \ell^2_{D=4-2\varepsilon}$ terms

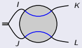
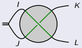
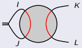
Four-dimensional on-shell methods yield form factors for $\mathcal{K}_6$

But: Only $\mathcal{K}$ is primary operator of the Konishi multiplet with known Konishi anomalous dimension!

[Nandan, Sieg, MW, Yang (2014)]

Group-theoretic decomposition of the contributions to $\langle \phi^K \phi^L | \operatorname{tr}(\phi^I \phi^K) | 0 \rangle$:



Can obtain  +  from $\mathcal{F}^{(\ell)}_{\operatorname{tr}\phi^{(I}\phi^{J)},2}$ and  from $\mathcal{F}^{(\ell)}_{\mathcal{K}_6,2}$

$\Rightarrow \mathcal{F}^{(\ell)}_{\mathcal{K},2}$

Yields correct one- and two-loop Konishi anomalous dimensions

Similar subtleties also arise for other operators and can be solved analogously.

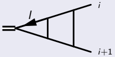
[Nandan, Sieg, MW, Yang (2014)]

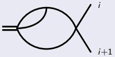
Single-trace operators built from $\uparrow = \phi_{14}$ and $\downarrow = \phi_{24}$

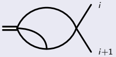
Two-loop minimal form factor can be computed via unitarity

Sums of densities of range two and three. E.g.

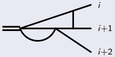
$$\langle \cdots \uparrow\downarrow \cdots | \, \mathrm{tr}(\cdots \downarrow\uparrow \ldots)|0\rangle$$

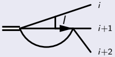
$= -$ 

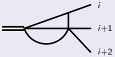
$s_{i\,i+1} s_{i\,l}$ $-$ 

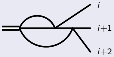
$-$ 

$$\langle \cdots \uparrow\uparrow\downarrow \cdots | \, \mathrm{tr}(\cdots \uparrow\downarrow\uparrow \ldots)|0\rangle$$

$= -$ 

$s_{i\,i+1}$ $-$ 

$s_{i\,l}$ $+$ 

$-$ 

$\Rightarrow$ Two-loop dilatation operator

[Loebbert, Nandan, Sieg, MW, Yang (2015)]

Remainder functions in the $\mathfrak{su}(2)$ sector

BDS remainder for form factors

$$R^{(2)} = \underline{\mathcal{I}}^{(2)}(\varepsilon) - \frac{1}{2}\left(\underline{\mathcal{I}}^{(1)}(\varepsilon)\right)^2 - f^{(2)}(\varepsilon)\underline{\mathcal{I}}^{(1)}(2\varepsilon) + \mathcal{O}(\varepsilon)$$

with $f^{(2)}(\varepsilon) = -2\zeta_2 - 2\zeta_3\varepsilon - 2\zeta_4\varepsilon^2$ [Bern, Dixon, Smirnov (2005)]

Properties

- Operator, given by sum of densities
- $\mathfrak{su}(2)$ Ward identities: $[J^A, R^{(2)}] = 0$
- Mixed tanscendentality, but highest transcendentality piece universal and agrees with the result of [Brandhuber, Penante, Travaglini, Wen (2014)] for $\mathrm{tr}(\phi_{14}^L)$

Conjecture

Highest transcendentality piece universal also beyond $\mathfrak{su}(2)$ sector
$\Rightarrow$ Extension of maximal transcendentality principle

[Loebbert, Nandan, Sieg, MW, Yang (2015)]

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Summary

- Study of form factors for generic operators in $\mathcal{N} = 4$ SYM
- Minimal tree-level form factors yield the spin chain of integrability in super spinor helicity variables
- Cut-constructible part of one-loop minimal form factor $\rightarrow$ Complete one-loop dilatation operator from generalised unitarity (includes all sectors), field-theoretic derivation of [Zwiebel (2011)]

[MW (2014)]

- Two-point two-loop form factor of the Konishi operator $\rightarrow$ Two-loop Konishi anomalous dimension
- Extension of the four-dimensional unitarity method

[Nandan, Sieg, MW, Yang (2014)]

- Minimal two-loop form factors in the $\mathfrak{su}(2)$ sector $\rightarrow$ Two-loop dilatation operator and remainder function

[Loebbert, Nandan, Sieg, MW, Yang (2015)]

Outlook

- Two-loop form factor for generic operator
  $\rightarrow$ Complete two-loop dilatation operator
- Twistor action for form factors
- On-shell diagrams, Graßmannians and Integrability for form factors [Frassek, Meidinger, Nandan, MW (last week)] $\rightarrow$ Poster

Thank you!

Bubble coefficient operator

$$\mathbb{B}_{i\,i+1}\hat{\mathcal{F}}^{(0)}_{\mathcal{O}}(\Lambda_1,\dots,\Lambda_L;q) =$$

$$-2\delta_{C_i,0}\int_0^{\pi/2} \mathrm{d}\theta \cot\theta \Big(\hat{\mathcal{F}}^{(0)}_{\mathcal{O}}(\Lambda_1,\dots,\Lambda_i,\Lambda_{i+1},\dots,\Lambda_L;q) - \hat{\mathcal{F}}^{(0)}_{\mathcal{O}}(\Lambda_1,\dots,\Lambda'_i,\Lambda'_{i+1},\dots,\Lambda_L;q)\Big)$$

with

$$\begin{pmatrix} \Lambda'_i \\ \Lambda'_{i+1} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \Lambda_i \\ \Lambda_{i+1} \end{pmatrix}, \quad \Lambda_i = (\lambda_i^\alpha, \tilde{\lambda}_i^{\dot{\alpha}}, \eta_i^A)$$

Polynomial in $\cos\theta$ and $\sin\theta$

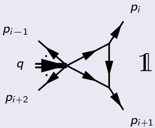
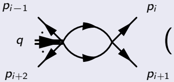
$\Rightarrow$ Evaluates to Euler β-function or harmonic number

$$\beta(x,y) = 2\int_0^{\frac{\pi}{2}} \mathrm{d}\theta (\sin\theta)^{2x-1}(\cos\theta)^{2y-1}$$

$$h(y) = 2\int_0^{\frac{\pi}{2}} \mathrm{d}\theta \cot\theta \left(1-(\cos\theta)^{2y}\right)$$

Single-trace operators built from $\uparrow = \phi_{24}$ and $\downarrow = \phi_{34}$

E.g. $\mathcal{O} = \text{tr}(\uparrow\downarrow\uparrow\downarrow) \Rightarrow$
$\hat{\mathcal{F}}^{(0)}_{\mathcal{O}}(1,2,3,4) = \delta^4(q - \sum_{i=1}^4 \lambda_i \tilde{\lambda}_i)\big(\eta_1^2\eta_1^4\eta_2^3\eta_2^4\eta_3^2\eta_3^4\eta_4^3\eta_4^4 + \text{cyclic}\big)$

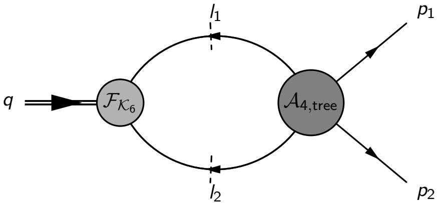
$\mathcal{I}^{(1)}_{i\,i+1} = -s_{i\,i+1}$  $\mathbb{1}_{i\,i+1} -$  $(\mathbb{1} - \mathbb{P})_{i\,i+1}$

UV divergence $\Rightarrow (\mathfrak{D}_2)_{i\,i+1} = 2(\mathbb{1} - \mathbb{P})_{i\,i+1}$

Hamiltonian of Heisenberg XXX spin chain

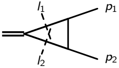
[MW(2014)]

Planar double cut

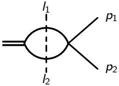


$$f^{(1)}_{\mathcal{K}_6,(\phi,\phi)}\Big|_{q^2} = \frac{F^{(1)}_{\mathcal{K}_6,(\phi,\phi)}}{F^{(0)}_{\mathcal{K}_6,(\phi,\phi)}}\Bigg|_{q^2} = \cdots = \int \mathrm{dPS}_{l_1,l_2}\left(\frac{\langle l_1 l_2\rangle\langle 12\rangle}{\langle 1 l_1\rangle\langle 2 l_2\rangle} + 6\,\frac{\langle l_1 2\rangle\langle l_2 1\rangle}{\langle 12\rangle\langle l_1 l_2\rangle}\right)$$

$$= -s_{12}$$

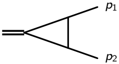


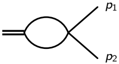
$$+\,6\,\frac{(l_1+p_2)^2}{s_{12}}$$



Lift and Passarino-Veltman reduction

$$f^{(1)}_{\mathcal{K}_6,(\phi,\phi)} = \underbrace{-2s_{12}\;[\text{triangle}]}_{f^{(2)}_{\mathrm{BPS},2}}\;-6\;[\text{bubble}]$$

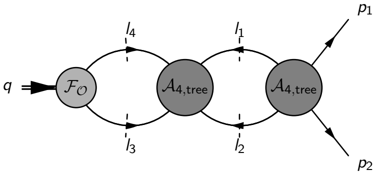




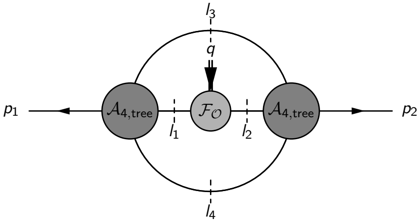
[Nandan, Sieg, MW, Yang (2014)]

Two-point Konishi form factor

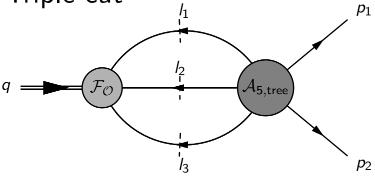
Planar double-double cut



Non-planar double-double cut

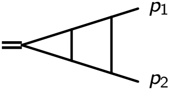


Triple cut

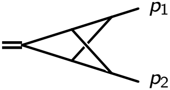


Final result:

$$f^{(2)}_{\mathcal{K}_6,2} = \underbrace{s_{12}^2 \Big(4$$

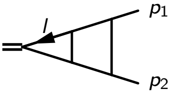


$$+$$

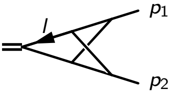


$$\Big)}_{f^{(2)}_{\mathrm{BPS},2}}$$

$$-\,6(l+p_1)^2(l+p_2)^2 \Big(4$$

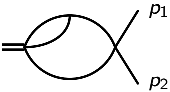


$$+$$



$$\Big)$$

$$+\,36$$



[Nandan, Sieg, MW, Yang (2014)]

Remainder functions in the $\mathfrak{su}(2)$ sector

Can be written in terms of three functions

$$
\begin{aligned}
(R_i^{(2)})^{XXX}_{XXX} =& -\mathrm{Li}_4(1-u_i) - \mathrm{Li}_4(u_i) + \mathrm{Li}_4\left(\frac{u_i-1}{u_i}\right) - \log\left(\frac{1-u_i}{w_i}\right)\left[\mathrm{Li}_3\left(\frac{u_i-1}{u_i}\right) - \mathrm{Li}_3(1-u_i)\right] \\
& - \log(u_i)\left[\mathrm{Li}_3\left(\frac{v_i}{1-u_i}\right) + \mathrm{Li}_3\left(-\frac{w_i}{v_i}\right) + \mathrm{Li}_3\left(\frac{v_i-1}{v_i}\right) - \frac{1}{3}\log^3(v_i) - \frac{1}{3}\log^3(1-u_i)\right] \\
& - \mathrm{Li}_2\left(\frac{u_i-1}{u_i}\right)\mathrm{Li}_2\left(\frac{v_i}{1-u_i}\right) + \mathrm{Li}_2(u_i)\left[\log\left(\frac{1-u_i}{w_i}\right)\log(v_i) + \frac{1}{2}\log^2\left(\frac{1-u_i}{w_i}\right)\right] \\
& + \frac{1}{24}\log^4(u_i) - \frac{1}{8}\log^2(u_i)\log^2(v_i) - \frac{1}{2}\log^2(1-u_i)\log(u_i)\log\left(\frac{w_i}{v_i}\right) \\
& - \frac{1}{2}\log(1-u_i)\log^2(u_i)\log(v_i) - \frac{1}{6}\log^3(u_i)\log(w_i) \\
& - \zeta_2\left[\log(u_i)\log\left(\frac{1-v_i}{v_i}\right) + \frac{1}{2}\log^2\left(\frac{1-u_i}{w_i}\right) - \frac{1}{2}\log^2(u_i)\right] \\
& + \zeta_3\log(u_i) + \frac{\zeta_4}{2} + G(\{1-u_i, 1-u_i, 1, 0\}, v_i) + (u_i \leftrightarrow v_i)
\end{aligned}
$$

[*Brandhuber, Penante, Travaglini, Wen*(2014)]

$$
\begin{aligned}
(R_i^{(2)})^{XYX}_{XXY} =& \left[\mathrm{Li}_3\left(-\frac{u_i}{w_i}\right) - \log(u_i)\mathrm{Li}_2\left(\frac{v_i}{1-u_i}\right) + \frac{1}{2}\log(1-u_i)\log(u_i)\log\left(\frac{w_i^2}{1-u_i}\right)\right. \\
& \left. - \frac{1}{2}\mathrm{Li}_3\left(-\frac{u_i v_i}{w_i}\right) - \frac{1}{2}\log(u_i)\log(v_i)\log(w_i) - \frac{1}{12}\log^3(w_i) + (u_i \leftrightarrow v_i)\right] \\
& - \mathrm{Li}_3(1-v_i) + \mathrm{Li}_3(u_i) - \frac{1}{2}\log^2(v_i)\log\left(\frac{1-v_i}{u_i}\right) + \frac{1}{6}\pi^2\log\left(\frac{v_i}{w_i}\right) - \frac{1}{6}\pi^2\log(-s_{i\,i+1\,i+2}) \\
& + \mathrm{Li}_2(1-u_i) + \mathrm{Li}_2(1-v_i) + \log(u_i)\log(v_i) - \frac{1}{2}\log(-s_{i+1\,i+2})\log\left(\frac{u_i}{v_i}\right) + 2\log(-s_{i\,i+1}) + \frac{\pi^2}{3} - 7
\end{aligned}
$$

$$
(R_i^{(2)})^{YXX}_{XXY} = \frac{1}{2}\log(-s_{i+1\,i+2})\log\left(\frac{u_i}{v_i}\right) - \mathrm{Li}_2(1-u_i) - \log(u_i)\log(v_i) + \frac{1}{2}\log^2(v_i) + \log(-s_{i+1\,i+2}) - 2\log(-s_{i\,i+1}) + \frac{7}{2}
$$

[Loebbert, Nandan, Sieg, MW, Yang (2015)]