Workshop on Gauge Theory and String Theory

ETH Zurich

July 2, 2008

Analytic Expressions for One-Loop Amplitudes

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- Unitarity method
- Unitarity cuts of amplitudes & master integrals
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- Proof that 4d coefficients are polynomials in u
- The D-dimensional algorithm, and masses

One-Loop Amplitudes

In four dimensions, Passarino-Veltman reduction brings the one-loop amplitude to the form

$$A_{n;1} = \sum_i d_i \, (\text{box}) + \sum_i c_i \, (\text{triangle}) + \sum_i b_i \, (\text{bubble}) + \sum_i a_i \, (\text{tadpole}) + \text{rational}$$

where expressions for scalar tadpole, scalar bubble, scalar triangle and scalar box integrals are known explicitly.

One-Loop Amplitudes

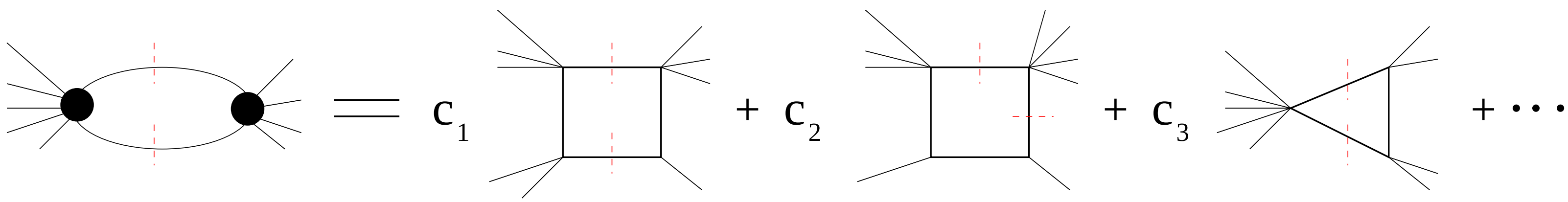
In $D = 4 - 2\epsilon$ dimensions, the result of reduction is

$$\begin{aligned} A_{n;1} \quad = \quad & \sum_i e_i \,(\text{pentagon}) + \sum_i d_i \,(\text{box}) + \sum_i c_i \,(\text{triangle}) \\ & + \sum_i b_i \,(\text{bubble}) + \sum_i a_i \,(\text{tadpole}) \end{aligned}$$

Amplitudes from unitarity cuts

$$C = \Delta A_n^{1-\text{loop}} = \sum c \, \Delta I$$

Tree level data.



Matching 4-dimensional cuts can suffice to determine reduction coefficients!

(Bern, Dixon, Dunbar, Kosower 1994)

But: we get several coefficients together in the same equation.

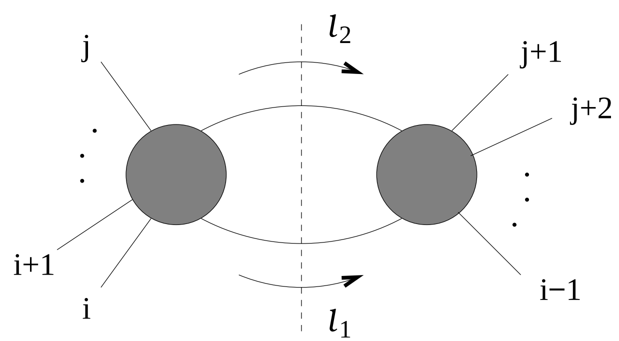
- We shall analyze the structure of the unitarity cut and distinguish all the coefficients.
- Let's start with massless fields. (No tadpoles.)

Unitarity Cuts

$$\Delta A^{1-\text{loop}} = \int d\mu \;\; A^{\text{tree}}_{\text{Left}} \;\times\; A^{\text{tree}}_{\text{Right}}$$

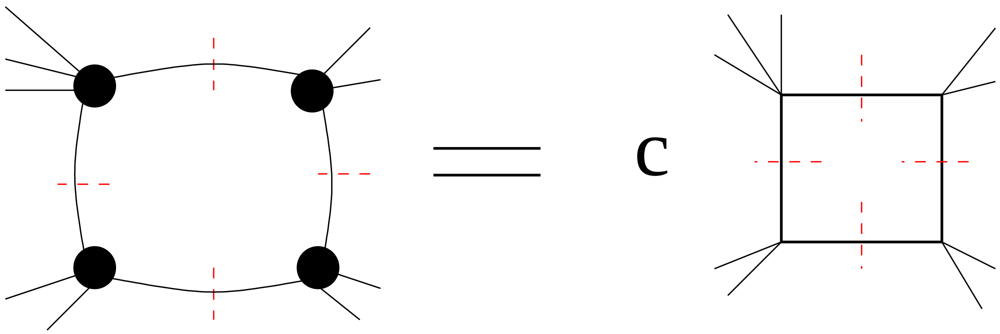
where

$$d\mu = d^4\ell_1 \, d^4\ell_2 \, \delta^{(4)}(\ell_1 + \ell_2 - K) \, \delta(\ell_1^2) \, \delta(\ell_2^2)$$



By unitarity, this is the discontinuity of the amplitude across a branch cut, in a kinematic region selecting the cut momentum K. (Cutkosky 1960)

Box Coefficients from Quadruple Cuts



Generalized Unitarity: Try replacing all four propagators by delta functions.

This operation isolates any given box.

In four dimensions, these four delta functions localize the integral completely. This computation is very easy!

The loop momentum solution

(RB, Cachazo, Feng)

The box coefficients computed from quadruple cuts are given by

$$c = \frac{1}{2} \sum_{\mathcal{S}} A_1^{\text{tree}} A_2^{\text{tree}} A_3^{\text{tree}} A_4^{\text{tree}}$$

$\mathcal{S}$ is the set of all solutions of the on-shell conditions for the internal lines.

$$\mathcal{S} = \{\, \ell \,|\, \ell^2 = 0, \quad (\ell - K_1)^2 = 0, \quad (\ell - K_1 - K_2)^2 = 0, \quad (\ell + K_4)^2 = 0 \}$$

Can these equations always be solved?

In complexified momentum space, there are exactly 2 solutions.

(Note: nonvanishing 3-point amplitudes.)

Applications of Generalized Unitarity

Triangle coefficients, real momentum:

Bern, Dixon, Dunbar, Kosower (x2); Bern, Dixon, Kosower (x2)

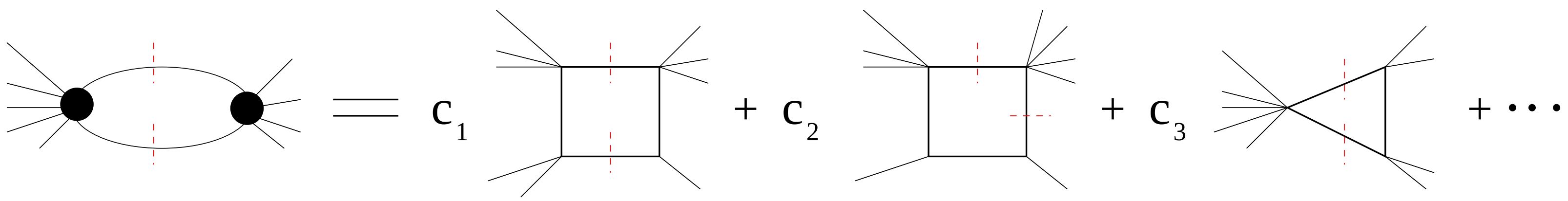
Complexified momentum:

RB, Cachazo, Feng; Mastrolia; Forde; Ossola, Papadopoulos, Pittau; Ellis, Giele, Kunszt; Kilgore; Giele, Kunszt, Melnikov

Specialized approaches to rational parts

Bern, Dixon, Kosower (x3); Forde, Kosower; Berger, Bern, Dixon, Forde, Kosower (x2); Xiao, Yang, Zhu; Binoth, Guillet, Heinrich; Ossola, Papadopoulos, Pittau; Badger

Integral Coefficients from Unitarity Cuts



RHS: cuts of master integrals.

Extract coefficients by matching cuts.

Start in four dimensions, neglecting rational terms.

Later we move on to D dimensions.

(Ordinary) Cut Integrals

A closer look at the cut integral:

$$\Delta A^{1-\text{loop}} = \int d\mu\, A^{\text{tree}}(-\ell, i, \ldots, j, \ell{-}K)\, A^{\text{tree}}(K{-}\ell, j{+}1, \ldots, i{-}1, \ell)$$

$$d\mu = d^4\ell\ \ \delta^+(\ell^2)\, \delta^+((\ell - K)^2)$$

Cachazo, Svrček, Witten (2004): Change to spinor variables with

$$\ell_{a\dot{a}} = t\, \lambda_a \tilde{\lambda}_{\dot{a}}.$$

t is real. The spinors λ and $\tilde{\lambda}$ are independent, homogeneous coordinates on two copies of CP^1. The integral is over the diagonal CP^1 defined by $\tilde{\lambda} = \bar{\lambda}$.

$$\int d^4\ell\, \delta^+(\ell^2)\, (\bullet) = \int_0^\infty dt\, t \int_{\tilde{\lambda}=\bar{\lambda}} \langle \lambda\, d\lambda \rangle\, [\tilde{\lambda}\, d\tilde{\lambda}]\, (\bullet)$$

Steps in spinor integration

(RB, Buchbinder, Cachazo, Feng; RB, Feng, Mastrolia; RB, Feng)

- Change variables, $\ell = t\lambda\tilde{\lambda}$, and use the spinor measure,

$$\int d^4\ell\ \delta(\ell^2)\ \delta((\ell - K)^2)\ (\bullet)$$
$$= \int_0^\infty dt\, t \int \langle\lambda\, d\lambda\rangle\, [\tilde{\lambda}\, d\tilde{\lambda}]\, \delta((t\lambda\tilde{\lambda} - K)^2)(\bullet)$$

- Use 2nd delta function to perform t-integral.

- Simplify denominators with spinor identities.

- Express result as a total spinor-derivative plus delta functions.
  Introduce one Feynman parameter (but only one, thanks to the spinor identities).

Recall the plan:

Use the equation

$$\Delta A^{1-\mathrm{loop}} = \sum c_i \, \Delta I_i$$

to extract the coefficients by matching logarithms.

Then, reconstruct the amplitude.

Input required: tree level amplitudes.

The discontinuities ΔI_i are immediately computable.

First I illustrate the cut of the scalar bubble and triangle.

These examples shows the most essential ideas that we need for the cut of the amplitude on the left hand side.

Prototype: cutting the bubble

$$\Delta I_2 \quad = \quad \int d^4\ell \ \ \delta^+(\ell^2) \ \ \delta^+((\ell - K)^2)$$

Substitute $\ell = t\lambda\tilde{\lambda}$ and the spinor measure.

Also use:

$$\begin{aligned}
\delta((\ell - K)^2) \quad &= \quad \delta(K^2 - 2K \cdot \ell) \\
&= \quad \delta(K^2 + t \left\langle \lambda | K | \widetilde{\lambda} \right]) \\
&= \quad \frac{1}{\left\langle \lambda | K | \widetilde{\lambda} \right]} \delta \left(t + \frac{K^2}{\left\langle \lambda | K | \widetilde{\lambda} \right]} \right)
\end{aligned}$$

- After t-integration we get

$$\Delta I_2 \quad = \quad \int \langle \lambda \, d\lambda \rangle \, [\widetilde{\lambda} \, d\widetilde{\lambda}] \frac{K^2}{\left\langle \lambda | K | \widetilde{\lambda} \right]^2}$$

- A key observation: (Cachazo, Svrček, Witten)

$$[\widetilde{\lambda} \, d\widetilde{\lambda}] \frac{1}{\left\langle \lambda | K | \widetilde{\lambda} \right]^2} = [d\widetilde{\lambda} \, \partial_{\widetilde{\lambda}}] \left(\frac{[\eta \, \widetilde{\lambda}]}{\langle \lambda | K | \eta] \left\langle \lambda | K | \tilde{\lambda} \right]} \right)$$

However, the contour of integration is where $\widetilde{\lambda}$ is the complex conjugate of λ, so there is a delta-function contribution, just as

$$\frac{\partial}{\partial \overline{z}} \frac{1}{(z-b)} = 2\pi \delta(z-b).$$

Find residues in the holomorphic variable.

- Applied to our case, we see that there is a contribution from the pole at

$$|\lambda\rangle = |K|\eta], \qquad \Rightarrow |\widetilde{\lambda}] = |K|\eta\rangle$$

- Final result for the cut bubble:

$$\begin{aligned}
\Delta I_2 &= \int \langle\lambda\, d\lambda\rangle\, [\widetilde{\lambda}\, d\widetilde{\lambda}] \frac{K^2}{\left\langle \lambda|K|\widetilde{\lambda}\right]^2} \\
&= -K^2 \left(\frac{[\eta\, \widetilde{\lambda}]}{\left\langle \lambda|K|\widetilde{\lambda}\right]} \right)\Bigg|_{|\lambda\rangle = |K|\eta]} \\
&= -1
\end{aligned}$$

Cutting the triangle

$$\Delta I_3 \quad = \quad \int d^4\ell \, \delta^+(\ell^2) \frac{\delta^+((\ell - K_1)^2)}{(\ell + K_3)^2}$$

- After t-integration:

$$\Delta I_3 \quad = \quad \int \langle \lambda \, d\lambda \rangle \, [\widetilde{\lambda} \, d\widetilde{\lambda}] \frac{1}{\left\langle \lambda | K_1 | \widetilde{\lambda} \right] \left\langle \lambda | Q | \widetilde{\lambda} \right]}$$

where $Q \equiv \frac{K_3^2}{K_1^2} K_1 + K_3$.

- Introduce a Feynman parameter:

$$\Delta I_3 \quad = \quad \int_0^1 dx \int \langle \lambda \, d\lambda \rangle \, [\widetilde{\lambda} \, d\widetilde{\lambda}] \frac{1}{\left\langle \lambda | (1 - x) K_1 + x Q | \widetilde{\lambda} \right]^2}$$

- Now we know how to do this integral:

$$\begin{aligned}\Delta I_3 &= -\int_0^1 dx \frac{1}{((1-x)K_1 + xQ)^2} \\ &= -\int_0^1 dx \frac{1}{ax^2 + bx + c}\end{aligned}$$

where $a = (Q - K_1)^2$, $b = 2((K_1 \cdot Q) - K_1^2)$ and $c = K_1^2$.

- The result is

$$\Delta I_3 = \frac{1}{\sqrt{\Delta_3}} \ln\left(\frac{2a + b - \sqrt{\Delta_3}}{2a + b + \sqrt{\Delta_3}}\right) - \frac{1}{\sqrt{\Delta_3}} \ln\left(\frac{b - \sqrt{\Delta_3}}{b + \sqrt{\Delta_3}}\right)$$

with

$$\Delta_3 = (K_1^2)^2 + (K_2^2)^2 + (K_3^2)^2 - 2K_1^2K_2^2 - 2K_2^2K_3^2 - 2K_3^2K_1^2$$

- The arguments of the logarithms (easily identified via the square root $\sqrt{\Delta_3}$) function as the signature of the triangle function.

Cuts of 4-d Master Integrals

$$\Delta I_2 \quad = \quad \int \langle \ell \, d\ell \rangle \, [\ell \, d\ell] \frac{K^2}{\langle \ell | K | \ell]^2}$$

$$\Delta I_3 \quad = \quad \int \langle \ell \, d\ell \rangle \, [\ell \, d\ell] \frac{1}{\langle \ell | K | \ell] \, \langle \ell | Q | \ell]}$$

$$\Delta I_4 \quad = \quad \int \langle \ell \, d\ell \rangle \, [\ell \, d\ell] \frac{1}{K^2} \frac{1}{\langle \ell | Q_1 | \ell] \, \langle \ell | Q_2 | \ell]}$$

$$Q_j \quad \equiv \quad -K_j + \frac{K_j^2}{K^2} K$$

Cutting the Amplitude in 4d

$$C \quad = \quad c \int d^4\ell \, \frac{\prod_{i=1}^{k+n}(-2\ell \cdot P_i)}{\prod_{j=1}^{k}(\ell - K_j)^2} \;\; \delta(\ell^2) \;\; \delta((\ell - K)^2)$$

We define the following vectors:

$$\begin{aligned} Q_j \quad &= \quad -K_j + \frac{{K_j}^2}{K^2} K, \\ R_i \quad &= \quad -P_i. \end{aligned}$$

Then the cut integral can be written as follows:

$$C \quad = \quad c \int \langle \ell \; d\ell \rangle [\ell \; d\ell] \, \frac{(K^2)^{n+1}}{\langle \ell | K | \ell]^{n+2}} \, \frac{\prod_{i=1}^{k+n} \langle \ell | R_i | \ell]}{\prod_{j=1}^{k} \langle \ell | Q_j | \ell]}$$

Finish by identifying poles and residues.

We have given the results in general form. (RB, Feng)

See also: Forde

Split the factors in the denominator with partial fractions.

$$\frac{\prod_{j=1}^{k-1} \langle\ell|R_j|\ell]}{\prod_{i=1}^{k} \langle\ell|Q_i|\ell]} = \sum_{i=1}^{k} \frac{1}{\langle\ell|Q_i|\ell]} \frac{\prod_{j=1}^{k-1} \langle\ell|R_j Q_i|\ell\rangle}{\prod_{m=1,m\neq i}^{k} \langle\ell|Q_m Q_i|\ell\rangle}$$

$$\frac{\prod_{j=1}^{n-1} \langle\ell|R_j|\ell]}{\langle\ell|K|\ell]^n \, \langle\ell|Q|\ell]} = \frac{\prod_{j=1}^{n-1} \langle\ell|R_j Q|\ell\rangle}{\langle\ell|KQ|\ell\rangle^{n-1}} \frac{1}{\langle\ell|K|\ell] \, \langle\ell|Q|\ell]}$$
$$+ \sum_{p=0}^{n-2} (-1)^{n-p} \frac{\prod_{j=1}^{n-p-2} \langle\ell|R_j Q|\ell\rangle \langle\ell|R_{n-p-1} K|\ell\rangle \prod_{t=n-p}^{n-1} \langle\ell|R_t|\ell]}{\langle\ell|K|\ell]^{p+2} \, \langle\ell|QK|\ell\rangle^{n-p-1}}$$

Box coefficients

$$C[K_r, K_s, K] = \frac{(K^2)^{2+n}}{2}\left(\frac{\prod_{j=1}^{k+n}\langle P_{sr,1}|R_j|P_{sr,2}]}{\langle P_{sr,1}|K|P_{sr,2}]^{n+2}\prod_{t=1,t\neq i,j}^{k}\langle P_{sr,1}|Q_t|P_{sr,2}]} + \{P_{sr,1}\leftrightarrow P_{sr,2}\}\right)$$

$$\begin{aligned}
P_{sr,1} &= Q_s + \left(\frac{-2Q_s\cdot Q_r + \sqrt{\Delta_{sr}}}{2Q_r^2}\right)Q_r \\
P_{sr,2} &= Q_s + \left(\frac{-2Q_s\cdot Q_r - \sqrt{\Delta_{sr}}}{2Q_r^2}\right)Q_r \\
\Delta_{sr} &= (2Q_s\cdot Q_r)^2 - 4Q_s^2Q_r^2
\end{aligned}$$

Triangle coefficients

$$C[K_s, K] = \frac{(K^2)^{1+n}}{2} \frac{1}{(\sqrt{\Delta_s})^{n+1}} \frac{1}{(n+1)!\langle P_{s,1}\, P_{s,2}\rangle^{n+1}}$$
$$\times \frac{d^{n+1}}{d\tau^{n+1}} \left(\frac{\prod_{j=1}^{k+n} \langle P_{s,1} - \tau P_{s,2} | R_j Q_s | P_{s,1} - \tau P_{s,2}\rangle}{\prod_{t=1, t\neq s}^{k} \langle P_{s,1} - \tau P_{s,2} | Q_t Q_s | P_{s,1} - \tau P_{s,2}\rangle} + \{P_{s,1} \leftrightarrow P_{s,2}\} \right) \Bigg|_{\tau=0} .$$

$$P_{s,1} = Q_s + \left(\frac{-2 Q_s \cdot K + \sqrt{\Delta_s}}{2K^2} \right) K$$
$$P_{s,2} = Q_s + \left(\frac{-2 Q_s \cdot K - \sqrt{\Delta_s}}{2K^2} \right) K$$
$$\Delta_s = (2Q_s \cdot K)^2 - 4Q_s^2 K^2$$

Bubble coefficients

$$C[K] = (K^2)^{1+n} \sum_{q=0}^{n} \frac{(-1)^q}{q!} \frac{d^q}{ds^q} \left(\mathcal{B}^{(0)}_{n,n-q}(s) + \sum_{r=1}^{k} \sum_{a=q}^{n} \left(\mathcal{B}^{(r;a-q;1)}_{n,n-a}(s) - \mathcal{B}^{(r;a-q;2)}_{n,n-a}(s) \right) \right) \Bigg|_{s=0}$$

$$\mathcal{B}^{(0)}_{n,t}(s) \equiv \frac{d^n}{d\tau^n} \left(\frac{1}{n![\eta|\eta' K|\eta]^n} \frac{(2\eta \cdot K)^{t+1}}{(t+1)(K^2)^{t+1}} \frac{\prod_{j=1}^{n+k} \langle \ell | R_j (K + s\eta) | \ell \rangle}{\langle \ell\, \eta \rangle^{n+1} \prod_{p=1}^{k} \langle \ell | Q_p (K + s\eta) | \ell \rangle} \right) \Bigg|_{|\ell\rangle \to |K - \tau\eta'|\eta], \tau=0}$$

$$\mathcal{B}^{(r;b;1)}_{n,t}(s) \equiv \frac{(-1)^{b+1}}{b! \sqrt{\Delta_r}^{\,b+1} \langle P_{r,1}\, P_{r,2} \rangle^b} \frac{d^b}{d\tau^b} \left(\frac{1}{(t+1)} \frac{\left\langle P_{r,1} - \tau P_{r,2} | \eta | P_{r,1} \right]^{t+1}}{\left\langle P_{r,1} - \tau P_{r,2} | K | P_{r,1} \right]^{t+1}} \right.$$

$$\left. \times \frac{\langle P_{r,1} - \tau P_{r,2} | Q_r \eta | P_{r,1} - \tau P_{r,2} \rangle^b \prod_{j=1}^{n+k} \langle P_{r,1} - \tau P_{r,2} | R_j (K + s\eta) | P_{r,1} - \tau P_{r,2} \rangle}{\langle P_{r,1} - \tau P_{r,2} | \eta K | P_{r,1} - \tau P_{r,2} \rangle^{n+1} \prod_{p=1, p\neq r}^{k} \langle P_{r,1} - \tau P_{r,2} | Q_p (K + s\eta) | P_{r,1} - \tau P_{r,2} \rangle} \right) \Bigg|_{\tau=0}$$

$$\mathcal{B}^{(r;b;2)}_{n,t}(s) \equiv \frac{(-1)^{b+1}}{b! \sqrt{\Delta_r}^{\,b+1} \langle P_{r,1}\, P_{r,2} \rangle^b} \frac{d^b}{d\tau^b} \left(\frac{1}{(t+1)} \frac{\left\langle P_{r,2} - \tau P_{r,1} | \eta | P_{r,2} \right]^{t+1}}{\left\langle P_{r,2} - \tau P_{r,1} | K | P_{r,2} \right]^{t+1}} \right.$$

$$\left. \times \frac{\langle P_{r,2} - \tau P_{r,1} | Q_r \eta | P_{r,2} - \tau P_{r,1} \rangle^b \prod_{j=1}^{n+k} \langle P_{r,2} - \tau P_{r,1} | R_j (K + s\eta) | P_{r,2} - \tau P_{r,1} \rangle}{\langle P_{r,2} - \tau P_{r,1} | \eta K | P_{r,2} - \tau P_{r,1} \rangle^{n+1} \prod_{p=1, p\neq r}^{k} \langle P_{r,2} - \tau P_{r,1} | Q_p (K + s\eta) | P_{r,2} - \tau P_{r,1} \rangle} \right) \Bigg|_{\tau=0}$$

D-dimensional unitarity

Orthogonal decomposition (Mahlon; Bern, Morgan) within Four Dimensional Helicity (FDH) scheme.

$$\int d^{4-2\epsilon}\ell_{4-2\epsilon} = \int d^{-2\epsilon}\ell_{-2\epsilon}\int d^4\ell_4$$

$$= \frac{(4\pi)^\epsilon}{\Gamma(-\epsilon)}\int_0^1 du\, u^{-1-\epsilon}\int d^4\widetilde{\ell}.$$

where $\ell^2_{-2\epsilon} = \frac{K^2}{4}u.$

The integral over u will remain. The u-dependence is controlled.

(Anastasiou, RB, Feng, Kunszt, Mastrolia)

$$\Delta A = \int_0^1 du\, u^{-1-\epsilon}\int d^4\ell\ \delta(\ell^2)\,\delta(\sqrt{1-u}\,K^2 - 2K\cdot\ell)$$

Problem is reduced to a standard 4-d cut integral.

(Cf. methods by Ossola, Papadopoulos, Pittau; Ellis, Giele, Kunszt; Kilgore; Giele, Kunszt, Melnikov)

Cuts of D-dimensional Master Integrals

$$\Delta I_2 \quad = \quad \int_0^1 du\, u^{-1-\epsilon} \int \langle \ell\, d\ell \rangle \, [\ell\, d\ell] \quad K^2 \sqrt{1-u} \frac{1}{\langle \ell | K | \ell]^2}$$

$$\Delta I_3 \quad = \quad \int_0^1 du\, u^{-1-\epsilon} \int \langle \ell\, d\ell \rangle \, [\ell\, d\ell] \quad \sqrt{1-u} \frac{1}{\langle \ell | K | \ell] \, \langle \ell | Q | \ell]}$$

$$\Delta I_4 \quad = \quad \int_0^1 du\, u^{-1-\epsilon} \int \langle \ell\, d\ell \rangle \, [\ell\, d\ell] \quad \frac{\sqrt{1-u}}{K^2} \frac{1}{\langle \ell | Q_1 | \ell] \, \langle \ell | Q_2 | \ell]}$$

$$\Delta I_4 = \frac{1}{2K^2}\frac{\sqrt{1-u}}{\sqrt{(Q_1\cdot Q_2)^2-Q_1^2Q_2^2}}\ln\frac{Q_1\cdot Q_2+\sqrt{(Q_1\cdot Q_2)^2-Q_1^2Q_2^2}}{Q_1\cdot Q_2-\sqrt{(Q_1\cdot Q_2)^2-Q_1^2Q_2^2}}$$

$$\begin{aligned}\Delta I_5 = \frac{\sqrt{1-u}}{(K^2)^2}\Bigg(&\frac{S[Q_3,Q_2,Q_1,K]}{4\sqrt{(Q_3\cdot Q_2)^2-Q_3^2Q_2^2}}\ln\frac{Q_3\cdot Q_2+\sqrt{(Q_3\cdot Q_2)^2-Q_3^2Q_2^2}}{Q_3\cdot Q_2-\sqrt{(Q_3\cdot Q_2)^2-Q_3^2Q_2^2}}\\ &+\frac{S[Q_3,Q_1,Q_2,K]}{4\sqrt{(Q_3\cdot Q_1)^2-Q_3^2Q_1^2}}\ln\frac{Q_3\cdot Q_1+\sqrt{(Q_3\cdot Q_1)^2-Q_3^2Q_1^2}}{Q_3\cdot Q_1-\sqrt{(Q_3\cdot Q_1)^2-Q_3^2Q_1^2}}\\ &+\frac{S[Q_2,Q_1,Q_3,K]}{4\sqrt{(Q_2\cdot Q_1)^2-Q_2^2Q_1^2}}\ln\frac{Q_2\cdot Q_1+\sqrt{(Q_2\cdot Q_1)^2-Q_2^2Q_1^2}}{Q_2\cdot Q_1-\sqrt{(Q_2\cdot Q_1)^2-Q_2^2Q_1^2}}\Bigg).\end{aligned}$$

$$S[Q_i,Q_j,Q_k,K]=\frac{T_1}{T_2}$$

$$T_1=-8\det\begin{pmatrix} K\cdot Q_k & Q_i\cdot K & Q_j\cdot K\\ Q_i\cdot Q_k & Q_i^2 & Q_i\cdot Q_j\\ Q_j\cdot Q_k & Q_i\cdot Q_j & Q_j^2\end{pmatrix};\quad T_2=-4\det\begin{pmatrix} Q_k^2 & Q_i\cdot Q_k & Q_j\cdot Q_k\\ Q_i\cdot Q_k & Q_i^2 & Q_i\cdot Q_j\\ Q_j\cdot Q_k & Q_i\cdot Q_j & Q_j^2\end{pmatrix}.$$

Cutting the Amplitude in D dimensions

$$C \;=\; c \int d^{4-2\epsilon}p \, \frac{\prod_{i=1}^{k+n}(-2p\cdot P_i)}{\prod_{j=1}^{k}(p-K_j)^2} \;\; \delta(p^2) \;\; \delta((p-K)^2)$$

Let us define the following four-vectors:

$$Q_j \;=\; -(\sqrt{1-u})K_j + \frac{{K_j}^2 - (1-\sqrt{1-u})(K_j\cdot K)}{K^2}K,$$

$$R_i \;=\; -(\sqrt{1-u})P_i - \frac{(1-\sqrt{1-u})(P_i\cdot K)}{K^2}K.$$

Then the cut integral can be written as follows:

$$C \;=\; \int_0^1 du\, u^{-1-\epsilon}\, c \int \langle \ell\, d\ell\rangle[\ell\, d\ell](\sqrt{1-u})\frac{(K^2)^{n+1}}{\langle \ell|K|\ell]^{n+2}} \frac{\prod_{i=1}^{k+n}\langle \ell|R_i|\ell]}{\prod_{j=1}^{k}\langle \ell|Q_j|\ell]}$$

Coefficient formulas look the same, now in terms of u-dependent Q_j, R_i, c.

We are assuming that all ϵ-dimensional dependence can be expressed in terms of u.

This assumption is sufficient to treat scalars in the loop, and so also gluon amplitudes via the supersymmetric decomposition.

We have checked 4- and 5-gluon examples to all orders in ϵ. (Anastasiou, RB, Feng, Kunszt, Mastrolia; RB, Feng, Yang. Original results from Bern, Dixon, Dunbar, Kosower.)

We have reproduced cuts for $gg \to gg$ and $gg \to gH$ with massive fermion loops, with initial expressions derived from Feynman diagrams. (RB, Feng, Mastrolia. Original results from Bern, Morgan; Rozowsky.)

Box coefficients

$$C[K_r, K_s, K] = \frac{(K^2)^{2+n}}{2} \left(\frac{\prod_{j=1}^{k+n} \langle P_{sr,1}|R_j|P_{sr,2}]}{\langle P_{sr,1}|K|P_{sr,2}]^{n+2} \prod_{t=1,t\neq i,j}^{k} \langle P_{sr,1}|Q_t|P_{sr,2}]} + \{P_{sr,1} \leftrightarrow P_{sr,2}\} \right)$$

$$P_{sr,1} = Q_s + \left(\frac{-2Q_s \cdot Q_r + \sqrt{\Delta_{sr}}}{2Q_r^2} \right) Q_r$$

$$P_{sr,2} = Q_s + \left(\frac{-2Q_s \cdot Q_r - \sqrt{\Delta_{sr}}}{2Q_r^2} \right) Q_r$$

$$\Delta_{sr} = (2Q_s \cdot Q_r)^2 - 4Q_s^2 Q_r^2$$

Pentagon coefficients

(RB, Feng, Yang)

If $k = 3$:

$$C[K_i, K_j, K_t, K] = (K^2)^{3+n} \prod_{s=1}^{n+3} \beta_s^{(q_i, q_j, q_t; p_s)}$$

If $k \geq 4$:

$$C[K_i, K_j, K_t, K] = (K^2)^{3+n} \frac{\prod_{s=1}^{n+3} \beta_s^{(q_i, q_j, q_t; p_s)}}{\prod_{w=1, w \neq i,j,t}^{k} \gamma_w^{(K_i, K_j; K_w, K_t)}}.$$

$$\alpha_j \equiv \frac{K_j^2 - K_j \cdot K}{K^2}$$

$$\beta_s^{(q_i, q_j, q_t; p_s)} \equiv (\beta_s - \sum_{h=i,j,k} a_h^{(q_i, q_j, q_t; p_s)} \alpha_h)$$

$$\gamma_s^{(K_i, K_j; K_s, K_t)} \equiv \frac{K_i^2 \epsilon(K, K_j, K_s, K_t) + K_j^2 \epsilon(K_i, K, K_s, K_t) + K_s^2 \epsilon(K_i, K_j, K, K_t) + K_t^2 \epsilon(K_i, K_j, K_s, K}{K^2 \epsilon(K_i, K_j, K, K_t).}$$

Triangle coefficients

$$C[K_s, K] = \frac{(K^2)^{1+n}}{2} \frac{1}{(\sqrt{\Delta_s})^{n+1}} \frac{1}{(n+1)!\langle P_{s,1}\, P_{s,2}\rangle^{n+1}}$$
$$\times \frac{d^{n+1}}{d\tau^{n+1}} \left(\frac{\prod_{j=1}^{k+n} \langle P_{s,1} - \tau P_{s,2} | R_j Q_s | P_{s,1} - \tau P_{s,2}\rangle}{\prod_{t=1, t\neq s}^{k} \langle P_{s,1} - \tau P_{s,2} | Q_t Q_s | P_{s,1} - \tau P_{s,2}\rangle} + \{P_{s,1} \leftrightarrow P_{s,2}\} \right)\Bigg|_{\tau=0} .$$

$$P_{s,1} = Q_s + \left(\frac{-2Q_s \cdot K + \sqrt{\Delta_s}}{2K^2} \right) K$$
$$P_{s,2} = Q_s + \left(\frac{-2Q_s \cdot K - \sqrt{\Delta_s}}{2K^2} \right) K$$
$$\Delta_s = (2Q_s \cdot K)^2 - 4Q_s^2 K^2$$

Bubble coefficients

$$C[K] = (K^2)^{1+n} \sum_{q=0}^{n} \frac{(-1)^q}{q!} \frac{d^q}{ds^q} \left(\mathcal{B}^{(0)}_{n,n-q}(s) + \sum_{r=1}^{k} \sum_{a=q}^{n} \left(\mathcal{B}^{(r;a-q;1)}_{n,n-a}(s) - \mathcal{B}^{(r;a-q;2)}_{n,n-a}(s) \right) \right) \Bigg|_{s=0}$$

$$\mathcal{B}^{(0)}_{n,t}(s) \equiv \frac{d^n}{d\tau^n} \left(\frac{1}{n![\eta|\eta' K|\eta]^n} \frac{(2\eta \cdot K)^{t+1}}{(t+1)(K^2)^{t+1}} \frac{\prod_{j=1}^{n+k} \langle \ell | R_j (K+s\eta) | \ell \rangle}{\langle \ell \, \eta \rangle^{n+1} \prod_{p=1}^{k} \langle \ell | Q_p (K+s\eta) | \ell \rangle} \right) \Bigg|_{|\ell\rangle \to |K-\tau\eta'|\eta], \tau=0}$$

$$\mathcal{B}^{(r;b;1)}_{n,t}(s) \equiv \frac{(-1)^{b+1}}{b! \sqrt{\Delta_r}^{\,b+1} \langle P_{r,1} \, P_{r,2} \rangle^b} \frac{d^b}{d\tau^b} \left(\frac{1}{(t+1)} \frac{\left\langle P_{r,1} - \tau P_{r,2} | \eta | P_{r,1} \right]^{t+1}}{\left\langle P_{r,1} - \tau P_{r,2} | K | P_{r,1} \right]^{t+1}} \right.$$
$$\left. \times \frac{\langle P_{r,1} - \tau P_{r,2} | Q_r \eta | P_{r,1} - \tau P_{r,2} \rangle^b \prod_{j=1}^{n+k} \langle P_{r,1} - \tau P_{r,2} | R_j (K+s\eta) | P_{r,1} - \tau P_{r,2} \rangle}{\langle P_{r,1} - \tau P_{r,2} | \eta K | P_{r,1} - \tau P_{r,2} \rangle^{n+1} \prod_{p=1, p\neq r}^{k} \langle P_{r,1} - \tau P_{r,2} | Q_p (K+s\eta) | P_{r,1} - \tau P_{r,2} \rangle} \right) \Bigg|_{\tau=0}$$

$$\mathcal{B}^{(r;b;2)}_{n,t}(s) \equiv \frac{(-1)^{b+1}}{b! \sqrt{\Delta_r}^{\,b+1} \langle P_{r,1} \, P_{r,2} \rangle^b} \frac{d^b}{d\tau^b} \left(\frac{1}{(t+1)} \frac{\left\langle P_{r,2} - \tau P_{r,1} | \eta | P_{r,2} \right]^{t+1}}{\left\langle P_{r,2} - \tau P_{r,1} | K | P_{r,2} \right]^{t+1}} \right.$$
$$\left. \times \frac{\langle P_{r,2} - \tau P_{r,1} | Q_r \eta | P_{r,2} - \tau P_{r,1} \rangle^b \prod_{j=1}^{n+k} \langle P_{r,2} - \tau P_{r,1} | R_j (K+s\eta) | P_{r,2} - \tau P_{r,1} \rangle}{\langle P_{r,2} - \tau P_{r,1} | \eta K | P_{r,2} - \tau P_{r,1} \rangle^{n+1} \prod_{p=1, p\neq r}^{k} \langle P_{r,2} - \tau P_{r,1} | Q_p (K+s\eta) | P_{r,2} - \tau P_{r,1} \rangle} \right) \Bigg|_{\tau=0}$$

Polynomial Proof

(RB, Feng, Yang)

Rewrite some definitions to make u-dependence manifest.

$$q_j \equiv K_j - \frac{K_j \cdot K}{K^2} K \qquad \alpha_j \equiv \frac{K_j^2 - K_j \cdot K}{K^2}$$

$$p_j \equiv P_j - \frac{P_j \cdot K}{K^2} K \qquad \beta_j \equiv -\frac{P_j \cdot K}{K^2}$$

$$\begin{aligned} Q_j(u) &= -(\sqrt{1-u})\, q_j + \alpha_j K, \\ R_j(u) &= -(\sqrt{1-u})\, p_j + \beta_j K \end{aligned}$$

Notice:

$$q_j \cdot K = p_j \cdot K = 0.$$

First, triangle. Then, bubble. Last, box/pentagon.

Triangle coefficients are polynomials

$$C[Q_s(u), K] = \frac{(K^2)^{1+n}}{2(\sqrt{\Delta(Q_s(u), K)(u)})^{n+1}} \frac{1}{(n+1)!\langle P_{(Q_s(u),K);1}(u)\; P_{(Q_s(u),K);2}(u)\rangle^{n+1}} \frac{d^{n+1}}{d\tau^{n+1}}$$

$$\left(\frac{\prod_{j=1}^{k+n} \langle P_{(Q_s(u),K);1}(u) - \tau P_{(Q_s(u),K);2}(u)|R_j(u)Q_s(u)|P_{(Q_s(u),K);1}(u) - \tau P_{(Q_s(u),K);2}(u)\rangle}{\prod_{t=1,t\neq s}^{k} \langle P_{(Q_s(u),K);1}(u) - \tau P_{(Q_s(u),K);2}(u)|Q_t(u)Q_s(u)|P_{(Q_s(u),K);1}(u) - \tau P_{(Q_s(u),K);2}(u)\rangle} \right.$$

$$\left. + \{P_{(Q_s(u),K);1}(u) \leftrightarrow P_{(Q_s(u),K);2}(u)\} \right) \Bigg|_{\tau\to 0}$$

$$y_{1,2}^{(s)} \equiv \pm \frac{\sqrt{(K_s \cdot K)^2 - K_s^2 K^2}}{K^2}.$$

$$P_{(Q_s(u),K);i}(u) = -\sqrt{1-u}(q_s + y_i^{(s)} K)$$

$$= -\sqrt{1-u}\, P_{(q_s,K);i}.$$

$$\sqrt{\Delta(Q_s(u), K)} = -\sqrt{1-u}\sqrt{-4q_s^2 K^2}$$

Triangle coefficients are polynomials

$$C[Q_s(u), K] = \frac{(K^2)^{1+n}}{2(-\sqrt{1-u})^{n+1}(\sqrt{\Delta(q_s, K)})^{n+1}} \frac{1}{(n+1)!\langle P_{(q_s,K);1}\, P_{(q_s,K);2}\rangle^{n+1}}$$

$$\frac{d^{n+1}}{d\tau^{n+1}} \left(\frac{\prod_{j=1}^{k+n} \langle P_{(q_s,K);1} - \tau P_{(q_s,K);2}|R_j(u)Q_s(u)|P_{(q_s,K);1} - \tau P_{(q_s,K);2}\rangle}{\prod_{t=1,t\neq s}^{k} \langle P_{(q_s,K);1} - \tau P_{(q_s,K);2}|Q_t(u)Q_s(u)|P_{(q_s,K);1} - \tau P_{(q_s,K);2}\rangle} + \{1 \leftrightarrow 2\} \right) \Bigg|_{\tau\to 0} .$$

$$\begin{aligned}
\langle \ell|Q_t(u)Q_s(u)|\ell\rangle &= \langle \ell|(Q_t(u) - \frac{\alpha_t}{\alpha_s}Q_s(u))Q_s(u)|\ell\rangle \\
&= -\sqrt{1-u}\langle \ell|(q_t - \frac{\alpha_t}{\alpha_s}q_s)Q_s(u)|\ell\rangle \\
\langle \ell|R_j(u)Q_s(u)|\ell\rangle &= \langle \ell|(R_j(u) - \frac{\beta_j}{\alpha_s}Q_s(u))Q_s(u)|\ell\rangle \\
&= -\sqrt{1-u}\langle \ell|(p_j - \frac{\beta_j}{\alpha_s}q_s)Q_s(u)|\ell\rangle .
\end{aligned}$$

$$\widetilde{q}_t \equiv (q_t - \frac{\alpha_t}{\alpha_s}q_s), \qquad \widetilde{p}_j \equiv (p_j - \frac{\beta_j}{\alpha_s}q_s)$$

Triangle coefficients are polynomials

$$C[Q_s(u), K] = \frac{(K^2)^{1+n}}{2(\sqrt{\Delta(q_s, K)})^{n+1}} \frac{1}{(n+1)!\langle P_{(q_s,K);1}\, P_{(q_s,K);2}\rangle^{n+1}}$$

$$\frac{d^{n+1}}{d\tau^{n+1}}\left(\frac{\prod_{j=1}^{k+n}\langle P_{(q_s,K);1} - \tau P_{(q_s,K);2}|\widetilde{p}_j Q_s(u)|P_{(q_s,K);1} - \tau P_{(q_s,K);2}\rangle}{\prod_{t=1,t\neq s}^{k}\langle P_{(q_s,K);1} - \tau P_{(q_s,K);2}|\widetilde{q}_t Q_s(u)|P_{(q_s,K);1} - \tau P_{(q_s,K);2}\rangle} + \{1 \leftrightarrow 2\}\right)\Bigg|_{\tau\to 0}$$

$$\begin{aligned}
E_1 &\equiv \langle P_{(q_s,K);2}|\widetilde{p}_j Q_s(u)|P_{(q_s,K);1}\rangle + \langle P_{(q_s,K);1}|\widetilde{p}_j Q_s(u)|P_{(q_s,K);2}\rangle \\
&= -\frac{\alpha_s K^2}{\sqrt{-q_s^2 K^2}}(2\widetilde{p}_j \cdot q_s)\langle P_{(q_s,K);1}\, P_{(q_s,K);2}\rangle \\
E_2 &\equiv \langle P_{(q_s,K);1}|\widetilde{p}_j Q_s(u)|P_{(q_s,K);1}\rangle\langle P_{(q_s,K);2}|\widetilde{p}_j Q_s(u)|P_{(q_s,K);2}\rangle \\
&= \langle P_{(q_s,K);1}\, P_{(q_s,K);2}\rangle^2 \left(q_s^2 \widetilde{p}_j^2 - (q_s \cdot \widetilde{p}_j)^2\right)\left(\frac{K^2\alpha_s^2}{q_s^2} + 1 - u\right)
\end{aligned}$$

Follow effect of derivatives. Degree is $[(n+1)/2]$.

Bubble coefficients are polynomials

$$C[K] = (K^2)^{1+n} \sum_{q=0}^{n} \frac{(-1)^q}{q!} \frac{d^q}{ds^q} \left(\mathcal{B}^{(0)}_{n,n-q}(s) + \sum_{r=1}^{k} \sum_{a=q}^{n} \left(\mathcal{B}^{(r;a-q;1)}_{n,n-a}(s) - \mathcal{B}^{(r;a-q;2)}_{n,n-a}(s) \right) \right) \Bigg|_{s=0}$$

$$\mathcal{B}^{(0)}_{n,t}(s) \equiv \frac{d^n}{d\tau^n} \left(\frac{1}{n![\eta|\eta' K|\eta]^n} \frac{(2\eta \cdot K)^{t+1}}{(t+1)(K^2)^{t+1}} \frac{\prod_{j=1}^{n+k} \langle \ell | R_j(u)(K+s\eta)|\ell\rangle}{\langle \ell\, \eta\rangle^{n+1} \prod_{p=1}^{k} \langle \ell | Q_p(u)(K+s\eta)|\ell\rangle} \right) \Bigg|_{\substack{|\ell\rangle \to |K-\tau\eta'|\eta] \\ \tau=0}}$$

$$\mathcal{B}^{(r;b;1)}_{n,t}(s,u) \equiv \frac{1}{b!(\sqrt{1-u})^{b+1} \sqrt{\Delta(q_r,K)}^{b+1} \langle P_{(q_r,K);1}\, P_{(q_r,K);2}\rangle^b}$$

$$\frac{d^b}{d\tau^b} \left(\frac{1}{(t+1)} \frac{\left\langle P_{(q_r,K);1} - \tau P_{(q_r,K);2} | \eta | P_{(q_r,K);1} \right]^{t+1}}{\left\langle P_{(q_r,K);1} - \tau P_{(q_r,K);2} | K | P_{(q_r,K);1} \right]^{t+1}} \right.$$

$$\times \frac{\langle P_{(q_r,K);1} - \tau P_{(q_r,K);2} | Q_r(u)\eta | P_{(q_r,K);1} - \tau P_{(q_r,K);2}\rangle^b}{\langle P_{(q_r,K);1} - \tau P_{(q_r,K);2} | \eta K | P_{(q_r,K);1} - \tau P_{(q_r,K);2}\rangle^{n+1}}$$

$$\left. \times \frac{\prod_{j=1}^{n+k} \langle P_{(q_r,K);1} - \tau P_{(q_r,K);2} | R_j(u)(K+s\eta) | P_{(q_r,K);1} - \tau P_{(q_r,K);2}\rangle}{\prod_{p=1,p\neq r}^{k} \langle P_{(q_r,K);1} - \tau P_{(q_r,K);2} | Q_p(u)(K+s\eta) | P_{(q_r,K);1} - \tau P_{(q_r,K);2}\rangle} \right) \Bigg|_{\tau=0}$$

Bubble coefficients are polynomials

Start over from spinor integrand. Coefficient is sum of residues of

$$\sum_{q=0}^{n} \frac{1}{q!} \frac{d^q B_{n,n-q}(s)}{ds^q}\bigg|_{s=0},$$

where

$$B_{n,t}(s) \equiv \frac{\langle \ell|\eta|\ell]^t}{\langle \ell|K|\ell]^{2+t}} \frac{\prod_{j=1}^{n+k} \langle \ell|R_j(K-s\eta)|\ell\rangle}{\langle \ell|\eta K|\ell\rangle^n \prod_{p=1}^{k} \langle \ell|Q_p(K-s\eta)|\ell\rangle}.$$

- Reduce to cases with $k \leq 2$.
- Expand power series in $\sqrt{1-u}$.
- Case by case, match odd powers to spurious terms of Ossola, Papdopoulos, Pittau. (Pure 4d analysis; correspond between spinor and vector forms of integrand.)

Box coefficients are polynomials

Reduce to cases $k = 2, k = 3$.

First, $k = 2$:

$$\frac{(K^2)^{2+n}}{2}\left(\frac{\prod_{s=1}^{n+2}\langle P_{(Q_j(u),Q_i(u));1}(u)|R_s(u)|P_{(Q_j(u),Q_i(u));2}(u)]}{\langle P_{(Q_j(u),Q_i(u));1}(u)|K|P_{(Q_j(u),Q_i(u));2}(u)]^{n+2}} + \{1 \leftrightarrow 2\}\right).$$

The spinors are the complicated part. Can expand $R(u)$ in other vectors:

$$\begin{aligned} R_s(u) \quad = \quad & -\sqrt{1-u}\, a_0^{(q_i,q_j,K;p_s)} q_0^{(q_i,q_j,K)} + a_i^{(q_i,q_j,K;p_s)} Q_i(u) \\ & + a_j^{(q_i,q_j,K;p_s)} Q_j(u) + \beta_s^{(q_i,q_j,K;p_s)} K \end{aligned}$$

First term in parentheses then looks like:

$$\sum_{h=0}^{n+2} C_h^{(q_i,q_j,K)} \frac{(-\sqrt{1-u})^h \langle P_{ji,1}(u)|q_0|P_{ji,2}(u)]^h}{\langle P_{ji,1}(u)|K|P_{ji,2}(u)]^h}$$

Box coefficients are polynomials

$$\frac{(-\sqrt{1-u})^h \langle P_{ji,1}(u)|q_0|P_{ji,2}(u)]^h}{\langle P_{ji,1}(u)|K|P_{ji,2}(u)]^h} + \frac{(-\sqrt{1-u})^h \langle P_{ji,2}(u)|q_0|P_{ji,1}(u)]^h}{\langle P_{ji,2}(u)|K|P_{ji,1}(u)]^h}$$

$$= \frac{(-\sqrt{1-u})^h \left[\left(2i(1-u)\sqrt{\Delta(Q_i(u), Q_j(u))} q_0^2 K^2 \right)^h + \left(-2i(1-u)\sqrt{\Delta(Q_i(u), Q_j(u))} q_0^2 K^2 \right)^h \right]}{\left(K^2(1-u)^2[(2q_i \cdot q_j)^2 - 4q_i^2 q_j^2] \right)^h}$$

Vanishes unless h is even!

$$\Delta(Q_i(u), Q_j(u)) \quad = \quad (1-u)\left\{ (1-u)[(2q_i \cdot q_j)^2 - 4q_i^2 q_j^2] + 4K^2[\alpha_i \alpha_j (2q_i \cdot q_j) - \alpha_i^2 q_j^2 - \alpha_j^2 q_i^2] \right\}.$$

Denominator factor is cancelled, and degree is $[(n+2)/2]$.

This was $k = 2$. Now do $k = 3$.

Box coefficients are polynomials

The case $k = 3$. Consider:

$$\frac{\prod_{s=1}^{n+3} \langle P_{ji,1}(u)|R_s(u)|P_{ji,2}(u)]}{\langle P_{ji,1}(u)|K|P_{ji,2}(u)]^{n+2} \langle P_{ji,1}(u)|Q_t(u)|P_{ji,2}(u)]} + \{P_{ji,1}(u) \leftrightarrow P_{ji,2}(u)\}.$$

Now,

$$\begin{aligned} \langle P_{ji,1}(u)|R_s(u)|P_{ji,2}(u)] &= a_t^{(q_i,q_j,q_t;p_s)} \langle P_{ji,1}(u)|Q_t(u)|P_{ji,2}(u)] \\ &\quad + \beta_s^{(q_i,q_j,q_t;p_s)} \langle P_{ji,1}(u)|K|P_{ji,2}(u)] \end{aligned}$$

So, the term is

$$\sum_{h=0}^{n+3} C_h^{(q_i,q_j,q_t)} \frac{\langle P_{ji,1}(u)|Q_t(u)|P_{ji,2}(u)]^h}{\langle P_{ji,1}(u)|K|P_{ji,2}(u)]^{h-1} \langle P_{ji,1}(u)|Q_t(u)|P_{ji,2}(u)]}$$

For $h > 0$, there is a cancellation, reducing to the $k = 2$ box case.

The $h = 0$ term is exactly the pentagon contribution.

Coefficients are polynomials in u

The (maximum) degrees are the following:

Pentagon: 0

Box: $[(n+2)/2]$

Triangle: $[(n+1)/2]$

Bubble: $[n/2]$

$$C = c\int d^{4-2\epsilon}p\,\frac{\prod_{i=1}^{k+n}(-2p\cdot P_i)}{\prod_{j=1}^{k}(p-K_j)^2}\delta(p^2)\delta((p-K)^2)$$

The terms in the polynomials

Analytically,

$$C(u) = \sum_{s=0}^{d} \left(\frac{1}{s!} \frac{d^s C(u)}{du^s} \bigg|_{u \to 0} \right) u^s .$$

Or, numerically,

1. Define

$$C_k \equiv C(u_k),$$

for $(k = 0, ..., d-1)$, where

$$u_k = e^{-2\pi i k/d} .$$

2. Then,

$$C(u) = \sum_{s=0}^{d} \left(\frac{1}{d} \sum_{k=0}^{d-1} C_k \, e^{2\pi i s k/d} \right) u^s .$$

D-dimensional unitarity algorithm

$$\Delta A \;=\; \int_0^1 du\, u^{-1-\epsilon} \int d^4\ell\;\; \delta(\ell^2)\, \delta(\sqrt{1-u}\, K^2 - 2K\cdot\ell)$$

1. 4d cut: get u-dependent coefficients of master integrals.

   u-dependence is polynomial. (RB, Feng, Yang; RB, Feng, Mastrolia)

2. Treat polynomial u-dependence of integrand. Two choices:

   (a) For each term in the polynomial, use shift identities to get coefficients of 4d master integrals.

   (b) Use dimensionally shifted master integrals.

   The u-integral is not done explicitly.

Here, u is like the $\widetilde{q}^2$ of Ossola, Papadopoulos, Pittau; or the s_e^2 of Giele, Kunszt, Melnikov.

Dimensional shift identities

(Anastasiou, RB, Feng, Kunszt, Mastrolia)

$$\mathrm{Bub}^{(n)} = F^{(n)}_{2\to 2}\mathrm{Bub}^{(0)}$$

$$\mathrm{Tri}^{(n)} = F^{(n)}_{3\to 3}\mathrm{Tri}^{(0)} + F^{(n)}_{3\to 2}\mathrm{Bub}^{(0)}$$

$$\mathrm{Box}^{(n)} = F^{(n)}_{4\to 4}\mathrm{Box}^{(0)} + F^{(n)}_{4\to 3}\mathrm{Tri}^{(0)} + F^{(n)}_{4\to 2}\mathrm{Bub}^{(0)}$$

$$F^{(n)}_{2\to 2} = \frac{(-\epsilon)_{\frac{3}{2}}}{(n-\epsilon)_{\frac{3}{2}}}, \quad F^{(n)}_{3\to 3} = \frac{-\epsilon}{n-\epsilon}(1-Z^2)^n,$$

$$F^{(n)}_{4\to 4} = \frac{(-\epsilon)_{\frac{1}{2}}}{(n-\epsilon)_{\frac{1}{2}}}\left(\frac{B}{A}\right)^n,$$

$$F^{(n)}_{3\to 2} = \frac{(-\epsilon)_{\frac{3}{2}}}{n-\epsilon}\sum_{k=1}^{n}\frac{2Z(1-Z^2)^{n-k}}{(k-\epsilon)_{\frac{1}{2}}}$$

$$F^{(n)}_{4\to j} = \frac{D+(Z^2-1)C}{(n-\epsilon)_{\frac{1}{2}}ZA}\sum_{k=1}^{n}\left(\frac{B}{A}\right)^{n-k}(k-1-\epsilon)_{\frac{1}{2}}F^{(k-1)}_{3\to j}$$

Incorporating Masses

(RB, Feng; RB, Feng, Mastrolia)

$$C = c < \int d^{4-2\epsilon}p \, \frac{\prod_{i=1}^{k+n}(-2p\cdot P_i)}{\prod_{j=1}^{k}((p-K_j)^2 - m_j^2)} \delta(p^2 - M_1^2)\delta((p-K)^2 - M_2^2)$$

We define the following four-vectors:

$$Q_j = -\left((1-2z) + \frac{M_1^2 - M_2^2}{K^2}\right) K_j + \frac{K_j^2 + M_1^2 - m_j^2 - 2zK\cdot K_j}{K^2} K$$

$$R_i = -\left((1-2z) + \frac{M_1^2 - M_2^2}{K^2}\right) P_i - \frac{z(2P_i\cdot K)}{K^2} K,$$

where

$$z = \frac{\alpha - \beta\sqrt{1-u}}{2}$$

$$\alpha = \frac{K^2 + M_1^2 - M_2^2}{K^2},$$

$$\beta = \frac{\sqrt{(K^2)^2 + (M_1^2)^2 + (M_2^2)^2 - 2K^2M_1^2 - 2K^2M_2^2 - 2M_1^2M_2^2}}{K^2}$$

Cut amplitude:

$$\int_0^1 du\, u^{-1-\epsilon} \int \langle \ell\, d\ell \rangle [\ell\, d\ell] \left((1-2z) + \frac{M_1^2 - M_2^2}{K^2} \right) \frac{(K^2)^{n+1}}{\langle \ell | K | \ell]^{n+2}} \frac{\prod_{j=1}^{n+k} \langle \ell | R_j | \ell]}{\prod_{i=1}^{k} \langle \ell | Q_i | \ell]}$$

Cut masters:

$$\begin{aligned}
\Delta I_2 &= \int_0^1 du\, u^{-1-\epsilon} \int \langle \ell\, d\ell \rangle [\ell\, d\ell] \left((1-2z) + \frac{M_1^2 - M_2^2}{K^2} \right) \frac{(K^2)}{\langle \ell | K | \ell]^2} \\
\Delta I_3 &= \int_0^1 du\, u^{-1-\epsilon} \int \langle \ell\, d\ell \rangle [\ell\, d\ell] \left((1-2z) + \frac{M_1^2 - M_2^2}{K^2} \right) \frac{1}{\langle \ell | K | \ell]\, \langle \ell | Q | \ell]} \\
\Delta I_4 &= \int_0^1 du\, u^{-1-\epsilon} \int \langle \ell\, d\ell \rangle [\ell\, d\ell] \left((1-2z) + \frac{M_1^2 - M_2^2}{K^2} \right) \frac{(K^2)^{-1}}{\langle \ell | Q_1 | \ell]\, \langle \ell | Q_2 | \ell]}
\end{aligned}$$

Again, the formulas for integral coefficients will look the same!

(See also: Kilgore)

Polynomial proof carries over. (RB, Feng, Mastrolia.)

Coefficients of tadpoles can be fixed by cutting an artificial extra propagator.

(Feng's talk)

Summary

- Systematic unitarity method exists at one loop, thanks to solid understanding of an integral basis.
- Spinor integration gives analytic formulas for integral coefficients.
- Generalizes to a D-dimensional unitarity method (with external lines in 4d).