Leading Singularities of the Two-Loop Six-Particle $\mathcal{N} = 4$ Yang-Mills Amplitude

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Z. Bern, L. Dixon, D. Kosower, R. Roiban, M. Spradlin, C. Vergu, AV, arxiv:0803.1465

F. Cachazo, M. Spradlin, AV, arxiv:0805.4832

also some work in progress: C. Kalousios, C. Wen

Scattering Amplitudes

- Gluon scattering amplitudes in QCD and supersymmetric gauge theories are very difficult to compute, so this is a fertile ground for new insights and methods.

- We have learned that Feynman diagrams are not the most efficient way to calculate scattering amplitudes: too messy+too many terms+hide the structure of amplitudes.

- There has been a lot of progress on tree amplitude calculations. [Bern, Dixon, Kosower] [Witten] [Cachazo, Svrcek, Witten] [Britto, Cachazo, Feng] [Roiban, Spradlin, AV] [Brandhuber, Spence, Travaglini] [Dixon, Glover, Khoze] [Bern, Dixon, Kosower] [Arkani-Hamed, Kaplan] [many others]

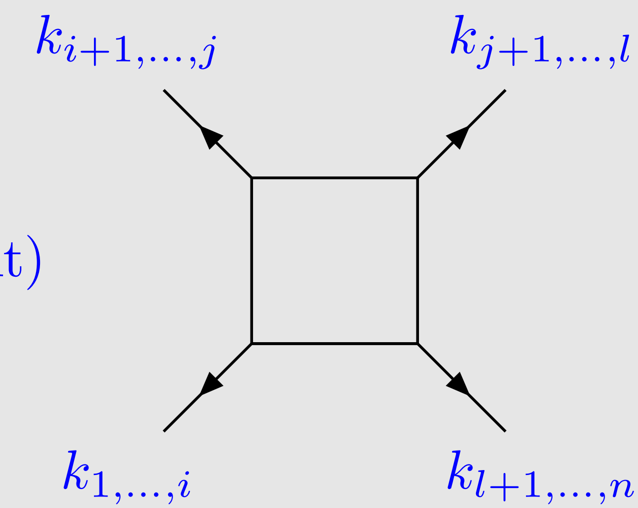
All tree level perturbative amplitides are under control.

One-Loop Amplitudes

In the $\mathcal{N} = 4$ theory, the problem of computing any one-loop amplitude can be reduced to that of computing tree amplitudes. (1990-2004)

Scalar box integrals provide a complete basis for all one-loop gluon amplitudes in $\mathcal{N} = 4$ [Bern, Dixon, Kosower].

$$\mathcal{A}^{1-\text{loop}} = \sum_{\text{boxes}} (\text{coefficient})$$



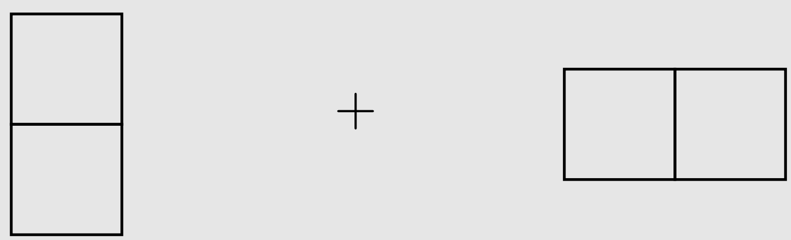
Generalized unitarity methods can be used to determine the coefficients for a desired amplitude [Bern, Dixon, Kosower][Britto, Cachazo, Feng].

Higher Loops

Unitarity based methods for computing the coefficients can be generalized to higher loop amplitudes [Bern, Dixon, Smirnov, 2005] [Buchbinder, Cachazo, 2005] [Bern, Czakon, Dixon, Kosower, Smirnov, 2006] [Bern, Carrasco, Johansson, Kosower, 2007]

Unfortunately a complete basis of integrals is not known even for all two-loop amplitudes...

For example, the two-loop four-particle amplitude is given by the sum of only two scalar integrals [Bern, Rozowsky, Yan, 1997]



But in general it is not trivial to determine which integrals contribute to any particular amplitude.

The Method

In my talk I will describe a method called the leading singularity method [Cachazo, 2008], [Cachazo, Spradlin, AV, 2008], which is a refinement of [Buchbinder, Cachazo, 2005], [Bern, Carrasco, Johansson, Kosower, 2007] [Cachazo, Skinner, 2008]. Via this set of techniques,

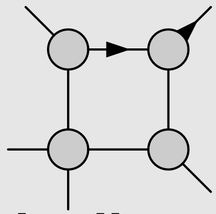
- a natural basis of integrals is provided (does not coincide with dual conformally invariant basis),

- the coefficients are determined by solving simple linear equations,

- and these linear equations are easy to write down by hand (for MHV at least).

Basic idea: Feynman diagrams possess singularity which must be reproduced by any representation of the amplitude in terms of simpler integrals.

For example: for one-loop, five-particles, there are 2×5 different points in $\mathbb{C}^4$ where the integrand has an order-4 pole. Diagramatically:



and four cyclic permutations

This diagram represents the set

$$S = \{\ell \in \mathbb{C}^4 : \ell^2 = 0, (\ell - k_1)^2 = 0, (\ell - k_1 - k_2)^2 = 0, (\ell + k_5)^2 = 0\}$$

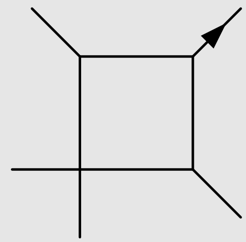
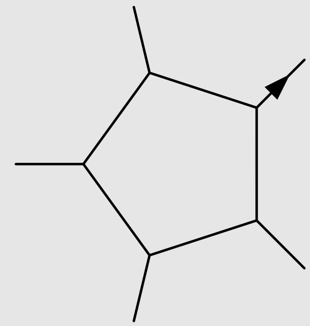
which consists of two distinct points $\{\ell^{(1)}, \ell^{(2)}\}$ (for generic external momenta).

The residue of the amplitude at any singularity is obtained by multiplying tree amplitudes, summed over all allowed internal helicities:

$$\sum_h A^{\text{tree}} A^{\text{tree}} A^{\text{tree}} A^{\text{tree}} \Big|_{l^{(1)}} = 0,$$

$$\sum_h A^{\text{tree}} A^{\text{tree}} A^{\text{tree}} A^{\text{tree}} \Big|_{l^{(2)}} = A_5^{\text{tree}}$$

By comparing to the ansatz

B  $+ P$ 

for some coefficients B and P, we find 2 equations

$$B + \frac{P}{(\ell^{(1)} + k_5 + k_4)^2} = 0, \qquad B + \frac{P}{(\ell^{(2)} + k_5 + k_4)^2} = A_5^{\text{tree}}$$

which determine the coefficients. [Cachazo, 2008]

From each pole we get an equation!

Reduction of a pentagon to a sum of boxes with particular coefficients!

[Bern, Dixon, Kosower, 1993]

The Target

Much of what I will say in this talk will be more general, but the specific target of our calculation is the two-loop six-particle MHV amplitude in $\mathcal{N} = 4$ super-Yang Mills theory.

The parity-even part of this amplitude was recently presented in [Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, AV, 2008]

The parity-odd part was presented in [Cachazo, Spradlin, AV, 2008]

In fact the helicity information (MHV versus non-MHV) appears only in the homogeneous terms of linear equations, so much of the work done for the MHV amplitude can be applied directly to NMHV [in progress]

Three-loop five-point amplitude [Spradlin, AV, Wen, to appear]

Background: Calculation of Amplitudes

Any L-loop scattering amplitude can, in principle, be obtained by summing over all Feynman diagrams:

$$\mathcal{A}^{(L)}(p) = \int d\ell_1 \cdots d\ell_L \sum_j F_j(p,\ell) \tag{1}$$

$p =$ external momenta

$\ell =$ loop momenta

However, in practice this is a hopeless exercise due to the enormously large number of Feynman diagrams and their complexity in Yang-Mills theory.

Background: Calculation of Amplitudes

$$\mathcal{A}^{(L)}(p) = \int d\ell_1 \cdots d\ell_L \sum_j F_j(p, \ell) \tag{2}$$

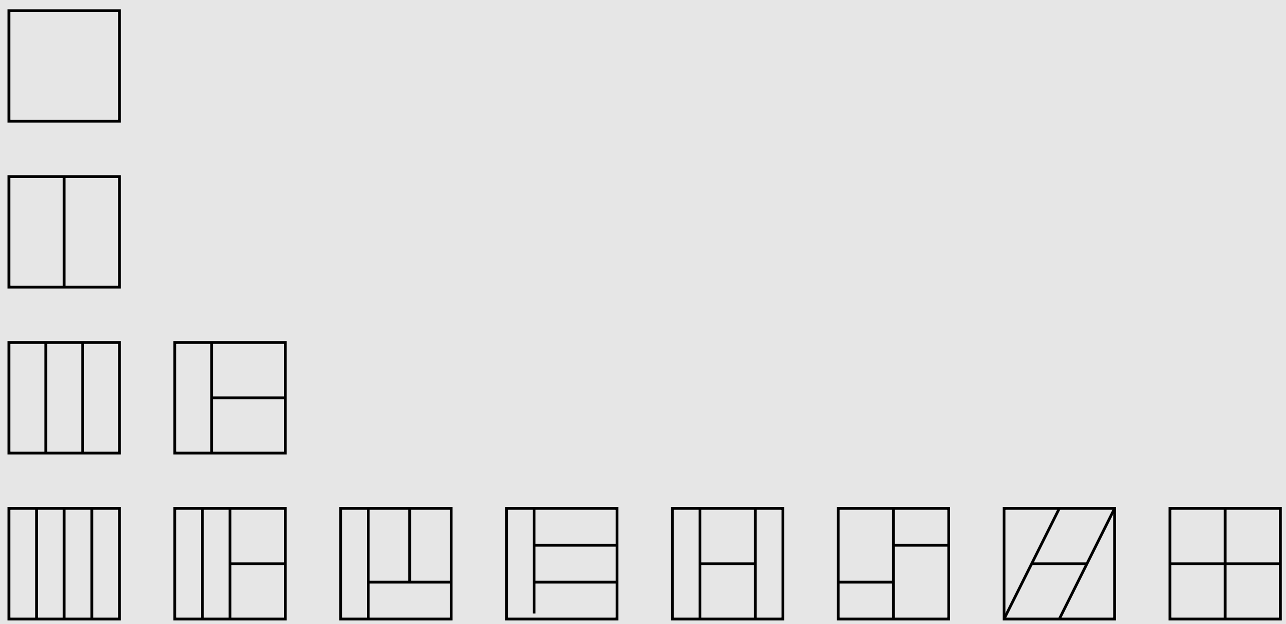
Rather, calculations typically proceed by first finding a representation of the amplitude in terms of a relatively simple basis of integrals $\{I_i\}$:

$$\mathcal{A}^{(L)}(p) = \sum_i c_i(p) \int d\ell_1 \cdots d\ell_L \, I_i(p, \ell) \tag{3}$$

where the coefficients $c_i(p)$ are computed by other means, such as the unitarity-based method [Bern, Dixon, Kosower, 1990s] or maximal cuts [Buchbinder, Cachazo] [Bern, Carrasco, Johansson, Kosower, 2007] .

Example: Four external particles

For example, unitarity based methods were used to express the four-particle amplitude in $\mathcal{N} = 4$ Yang-Mills as the sum of the following scalar integrals:



[Bern, Rozowsky, Yan, 1997] [Bern, Dixon, Smirnov, 2005]

[Bern, Czakon, Dixon, Kosower, Smirnov, 2006]

A Difficulty

One important difficulty is that there is no known basis of integrals in the general case. Only in some special cases is a basis known:

• one-loop, any number of external particles;

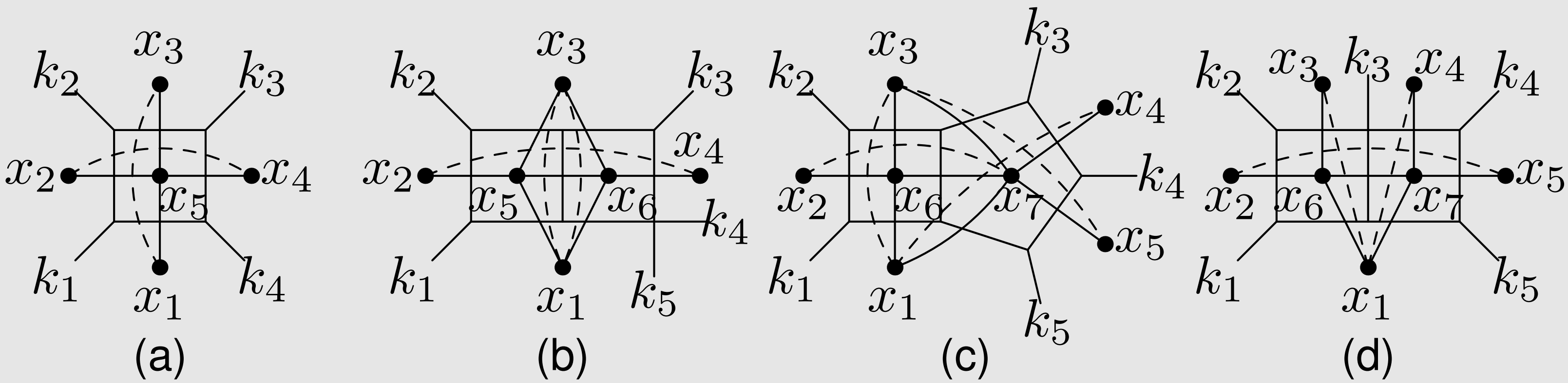
⇒ scalar box integrals, as discussed above...

• and a very plausible conjecture exists for four particles at any number of loops which has emerged from the work of [Bern, Czakon, Dixon, Drummond, Henn, Korchemsky, Kosower, Smirnov, Sokatchev, and others, 2006-2007].

⇒ dual conformal integrals

• higher number of particles? Parity odd parts of two-loop amplitudes are not given in terms of dual conformally invariant integrals. [Bern, Dixon, Kosower, Roiban, Smirnov] [Cachazo, Spradlin, AV]

Examples of Dual Conformal Integrals

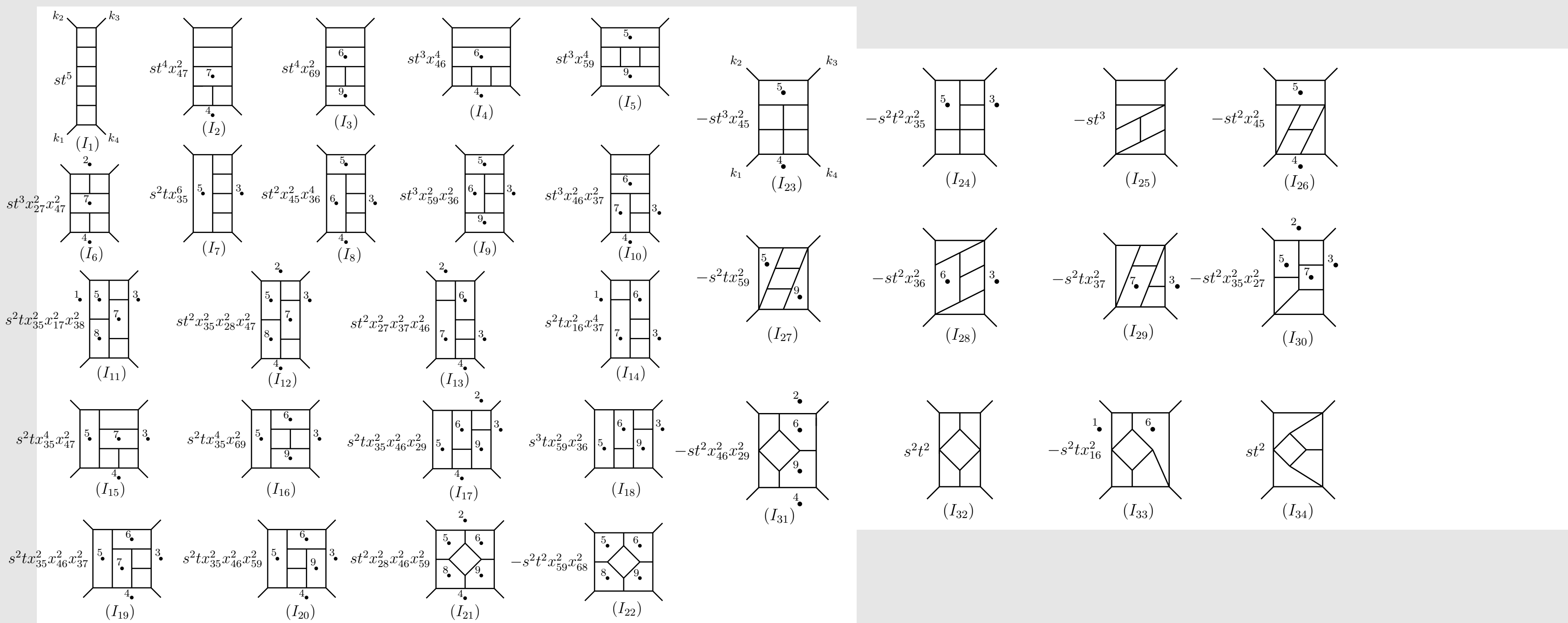


- **Points x_i label the vertices of the dual graph, a solid line connecting two points x_i and x_j corresponds to a factor of $1/x_{ij}^2$, while a dashed line corresponds to a factor of x_{ij}^2.**

- **An integral is dual conformal invariant if the difference between the number of solid lines and dashed lines at a vertex equals 4 at the internal vertices and 0 at the external vertices.**

Dual Conformal Invariant Diagrams at Five Loops

[Bern, Carrasco, Johansson, Kosower, 2007]



Leading Singularity Method

Each individual singularity provides enough linear equations to determine higher-point and multi-particle equations:

each pole gives a different equation!

- **Contours: inhomogenous part of equations**
- **Geometric integrals: homogeneous part of equations**

Determining the Integrand

The idea is to look at the equation

$$\sum_i c_i(p) \int d\ell_1 \cdots d\ell_L \, I_i(p,\ell) = \int d\ell_1 \cdots d\ell_L \sum_j F_j(p,\ell) \qquad (4)$$

at the level of the integrand, and instead of integrating over the real ℓ-axis in $\mathbb{C}^{4L}$ we integrate over closed contours $\Gamma \subset \mathbb{C}^{4L}$ to obtain linear equations for the desired coefficients!

$$\sum_i c_i(p) \int_\Gamma I_i(p,\ell) = \int_\Gamma \sum_j F_j(p,\ell) \qquad (5)$$

We can require this to be true for any contour Γ. By choosing many different contours, we get many different linear equations!

Each contour computes the residue of the integrand on only one of the many isolated singularities.

Choice of Contours

Which contours give the most useful equations?

If we choose a random contour in $\mathbb{C}^{4L}$, we get the useless equation

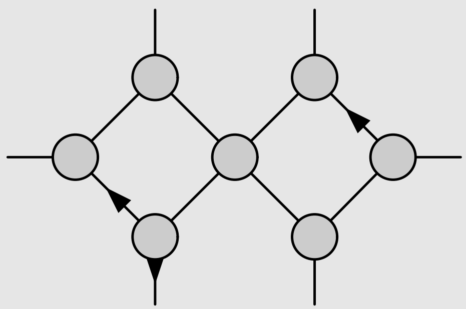
$$0 = 0$$

In order to get useful equations, we should identify the isolated poles of the integrand, and for each one we consider a contour $\Gamma \subset \mathbb{C}^{4L}$ so that integrating over this contour computes the residue at the corresponding pole.

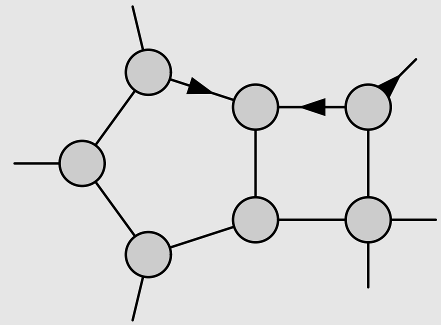
Fortunately it is easy to identify the isolated poles of the integrand:

The poles in Feynman diagrams occur when internal propagators go on-shell.

For the two-loop six-particle amplitude there are five obvious topologies associated with singularities where eight different propagators are going simultaneously on-shell, and hence can be associated with T^8 contours in $\mathbb{C}^8$:



(A)



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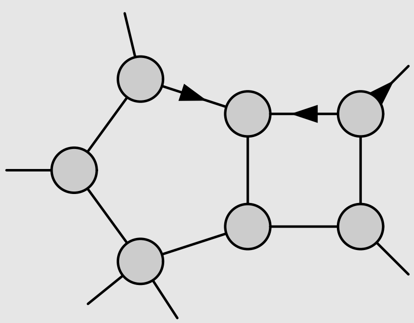

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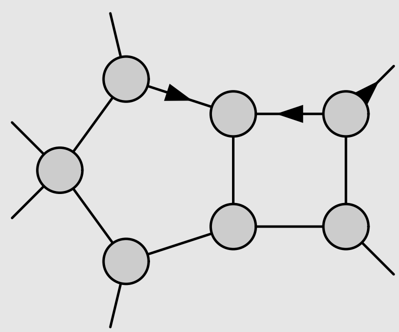

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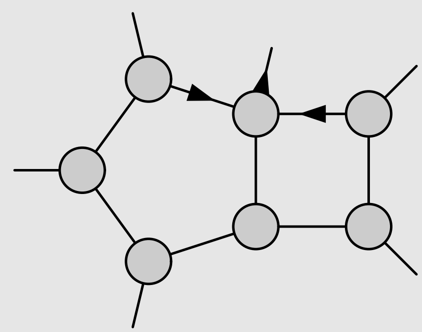
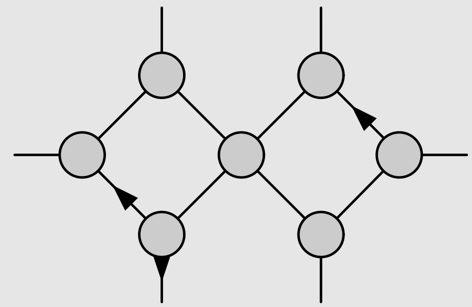


(E)

For example, if we look at the first diagram:



(A)

it represents the sum over the subset of all Feynman diagrams which contain all eight of the indicated propagators.

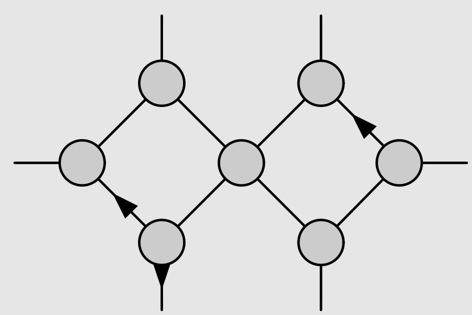
This set of Feynman diagrams has isolated poles at

$$
\begin{aligned}
S \quad = \quad \{(\ell_1, \ell_2) \in \mathbb{C}^8 : \ & \ell_1^2 = 0, (\ell_1 + p_1)^2 = 0, (\ell_1 - p_2)^2 = 0, \\
& (\ell_1 - p_2 - p_3)^2 = 0, \ell_2^2 = 0, (\ell_2 - p_4)^2 = 0, \\
& (\ell_2 + p_5)^2 = 0, (\ell_2 + p_5 + p_6)^2 = 0\}
\end{aligned}
$$

For generic momenta p_i this consists of four distinct points in $\mathbb{C}^8$.

At each of these four points, the amplitude has an isolated order-8 pole.

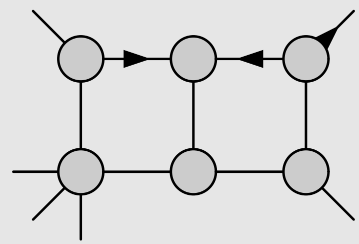
To calculate the residue at this pole (i.e., the result of integrating over the corresponding contour Γ) is simple: just take the product of seven on-shell tree-level amplitudes, at each of the grey circles, and evaluate this product at the corresponding solution (ℓ_1, ℓ_2). [Bern, Dixon, Kosower] [Buchbinder, Cachazo] [Bern, Carrasco, Johansson, Kosower]



(A)

$\Rightarrow$ From this topology, we get four different equations; one from each of the four poles.

The less obvious contours with only seven propagators have more subtle leading singularities. To see how these singularities arise, consider the topology:

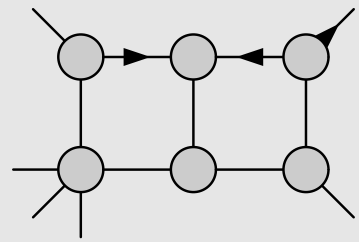


(F)

Although it looks like there is only a pole of order 7, not 8, there is in fact another hidden singularity.

We need the 8th condition because without it we would still have one unfixed loop integral; for $n = 4$ particle amplitudes this last integral is trivial but for $n > 4$ it is not! [Bern, Carrasco, Johansson, Kosower, 2007])

To expose it, consider a contour integral which computes the residue at either of the two singularities of the right-hand box:



(F)

To expose it, consider a contour integral which computes the residue at either of the two singularities of the right-hand box:

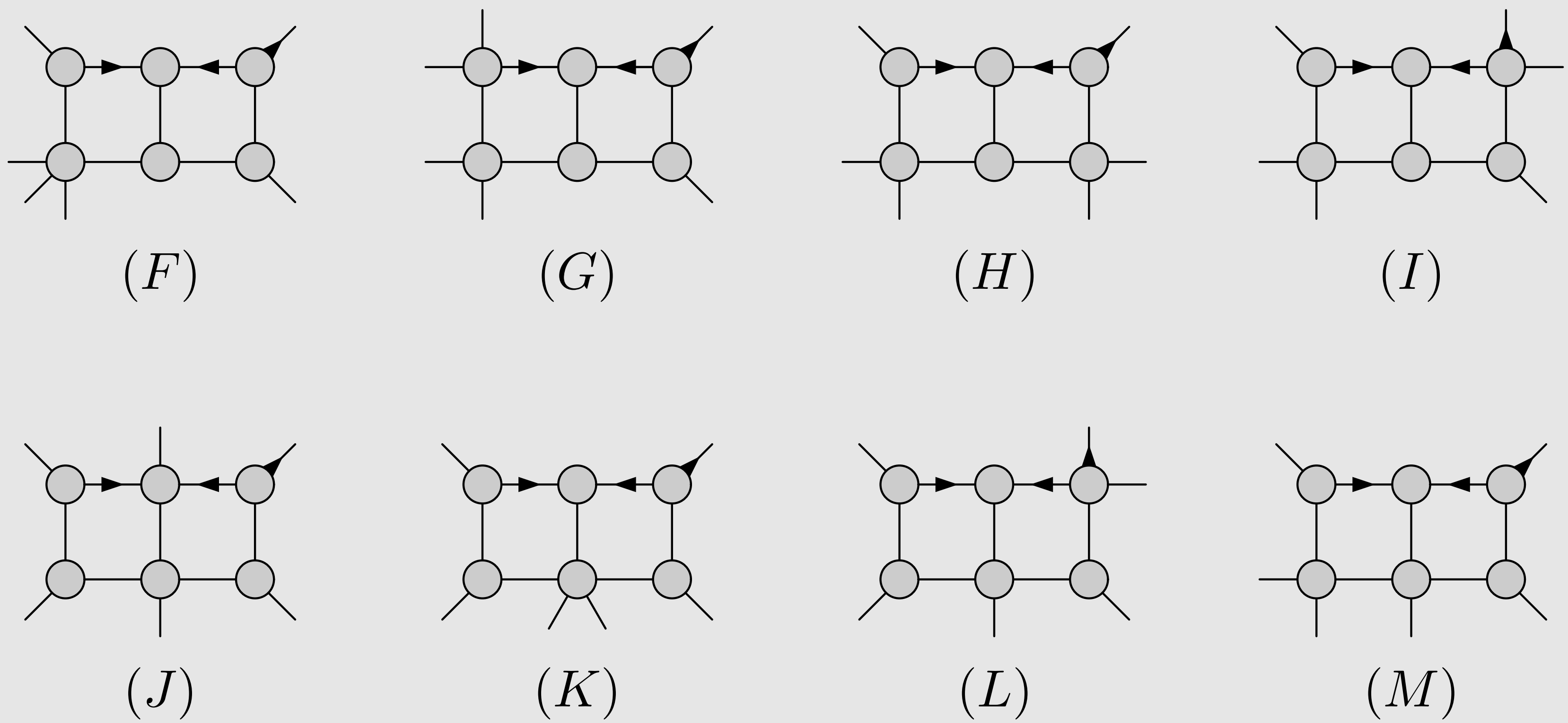
$$\int_\Gamma d^4\ell_2 \frac{1}{\ell_2^2(\ell_1+k_1)^2(\ell_1+k_1+k_2)^2(\ell_1+\ell_2)^2} = \frac{1}{2}\frac{1}{(k_1+k_2)^2(\ell_2-k_1)^2}$$

where the right-hand side is just the Jacobian evaluated at the location of the singularity. Now this Jacobian has itself another singularity

$1/(\ell_2-k_1)^2$.

The conclusion is that there do exist isolated poles of order 8 in such topologies. The residues at these poles can be computed by integrating over appropriate contours Γ.

There are a total of 8 different topologies of this type:



As before, each of these topologies represents several different leading singularities (solutions of the 8 on-shell conditions), each of which gives rise to a different linear equation ...

Contructing a Basis of Integrals

Next we need to construct a set of integrals $\{I_i\}$ in terms of which to express the amplitude.

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The construction proceeds as follows:

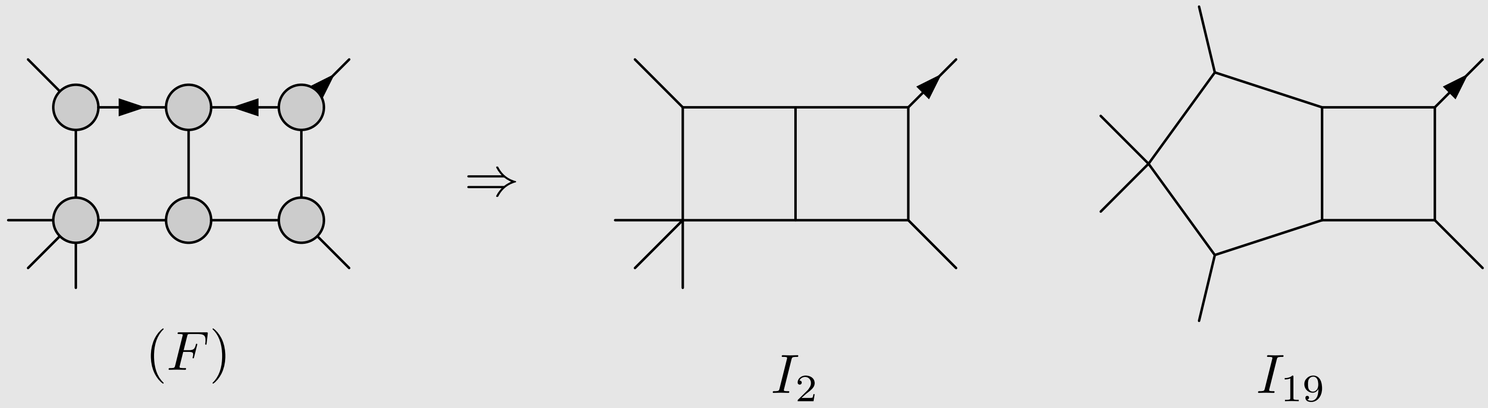
We begin with a set that just contains the 13 scalar integrals appropriate to the 13 different topologies shown on the previous slides.

It turns out that with just this set of integrals, the linear equations have no solution, so we must add additional integrals to the set $\{I_i\}$

There is a systematic procedure to do this, which ends when one is able to solve all of the equations...

We can determine all the integrals and coefficients except the terms which vanish in $D = 4$ ($\mu-$terms).

Example: Topology F



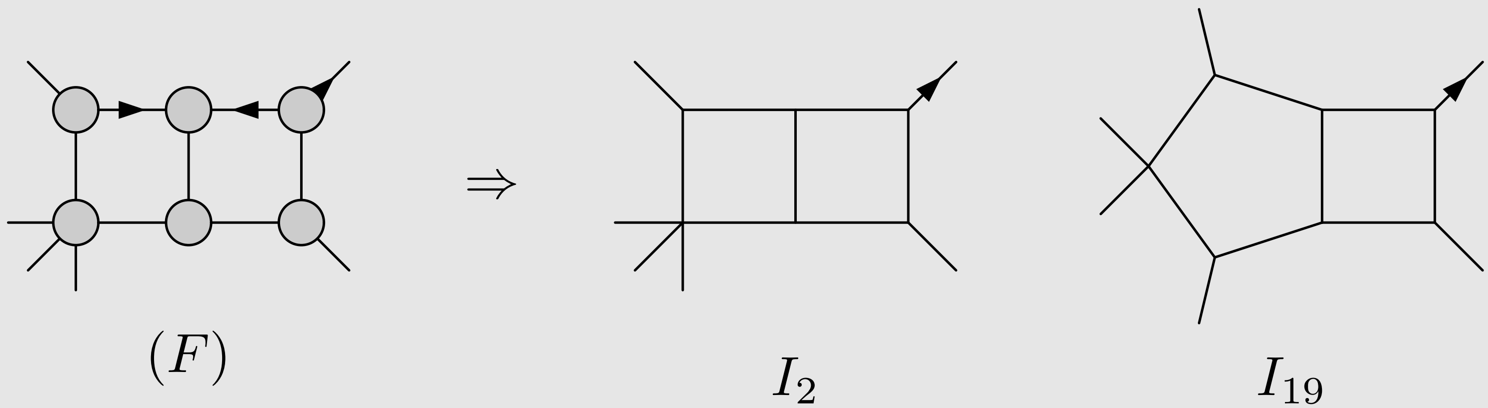
This topology gives two different equations:

$$4s_{12}^2 s_{61} = c_2 + \frac{c_{19}}{(p^{(1)} + k_{456})^2}\,, \qquad 0 = c_2 + \frac{c_{19}}{(p^{(2)} + k_{456})^2}\,. \tag{6}$$

$$p^{(1)} = \frac{\langle 2,1\rangle}{\langle 2,6\rangle}\lambda_6\widetilde{\lambda}_1, \quad p^{(2)} = \frac{[2,1]}{[2,6]}\lambda_1\tilde{\lambda}_6.$$

Note that if we did not have c_{19} there would be a contradiction (no solution)!

Example: Topology F



Solving this 2×2 linear system gives the coefficients c_2 and c_{19}. Note that the even and odd-parity parts are determined simultaneously:

$$\frac{1}{2}(c_2 + \bar{c}_2) = -2 s_{16} s_{12}^2 \,, \qquad \frac{1}{2}(c_2 - \bar{c}_2) = 2 s_{16} s_{12}^2 \left(\frac{a+1}{a-1}\right) \,,$$

$$\frac{1}{2}(c_{19} + \bar{c}_{19}) = 0 \,, \qquad \frac{1}{2}(c_{19} - \bar{c}_{19}) = 4 s_{12}^2 s_{61} \frac{(p^{(1)} + k_{456})^2}{1-a} \,. \tag{7}$$

where $a = \frac{(p^{(1)}+k_{456})^2}{(p^{(2)}+k_{456})^2}$

Constructing a Basis of Integrals

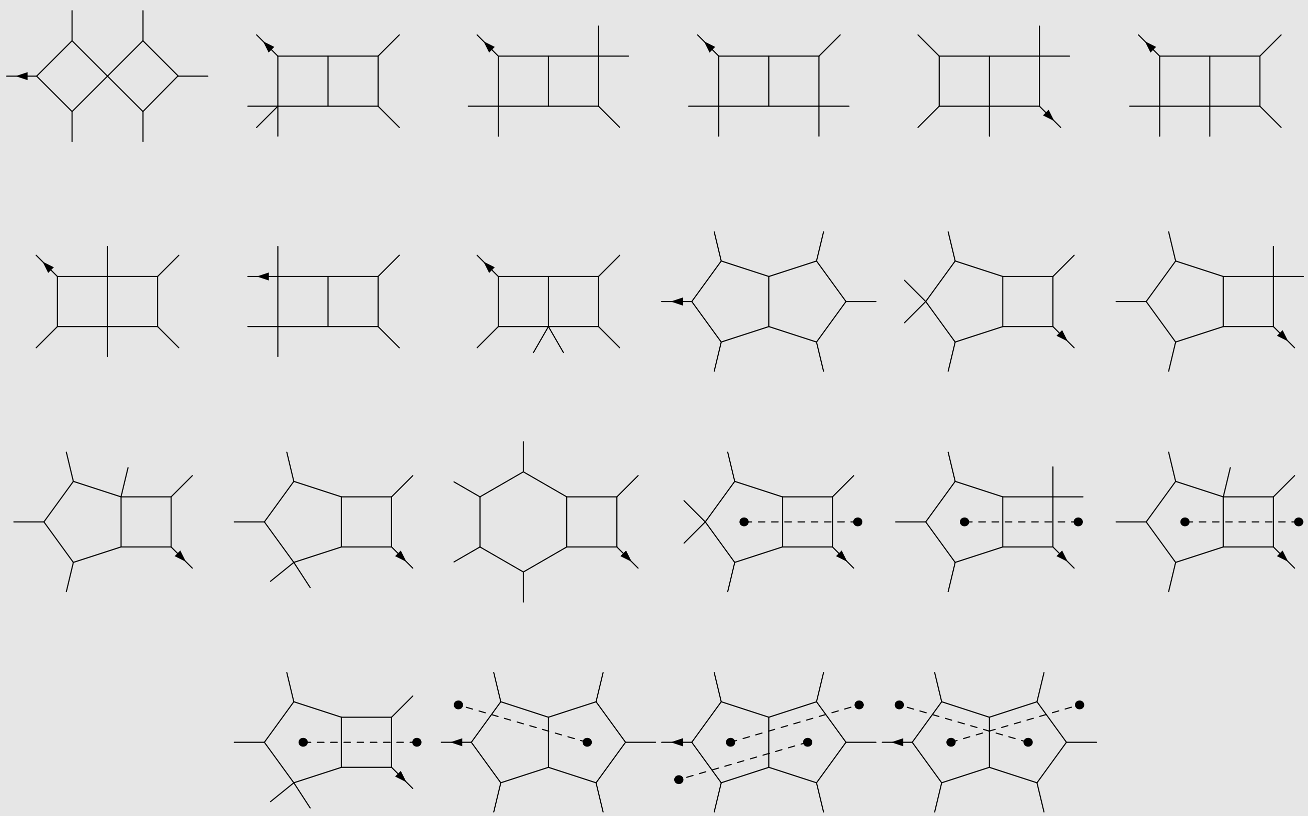
It can happen that when this procedure finishes, one ends up with a set of integrals $\{I_i\}$ that is overcomplete.

This happens because loop integrals for 6 or more external particles can frequently be expressed as linear combinations of other integrals. [van Neerven and Vermaseren, 1984][Bern, Dixon, Kosower, 1993] .

If this happens, then the equations do not have a unique solution: given any solution $\{c_i\}$, one can add any set of coefficients $\{\tilde{c}_i\}$ that is actually zero due to a reduction identity.

Result

We find a representation of the 2-loop six-particle MHV amplitude in terms of



(Several of these can actually be set to zero using reduction identities).

Result

The parity-even part of the amplitude agrees with the recent result of [Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, AV, 2008].

The parity-odd part was presented in [Cachazo, Spradlin, AV, 2008]

Note that the full coefficients, both the parity even and parity odd parts, emerge from solving the same linear equations—in in fact it is unnatural to separate the two parts, and we have only done this in order to make the comparison and check our results.

Only the parity even part can be written in a basis with only dual conformal integrals. The leading singularity method naturally provides a set of geometric integrals , which includes also non-dual conformal integrals (these appear in the parity-odd part).

The ABDK/BDS Conjecture

One reason for the recent interest in multi-loop amplitudes in $\mathcal{N} = 4$ super-Yang Mills theory is the ABDK/BDS conjecture, which at two-loops takes the form [Anastasiou, Bern, Dixon, Kosower, 2003]

$$M_n^{(2)}(\epsilon) = \frac{1}{2}(M^{(1)}(\epsilon))^2 - (\zeta(2) + \zeta(3)\epsilon + \zeta(4)\epsilon^2 + \cdots)M^{(1)}(2\epsilon) - \frac{\pi^4}{72} + \mathcal{O}(\epsilon)$$

in dimensional regularization to $D = 4 - 2\epsilon$.

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in dimensional regularization to $D = 4 - 2\epsilon$.

This form is based on explicit computations of two-loop amplitudes for four particles. For five-point amplitude, it has been confirmed by direct calculation [Cachazo, Spradlin, AV, 2006] [Bern, Dixon, Kosower, Roiban, Smirnov, 2006].

BDS Iteration Relations for Multiloop Amplitudes

- This iterative structure together with the exponential nature of IR divergences suggests an all-order resummation should be possible. [Bern, Dixon, Smirnov, 2005]

- Indeed, BDS verified the three-loop generalization for four-particle amplitude by direct calculation, guiding the all-loop order proposal

$$\ln M_n = \sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + \mathcal{O}(\epsilon)\right)$$

where

$$M_n = \sum_{L=0}^{\infty} a^L M_n^{(L)}(\epsilon),$$

$$f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)},$$

$$a = \frac{\lambda}{8\pi^2}(4\pi e^{-\gamma})^{\epsilon}.$$

Why compute two-loop six-point amplitude?

- Alday and Maldacena (2007) have given a prescription for using AdS/CFT to calculate gluon scattering amplitudes at strong coupling.

- Their calculations confirmed the strong coupling prediction from the BDS iteration ansatz for the four-point amplitude and suggested disagreement in the limit of a large number of legs, between the Wilson loop calculation and BDS asnatz.

- Drummond, Henn, Korchemsky, Sokatchev and Brandhuber, Heslop, Travaglini (2007) showed that lowest-order contributions to a light-like rectangular Wilson loop agrees with BDS ansatz for gauge theory amplitudes.

- Either the connection between Wilson loops and the amplitudes breaks down? Or BDS ansatz breaks down beyond five-point amplitudes? To answer, one needs six-point Wilson loop and amplitude calculations!

Two-loop six-point amplitude: Even part

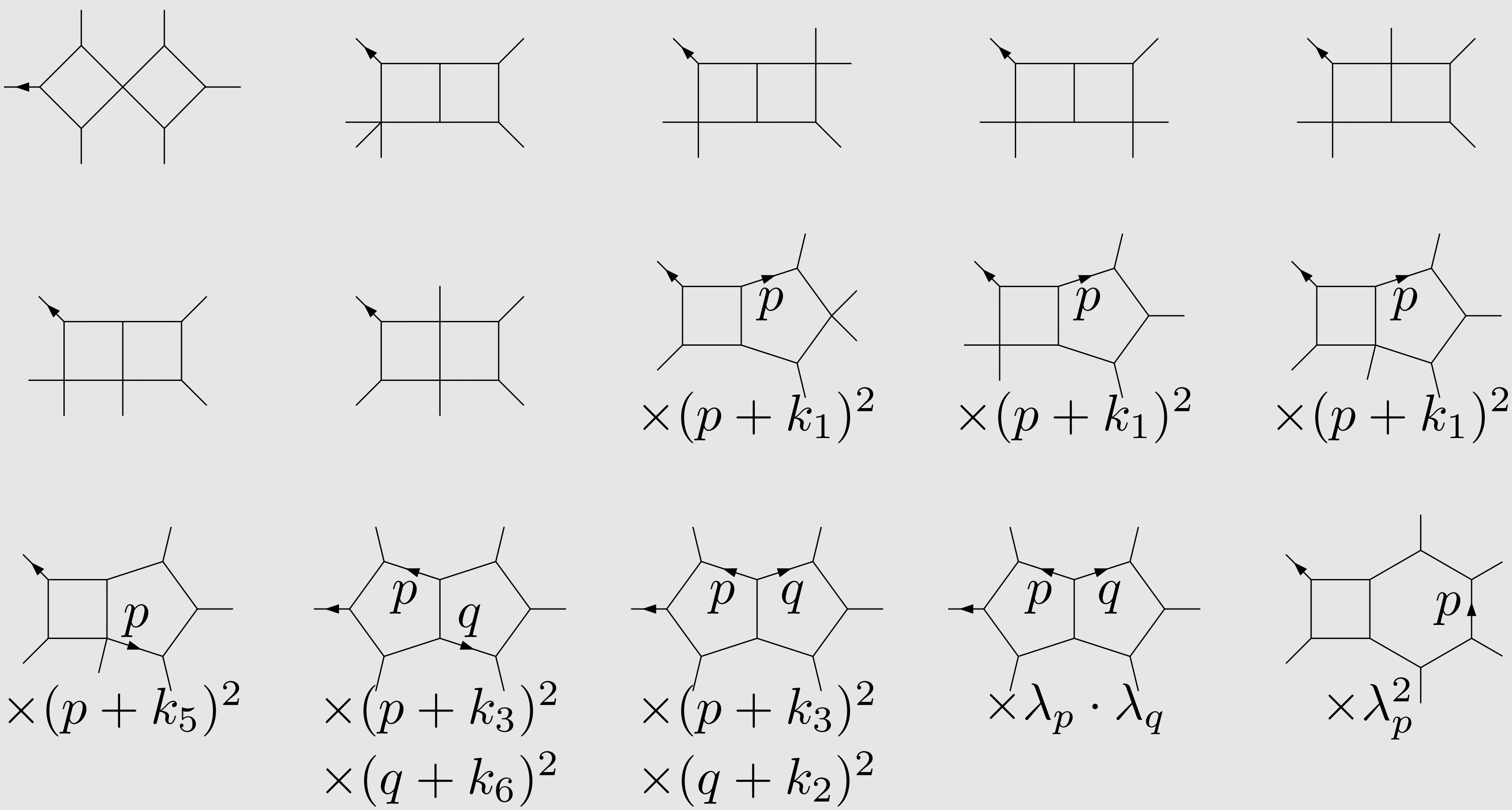
We find the complete expression for the parity-even part of the two-loop six-particle amplitude [Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, AV, 2008]

We performed the calculation using the unitarity-based method, employing a variety of cuts to express the amplitude in terms of selected set of six-point two-loop Feynman integrals.

$$M_6^{(2),D=4-2\epsilon}(\epsilon) = \frac{1}{16}\sum_{i=1}^{15} c_i I^{(i)}(\epsilon)$$

We evaluated the integrals using AMBRE and MB packages and computed the amplitude numerically against BDS ansatz, and against values for the corresponding Wilson loop.

Two-loop six-point amplitude: integrals



Two-loop six-point amplitude: coefficients

$$
\begin{aligned}
c_1 &= s_{61}s_{34}s_{123}s_{345} + s_{12}s_{45}s_{234}s_{345} + s_{345}^2(s_{23}s_{56} - s_{123}s_{234})\,, \\
c_2 &= 2s_{12}s_{23}^2\,, \\
c_3 &= s_{234}(s_{123}s_{234} - s_{23}s_{56})\,, \\
c_4 &= s_{12}s_{234}^2\,, \\
c_5 &= s_{34}(s_{123}s_{234} - 2s_{23}s_{56})\,, \\
c_6 &= -s_{12}s_{23}s_{234}\,, \\
c_7 &= 2s_{123}s_{234}s_{345} - 4s_{61}s_{34}s_{123} - s_{12}s_{45}s_{234} - s_{23}s_{56}s_{345}\,, \\
c_8 &= 2s_{61}(s_{234}s_{345} - s_{61}s_{34})\,, \\
c_9 &= s_{23}s_{34}s_{234}\,, \\
c_{10} &= s_{23}(2s_{61}s_{34} - s_{234}s_{345})\,, \\
c_{11} &= s_{12}s_{23}s_{234}\,, \\
c_{12} &= s_{345}(s_{234}s_{345} - s_{61}s_{34})\,, \\
c_{13} &= -s_{345}^2s_{56}\,, \\
c_{14} &= -2s_{126}(s_{123}s_{234}s_{345} - s_{61}s_{34}s_{123} - s_{12}s_{45}s_{234} - s_{23}s_{56}s_{345})\,, \\
c_{15} &= 2s_{61}(s_{123}s_{234}s_{345} - s_{61}s_{34}s_{123} - s_{12}s_{45}s_{234} - s_{23}s_{56}s_{345})\,.
\end{aligned}
\qquad (8)
$$

An example of an integral

$$
\begin{aligned}
I^{(12)} \quad = \quad & \frac{(-1)^{1+2\eta} e^{2\epsilon\gamma}}{\Gamma(-1-2\epsilon-\eta)\Gamma(\eta)} \int_{-i\infty}^{+i\infty} \cdots \int_{-i\infty}^{+i\infty} \prod_{j=1}^{18} \frac{dz_j}{2\pi i} \Gamma(-z_j) \\
& \times \frac{\Gamma(3+\epsilon+\eta+z_{1,2,3,4,5,6,7,8,9,10})}{\Gamma(4+\epsilon+\eta+z_{1,2,3,4,5,6,7,8,9,10})} \Gamma(1+z_{3,5,9}) \\
& \times (-s_{12})^{z_{8,13}} (-s_{23})^{z_{14}} (-s_{34})^{z_{1,18}} (-s_{45})^{z_{3,15}} (-s_{61})^{z_{11}} \\
& \times (-s_{123})^{z_{9,16}} (-s_{234})^{z_{17}} (-s_{345})^{z_{2,12}} \\
& \times (-s_{56})^{-5-2\epsilon-2\eta-z_{1,2,3,8,9,11,12,13,14,15,16,17,18}} \\
& \times \frac{\Gamma(-3-\epsilon-z_{1,2,3,4,5,6,7})}{\Gamma(-3-3\epsilon-2\eta-z_{1,2,3,8,9,10})} \\
& \times \frac{\Gamma(5+2\epsilon+2\eta+z_{1,2,3,8,9,10,11,12,13,14,15,16,17,18})}{\Gamma(1-z_4)\Gamma(\eta-z_5)\Gamma(-z_6)\Gamma(1-z_7)} \\
& \times \Gamma(-5-2\epsilon-2\eta-z_{1,2,3,6,8,9,10,11,12,13,14,15,16}) \\
& \times \Gamma(-1-\epsilon-\eta+z_{4,5,6,7}-z_{11,12,14,15,17,18}) \\
& \times \Gamma(-2-\epsilon-\eta-z_{1,2,3,8,9,10}) \Gamma(\eta-z_5+z_{14,15,16}) \\
& \times \Gamma(1-z_4+z_{12,13,18}) \Gamma(1+z_{1,2,4,8}) \\
& \times \Gamma(1-z_7+z_{11,12,15}) \Gamma(1+z_{1,6,10}) \Gamma(1+z_{2,3,7}) \Gamma(1+z_{11,14,17})
\end{aligned} \tag{9}
$$

An example of an integral

$$
\begin{aligned}
I^{(12)} = & -\frac{1}{\epsilon^4}\left[\frac{3s_{123}}{(-s_{12})^{1+2\epsilon}\, s_{61}s_{34}s_{45}} + \frac{s_{23}s_{56}}{s_{12}s_{61}s_{34}s_{45}(-s_{234})^{1+2\epsilon}}\right. \\
& \left. + \frac{1}{s_{61}s_{34}(-s_{345})^{1+2\epsilon}}\right] + \frac{1}{\epsilon^3}\left[\frac{s_{123}}{s_{12}s_{61}s_{34}s_{45}} \ln\left(\frac{s_{234}^2 s_{345}^6}{s_{23}s_{34}^3 s_{45}^3 s_{56}}\right)\right. \\
& + \frac{s_{23}s_{56}}{s_{12}s_{61}s_{34}s_{45}s_{234}} \ln\left(\frac{s_{23}s_{56}s_{345}^2}{s_{12}^2 s_{34}s_{45}}\right) + \frac{1}{s_{61}s_{34}s_{345}} \ln\left(\frac{s_{45}s_{234}s_{345}}{s_{23}s_{34}s_{56}}\right) \\
& + \frac{1}{s_{61}s_{34} - s_{234}s_{345}}\, \frac{s_{45}s_{234}s_{12} + 2s_{345}s_{23}s_{56}}{s_{45}s_{234}s_{12}s_{345}} \ln\left(\frac{s_{61}s_{34}}{s_{234}s_{345}}\right) \\
& \left. + \frac{s_{12}s_{45}s_{234} + (s_{23}s_{56} + 3s_{123}s_{234})s_{345}}{s_{12}s_{61}s_{34}s_{45}s_{234}s_{345}} \ln\left(\frac{s_{12}}{s_{61}}\right)\right] + O(\epsilon^{-2}),
\end{aligned}
$$

Two-loop six-point amplitude: Results

[Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, AV, 2008]

- Discrepancy with BDS ansatz.

- But agreement with Wilson loop calculations by [Drummond, Henn, Korchemsky and Sokachev, 2008] !!!

kinematics	(u_1, u_2, u_3)	Δ_A	Δ_W
$K^{(1)}$	$(1/4, 1/4, 1/4)$	-0.0181 ± 0.017	$< 10^{-5}$
$K^{(2)}$	$(0.547253,\ 0.203822,\ 0.88127)$	-2.753 ± 0.012	-2.7553
$K^{(3)}$	$(28/17, 16/5, 112/85)$	-4.74445 ± 0.00653	-4.7446
$K^{(4)}$	$(1/9, 1/9, 1/9)$	4.1161 ± 0.10	4.0914
$K^{(5)}$	$(4/8, 4/81, 4/81)$	9.9963 ± 0.50	9.7255

Two-loop six-point amplitude: Odd part

ABDK/BDS ansatz breaks down for parity-even part of the two-loop six-particle MHV amplitude. [Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, AV, 2008]

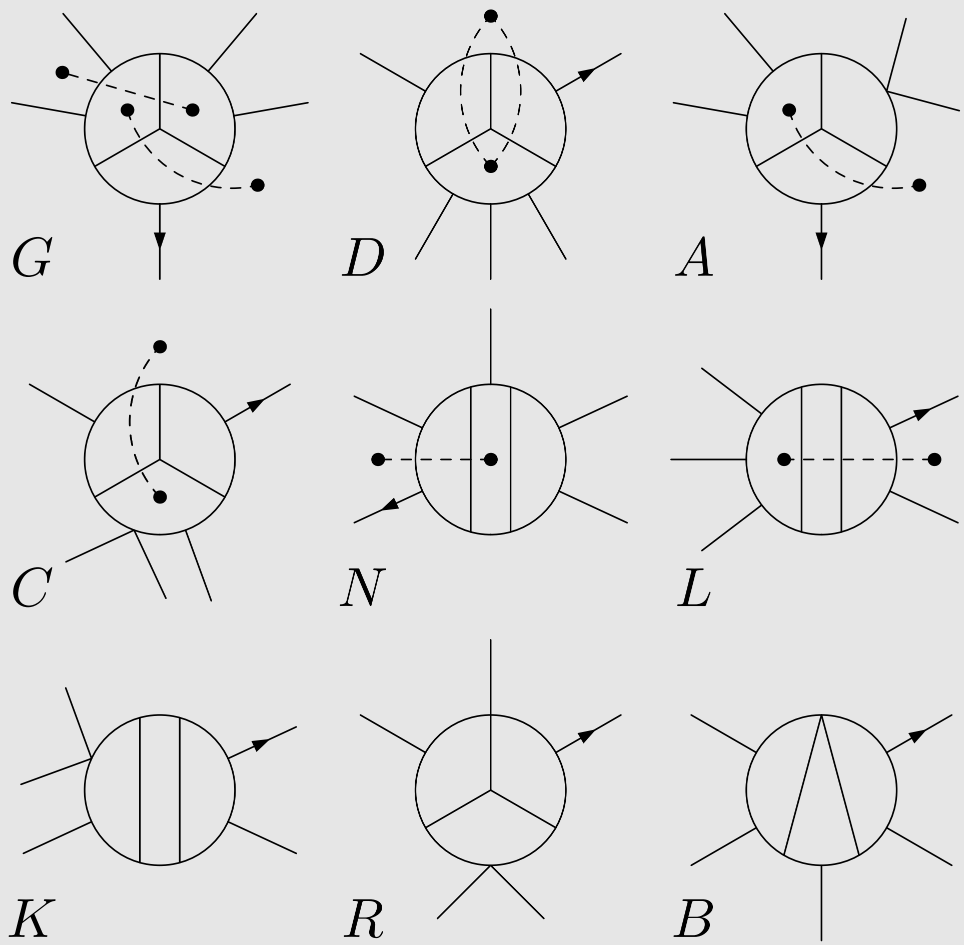
The parity-odd part of the two-loop six-particle amplitude does in fact satisfy ABDK/BDS. [Cachazo, Spradlin, AV, 2008]

For instance, denoting the left- and right-hand sides of ABDK/BDS ansatz by L and R respectively, at randomly generates momenta, we find

$$\begin{aligned} L(\epsilon) &= -\frac{4\times 10^{-16}}{\epsilon^4} + \frac{4\times 10^{-15}}{\epsilon^3} + \frac{1(2)\times 10^{-11}}{\epsilon^2} - \frac{0.430(7)}{\epsilon} - 0.9(1) + \cdots , \\ R(\epsilon) &= -\frac{0.428(2)}{\epsilon} - 0.92(1) + \mathcal{O}(\epsilon) \end{aligned} \tag{10}$$

(It is in fact reasonable to believe that the parity-odd part always satisfies ABDK/BDS, but this remains unproven.)

Three-loop five-point amplitude



The first seven enter the amplitude with coefficient $+1$, the last two with coefficient -1. [Spradlin, AV, Wen, to appear]

Three-loop five-point amplitude

The one, two and three loop "obstructions" are

$$M^{(1)} = -\frac{5}{2}\frac{1}{\epsilon^2} + \frac{5\pi^2}{8} + \frac{179\zeta(3)}{24}\epsilon + \frac{97\pi^4}{1440}\epsilon^2 - \left(\frac{51\pi^2\zeta(3)}{32} - \frac{137\zeta(5)}{8}\right)\epsilon^3 - \cdots$$

$$M^{(2)} = \frac{25}{8}\frac{1}{\epsilon^4} - \frac{35\pi^2}{24}\frac{1}{\epsilon^2} - \frac{865\zeta(3)}{48}\frac{1}{\epsilon} - \frac{97\pi^4}{1152} + \cdots$$

$$M^{(3)} = -\frac{125}{48}\frac{1}{\epsilon^6} + \frac{325\pi^2}{192}\frac{1}{\epsilon^4} + \frac{4175\zeta(3)}{192}\frac{1}{\epsilon^3} + \frac{499\pi^4}{10368}\frac{1}{\epsilon^2} + \cdots$$

These obstructions satisfy the expected BDS relation

$$M^{(3)}(\epsilon) = -\frac{1}{3}(M^{(1)}(\epsilon))^3 + M^{(1)}(\epsilon)M^{(2)}(\epsilon) + f^{(3)}M^{(1)}(3\epsilon) + C^{(3)} + \mathcal{O}(\epsilon)$$

with

$$f^{(3)} = \frac{11\pi^4}{180} + \left(\frac{5\pi^2\zeta(3)}{6} + 6\zeta(5)\right)\epsilon + N_1\epsilon^2, \qquad C^{(3)} = N_2.$$

with N_1 and N_2 to be announced...

Conclusions

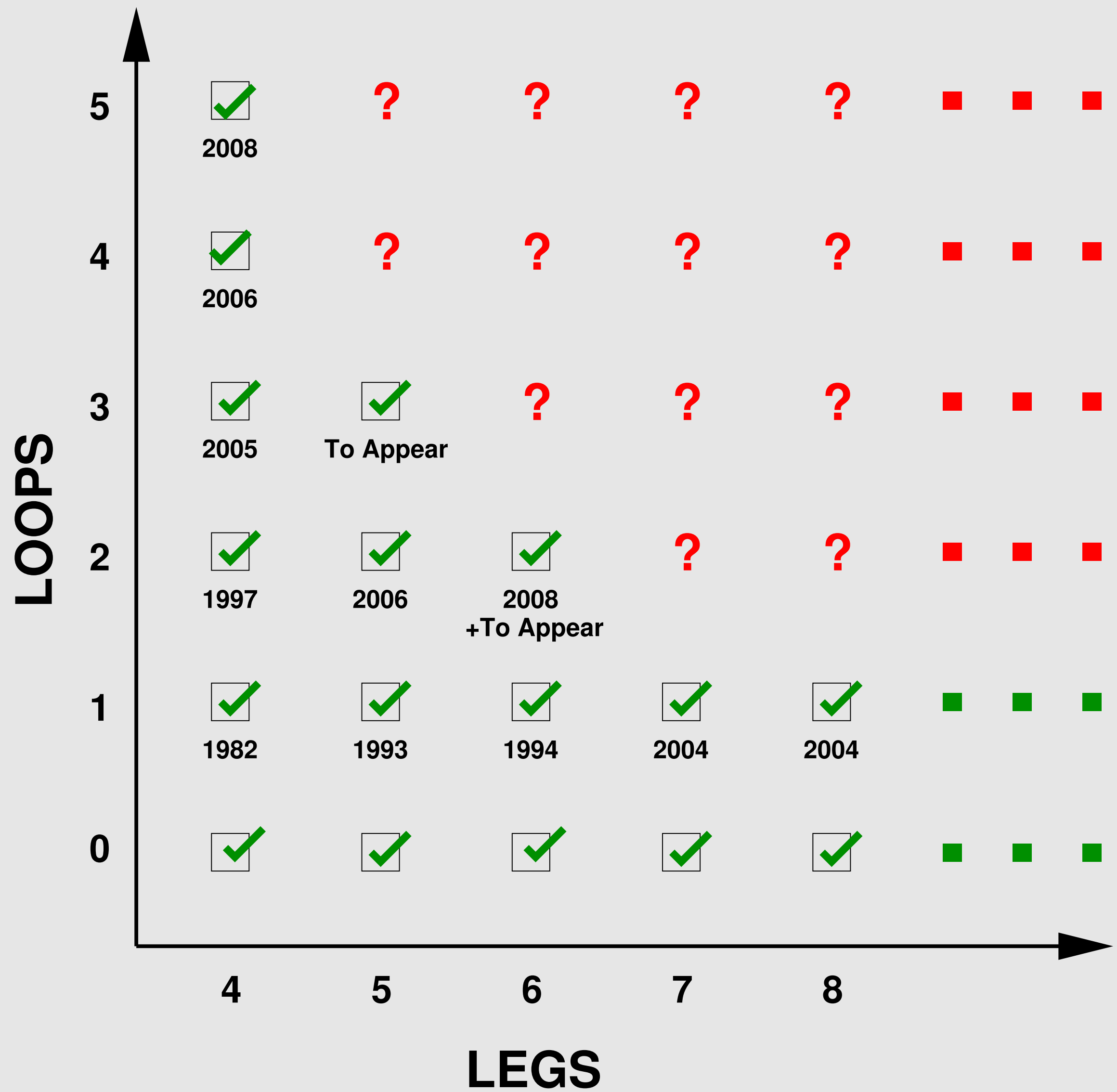
The motivation for our work was two-fold

- To unlock previously hidden mathematical richness lurking deep inside multi-loop gluon amplitudes in $\mathcal{N} = 4$ SYM, and
- To exploit that structure to help simplify otherwise formidable computations.

The leading singularity method provides a relatively simple way to find representations of complicated amplitudes in terms of a simple basis of integrals by just solving linear equations.

One final comment is that the helicity information (MHV versus non-MHV) appears only in the inhomogeneous terms (the right-hand side) of the equations.

$\mathcal{N} = 4$ Yang-Mills Status Report



Open Questions

How far need one calculate before unlocking all the structure?

How much is gained by adding one more loop, or one more leg?

Every new calculation has led to a new surprise!

In the case of loops, there were strong reasons to suspect that special things would start happening at four loops (and they did!) so there was great interest in the calculation of the four-loop cusp anomalous dimension. Five loops: cancellation of $\zeta(6,2)$?

In the case of legs, starting at six-points BDS ansatz breaks down while Wilson loop/amplitude duality holds, suggesting that there should be an additional mechanism besides dual conformal symmetry.

Resummations, non-MHV, non-planar, connection to integrability, other quantities, string theory side, etc.