Comments on Wilson loops

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Introduction and motivation

I've spent a lot of time studying Wilson loop operators. Those are very interesting Gauge invariant non-local operators in non-Abelian gauge theories.

$$W[C] = \mathrm{Tr}\,\mathcal{P}\exp\left[i\oint_C A_\mu\,dx^\mu\right].$$

Can be used to characterize the phases of the theory. For two parallel lines of seperation R and length $T\to\infty$, we get

$$\lim_{T\to\infty}\langle W\rangle = e^{-T\,V(R)}\,.$$

$V(R)$ is the potential between two probe charges and depends on the phase.

Main examples

Phase	$V(R)$	
Coulomb	$1/R$	Conformal
Free electric	$1/R\log(\Lambda R)$	Massless electric charges
Free magnetic	$\log(\Lambda R)/R$	Massless magnetic charges
Higgs	constant	Electric charges condense
Confining	σR	Magnetic charges condense

In the case of $\mathcal{N}=4$ there is no question of phase, it's conformal. Still those are extremely interesting observables in gauge theories and should be studied here too.

One approach is to expand them in terms of local operators. This is an infinite expansion that for most purposes is not very useful. Want better ways to study those operators.

Outline

- Introduction and motivation
- Maldacena-Wilson loops in $\mathcal{N} = 4$ SYM.
- 1/2 BPS loops:
  - Line, circle.
  - Gauge theory and the matrix model.
  - $AdS_5 \times S^5$, strings, D3s and D5s.
- 1/4 BPS loops:
  - Zarembo's construction.
  - Hybrid circle: Matrix model, unstable saddle point...
- More general loops:
  - Integrability in AdS.
  - Nearly circular loops and spin-chains.
- Discussion

Maldacena-Wilson loops

In $\mathcal{N} = 4$ it's very natural to consider the operators

$$W = \frac{1}{N} \mathrm{Tr}\, \mathcal{P} \exp \left[\int \left(A_\mu(x(t)) \dot{x}^\mu(t) + i |\dot{x}(t)| \Theta^I(t) \Phi_I(x(t)) \right) dt \right] ,$$

where $x^\mu(t)$ is an arbitrary path, Φ_I are the six scalars and $\Theta^I(t)$ arbitrary scalar couplings. If Θ^I have norm one this operator is locally supersymmetric and for a smooth curve seems to be finite. It's possible to include also couplings to the Fermi fields.

In AdS the Wilson loop is normally described by a macroscopic string extending to the boundary. The expectation value of the Wilson loop is the partition function of the string satisfying boundary conditions given by x^μ (Dirichlet) and by Θ^I (Neumann), which is usually approximated by a classical saddle point.

$$\langle W \rangle = \int \mathcal{D}X \, e^{-S[X]} \sim e^{-S[X_{\text{classical}}]} .$$

$1/2$ BPS loops

Line

In special cases the local supersymmetry is preserved globally. The simplest example is the straight line coupled to a single scalar

$$W = \frac{1}{N}\mathrm{Tr}\,\mathcal{P}\exp\left[i\int\left(A_t(t)+\Phi_6(t)\right)dt\right]\,,$$

This is annihilated by a linear combination of Q and $\bar{Q}$ and preserves half the supercharges. The symmetry of the loop is the supergroup $OSp(4^\star|4)$ subgroup of the $PSU(2,2|4)$ symmetry of the vacuum. Its even subgroup is $SL(2,\mathbb{R})\times SO(3)\times SO(5)$

$SL(2,\mathbb{R})$ includes time translations, dilation and a conformal transformation. $SO(3)$ are rotations around the line and $SO(5)$ is the unbroken R-symmetry.

Circle

By a conformal transformation we can map the line to the circle

$$W = \frac{1}{N} \mathrm{Tr}\, \mathcal{P} \exp \left[i \int \left(A_\psi(\psi) + iR\Phi_6(\psi) \right) d\psi \right] ,$$

where R is its radius and ψ is an angular coordinate in the plane. This also preserves half the supersymmetries, but a different combination, now involving also S and $\bar{S}$.

It's slightly harder to see how $OSp(4^*|4)$ (or its even subgroup) acts, but this is clearly still the symmetry of the operator. We will come back to this symmetry later when studying arbitrary non-supersymmetric loops, which we write as deformations of the line/circle and are best classified by representations of this supergroup.

Gauge theory and the matrix model

The line seems to have a trivial expectation value, but the circle is more interesting. Expanding to second order

$$\langle W\rangle = 1 - \frac{1}{2N}\,\mathrm{Tr}\int d\psi_1\,d\psi_2\left[\langle A_\psi(\psi_1)A_\psi(\psi_2)\rangle - R^2\langle\Phi_6(\psi_1)\Phi_6(\psi_2)\rangle\right]$$

The scalar propagator is a constant divided by the distance, $4\sin^2\frac{\psi_2-\psi_1}{2}$. The gauge field propagator (in the Feynman gauge is the same, times the product of the two tangent vectors $\cos(\psi_2-\psi_1)$. Together one finds the combination

$$\frac{\cos(\psi_2-\psi_1)-1}{4\sin^2\frac{\psi_2-\psi_1}{2}} = -\frac{1}{2}\,.$$

This is multiplied by $g^2N^2/16\pi^2$ and integrated over the two angles.

At two loop order the interacting graphs vanish and assuming that this continues to higher loop orders, the full result for the circle is given by the sum over free propagators. Since the combination of gauge field and scalar propagators has no spatial dependence, and can therefore be summarized by a 0-d Hermitean matrix model

$$\langle W\rangle = \left\langle \frac{1}{N}\mathrm{Tr}\, e^{M}\right\rangle_{\mathrm{M.M.}} = \frac{1}{Z}\int dM\, \frac{1}{N}\mathrm{Tr}\, e^{M}\, e^{-\frac{2}{g^2}\mathrm{Tr}\, M^2}\,.$$

At the planar limit this is given by the Wigner distribution

$$\langle W\rangle = \frac{2}{\pi\lambda}\int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} dx\sqrt{\lambda - x^2}\, e^{x} = \frac{2}{\sqrt{\lambda}} I_1\left(\sqrt{\lambda}\right) \sim \frac{\sqrt{2}}{\sqrt{\pi}\lambda^{3/4}} e^{\sqrt{\lambda}}\,.$$

The matrix model may be calculated exactly, including non-planar contributions. For example if one considers a loop wrapped k times with $k \sim N$ the matrix model gives (another expression was calculated for loop in the k'th antisymmetric representation).

$$\langle W_k\rangle \sim e^{2N\left(\kappa\sqrt{1+\kappa^2}+\mathrm{arcsinh}\,\kappa\right)}\,, \qquad \kappa = \frac{k\sqrt{\lambda}}{4N}\,.$$

Those expressions will be reproduced from string theory below.

String in AdS_5

With the AdS_5 metric

$$ds^2 = \frac{1}{y^2}(dy^2 + d\vec{x}^2)\,,$$

the solution describing the line is given by the surface spanned by y and $x_0 = t$. The bulk action is

$$S_{\rm bulk} = \frac{\sqrt{\lambda}}{2\pi}\int dy\,dt\,\frac{1}{y^2}\,,$$

which diverges. But there is a boundary term ensuring that the boundary conditions on y and the sphere directions are Neumann. This is

$$S_{\rm bndry} = \int dt p_y\,, \qquad p_y = \frac{\delta S}{\delta \partial_\sigma y}\,.$$

The combined action vanishes.

For the circle the resulting surface satisfies the relation $y^2 + r^2 = y^2 + x_1^2 + x_2^2 = R^2$, a constant. The bulk action is

$$S_{\text{bulk}} = \frac{\lambda}{2\pi} \int dr\, r\, d\psi\, \frac{\sqrt{1+y'^2}}{y^2}\,, \qquad y' = \frac{dy}{dr} = -\frac{r}{y}\,.$$

The integral is

$$S_{\text{bulk}} = \sqrt{\lambda} \int_0^R dr\, r\, \frac{\sqrt{1+y'^2}}{y^2} = \sqrt{\lambda} \int_0^R dr\, \frac{rR}{(R^2-r^2)^{3/2}}$$
$$= \sqrt{\lambda}\left(-1 + \frac{R}{y_0}\right)\,,$$

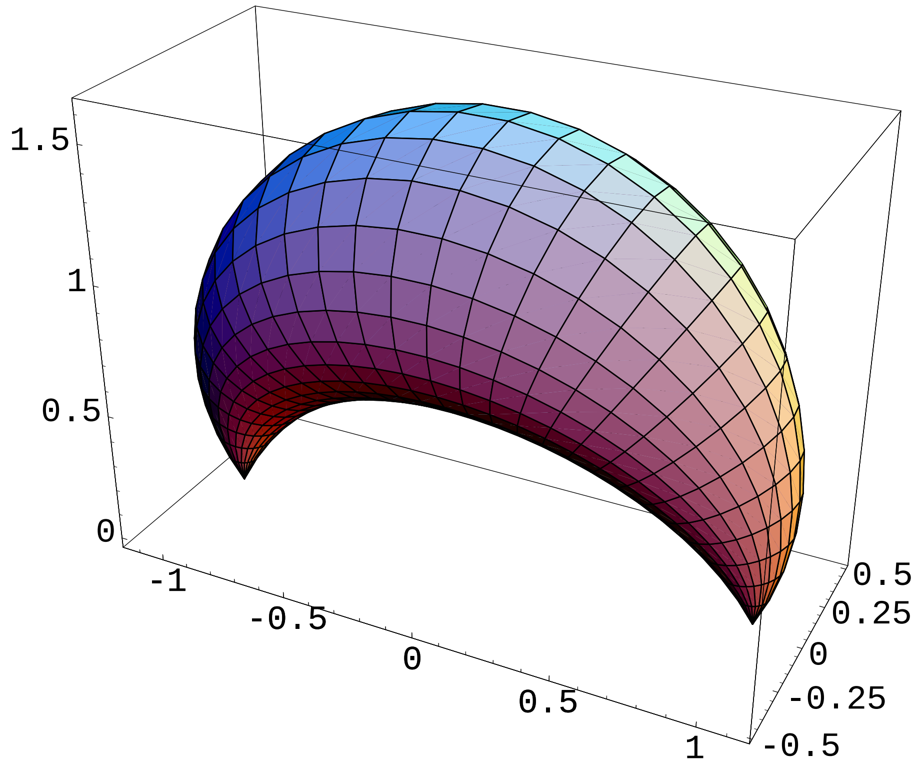
with a cutoff y_0. The Legendre transform removes this term leaving us with the result that matches the large λ expression from the matrix model

$$\langle W \rangle \sim e^{\sqrt{\lambda}}\,.$$

D3 and D5 branes

For a multiply wound Wilson loop the AdS description would involve a large number of coincident fundamental strings. This may lead to a Myers effect, where the strings blow up to a D-brane. For a Wilson loop in the antisymmetric representation of dimension $k \propto N$ the appropriate description is in terms of a D5-brane along the same AdS_2 subspace of AdS_5 and wrapping an $S^4 \subset S^5$.

A multiply wrapped loop seems to be given by a D3-brane wrapping an $AdS_2 \times S^2$ entirely inside AdS_5. In both cases those branes will carry k units of electric flux, representing the fundamental string charge. The radius of the S^4 or of the S^2 will be fixed by the ratio k/N.



Calculating the action is quite a subtle affair. There are again divergences and one has to worry about defining the correct boundary problem. Like before one has to Legendre transform from y to the conjugate momentum p_y. Then it's also necessary to replace the gauge field with the conjugate π_A.

After that, the resulting action is finite and in both cases is a function of k and N. For the antisymmetric representation

$$\langle W_{\mathrm{asym}} \rangle = e^{\sqrt{\lambda}\frac{2N}{3\pi}\sin^3\theta_k} \,.$$

For the multiply wrapped Wilson loop

$$\langle W_{k\text{-wrapped}} \rangle = e^{2N\left(\kappa\sqrt{1+\kappa^2}+\mathrm{arcsinh}\,\kappa\right)} \sim e^{k\sqrt{\lambda}+\frac{\lambda^{3/2}k^3}{96N^2}} \,.$$

The same answer is found for the related matrix model observables.

An interesting fact is that at large λ the matrix model gives the same result for the multiply wrapped loop and the *symmetric* representation, so both may be described by the D3-brane.

$1/4$ BPS loops

To see the supersymmetry of the straight line it's convenient to use the notations of SYM in ten dimensions, then the variation of $A_t + \Phi_6$ is $(\Gamma_t + \Gamma_6)\Psi$ and this combination of gamma matrices has half its eigenvalues zero.

It was then noticed that it is possible to construct $1/4$ BPS, $1/8$ BPS and $1/16$ BPS loops for arbitrary curves in a 2-plane, 3-plane or in all of $\mathbb{R}^4$. In the planar case one has to associate to each direction in the plane a scalar and if the tangent vector to the curve is $a_1\hat{x}^1 + a_2\hat{x}^2$, it should be coupled to the scalar $a_1\Phi_1 + a_2\Phi_2$. Then the Wilson loop will be invariant under the overlap of the supersymmetries of the line with$A_1 + i\Phi_1$ and the line with $A_2 + i\Phi_2$.

All those loops seem to have trivial expectation values and recently the surfaces in $AdS_5 \times S^5$ associated with them were studied.

"Hybrid" loop

These loops are quite interesting, particularly the fact that they are so many of them. But there are many more supersymmetric loops. First, like in the case of the $1/2$ BPS circle, if the loops are confined to a sphere rather than a plane they can be made invariant under other supersymmetries, but expectation value does not need to be trivial (this has not been studied, as far as I know).

Another case found recently is a mixture between the $1/2$ and $1/4$ BPS circles. Consider the Wilson loop with combination

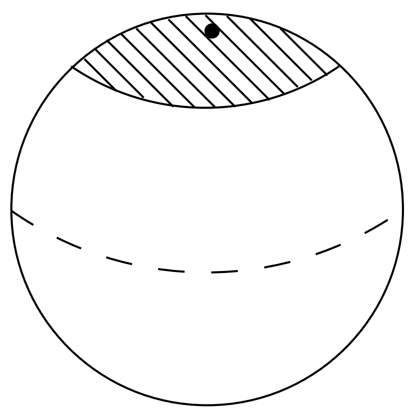
$$iR\left[A_1\cos\tau + A_2\sin\tau + i\left(\sin\theta_0(\Phi_1\cos\tau + \Phi_2\sin\tau) + \cos\theta_0\Phi_3\right)\right] .$$

For $\theta_0 = 0$ it couples only to Φ_3, ($1/2$ BPS) while for $\theta_0 = \pi/2$ it couples periodically to the two other scalars ($1/4$). In a few lines it's possible to show that for all $\theta_0 \neq 0$ this is invariant under a combination of a quarter of the Q, $\bar{Q}$, S, and $\bar{S}$ SUSYs.

At one loop the periodic scalars proportional to $\sin^2\theta_0$ will cancel part of the gauge contribution, leaving the third scalar and gauge field terms both proportional to $\cos^2\theta_0$. Thus all ladder diagrams are constants, like the $1/2$ BPS circle with $\lambda \to \lambda' = \lambda\cos^2\theta_0$.

At two loop order the interacting graphs cancel again, so we may guess that these loops with arbitrary θ_0 are given by ladder diagrams to all orders. The result will be the same matrix model as before with the above replacement in the coupling.

In $AdS_5 \times S^5$ the boundary conditions on the string map it to a circle on the boundary wrapping a parallel at angle θ_0 from the north-pole in an $S^2 \subset S^5$.



The solution has the AdS_5 part with action $S_{AdS} = -\sqrt{\lambda}$. The sphere contributes the area the surface contracting the surface over the pole, $S_S = \sqrt{\lambda}(1 - \cos\theta_0)$. Together this is $S = -\sqrt{\lambda}\cos\theta_0$.

This is exactly the same as the replacement $\lambda \to \lambda' = \lambda\cos^2\theta_0$ discussed before!

But we were not too careful, since there are in fact there are two ways to contract the circle. One over the north pole and the other over the south pole, the second solution has $\cos\theta \to -\cos\theta$, so the two saddle points contribute

$$S = \pm\sqrt{\lambda'}\,,$$

and the one with positive action is unstable (it can "slip" off the sphere in the three other directions of S^5).

Recall the planar result of the matrix model it was (after the modification)

$$\langle W\rangle_{\mathrm{M.M.}}=\frac{2}{\sqrt{\lambda'}}I_1\left(\sqrt{\lambda'}\right)\,.$$

The asymptotic expansion of the Bessel function at large argument includes the saddle point written before with $e^{\sqrt{\lambda'}}$, but also another saddle point with $e^{-\sqrt{\lambda'}}$

$$\langle W\rangle\sim\frac{\sqrt{2}}{\sqrt{\pi}\lambda'^{3/4}}e^{\sqrt{\lambda}'}(1+O(1/\sqrt{\lambda'}))-i\frac{\sqrt{2}}{\sqrt{\pi}\lambda'^{3/4}}e^{-\sqrt{\lambda}'}(1+O(1/\sqrt{\lambda'}))\,.$$

This agrees with the two solutions found from string theory! The subleading term is imaginary, which is expected for an unstable saddle point with an odd number of tachyonic directions.

This is a very interesting system where there is an unstable string solution the preserves supersymmetry and may other peculiar properties...

Furthermore we can consider a BMN-like limit where λ is large but λ' is small. This is a loop that is only slightly removed from the equator where $\cos\theta_0 = 0$. At the equator itself the system has three zero modes parameterizing an S^3 with measure

$$d\Omega_3 = \frac{1}{2\pi^2}\,d\alpha\sin^2\alpha\,d\Omega_2\,,$$

where α is in $[0,\pi]$ and $d\Omega_2$ is the measure on S^2. Turning on a small λ' leads to a potential $\cos\alpha\cos\theta_0\sqrt{\lambda}$ while preserving the symmetry of the S^2, so the integration over the broken zero modes gives for the Wilson loop

$$\langle W\rangle = \frac{2}{\pi}\int_0^\pi d\alpha\,\sin^2\alpha\,e^{-\cos\alpha\,\sqrt{\lambda'}} = \frac{2}{\sqrt{\lambda'}}I_1\left(\sqrt{\lambda'}\right)\,.$$

So doing the path integral only over the broken zero modes gives the full planar result!

More general loops

So far I talked about a very restricted class of supersymmetric Wilson loops, but we want to be able to describe arbitrary loops. Here I describe some steps in this direction.

Integrability in AdS

As mentioned before, the expectation values of Wilson loops at strong coupling are given by the action of minimal surfaces in $AdS_5 \times S^5$. That is, one has to solve the classical equations of motion for a string in this background with prescribed boundary conditions.

The σ-model on $AdS_5 \times S^5$ is classically integrable, which allows in principle to solve this problem. Indeed we did this for a wide class of Wilson loops that have periodic couplings. That includes lines, circles, helices with different scalar couplings. If one takes a symmetric ansatz for the embedding, the system reduces to a one-dimensional integrable system.

From studying different examples there is generally a very complicated phase structure of solutions, where for certain range of parameters some solutions exist and for others not. Then the different saddle points dominate in different regimes, meaning that while doable in principle, in practice the calculations are very hard.

Then if we try to study an arbitrary Wilson loop at weak coupling we were not able to find similar simplifying structure. Usually the integrability in the gauge theory side is given by some form of spin-chain that that doesn't show up from studying arbitrary periodic Wilson loops. The exception are the $1/4$ BPS loops presented before.

Nearly circular Wilson loops

So instead of studying Wilson loops with arbitrary shapes, let us look at loops that are nearly circular (or straight). The reason is that the starting point is a symmetric object, and then we can study the deformed one in terms of representations of the broken symmetry group.

The circle has an $OSp(4^*|4)$ symmetry which includes a non-compact $SL(2,\mathbb{R})$. The representations of this group are labeled by a continuous parameter allowing it to get perturbative corrections, hence it may be an interesting quantity to calculate.

A loop that is close to the circle may be written as a circle with insertions into it, like

$$W[\mathcal{O}_p(\psi_p)\cdots\mathcal{O}_1(\psi_1)] = \frac{1}{N}\mathrm{Tr}\,\mathcal{P}\left[\mathcal{O}_p(\psi_p)\cdots\mathcal{O}_1(\psi_1)\,e^{i\int(A_\psi+iR\Phi_6)d\psi}\right].$$

where $\mathcal{O}_p$ are local operators in the anjoint of the gauge group.

Simple examples of such insertions are

$$(\Phi_1+i\Phi_2)^J\,, \qquad \Phi_1 D_\mu\Phi_2\,, \qquad F_{\mu\nu}\,, \qquad F^2\,, \qquad \cdots$$

Because the symmetric loop preserves $SL(2,\mathbb{R})$, this may be thought of as the correlator of those adjoint operators and it will satisfy all the axioms of a conformal field theories. We can regard the Wilson loop with p adjoint insertions as a definition of the p-point function of adjoint operators. The 2-point function will be related to the conformal dimension of the operators. Then it is possible to separate the operators into primaries, those annihilated by K_t, and descendants.

One can go further and study the 3-point function, which will give the structure constants of this conformal field theory. Then the 4-point function and so on.

In what follows I focus on the 2-point function and furthermore restrict to the very simple class of insertions made only of the two scalars

$$Z = \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2)\,, \qquad X = \frac{1}{\sqrt{2}}(\Phi_3 + i\Phi_4)\,.$$

The conformal dimension

For the line, the Ward identity of broken dilatation symmetry is

$$(\Delta_1 + \Delta_2 + t\,\partial_t)W[\mathcal{O}_2(t)\mathcal{O}_1(0)] = 0\,.$$

This equation is solved by

$$W[\mathcal{O}_2(t)\mathcal{O}_1(0)] \propto \frac{1}{t^{\Delta_1+\Delta_2}}\,.$$

This is then used to define the conformal dimension of those insertions.

For the circle the equation is

$$(\Delta_1 + \Delta_2\cos\psi + \sin\psi\,\partial_\psi)W[\mathcal{O}_2(\psi)\mathcal{O}_1(0)] = 0\,.$$

This equation is solved by

$$W[\mathcal{O}_2(\psi)\mathcal{O}_1(0)] \propto \frac{\cos^{|\Delta_1-\Delta_2|}(\psi/2)}{\sin^{\Delta_1+\Delta_2}(\psi/2)}\,.$$

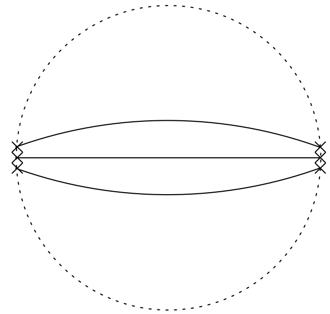
Gauge theory calculation

Perturbation theory

At tree level each Z will have to be contracted with a $\bar{Z}$ and each X with a $\bar{X}$. In a convenient gauge the holonomy in the Wilson loop will not contribute and the 2-point function is given by

$$\left\langle W[\mathcal{O}'^\dagger(t)\,\mathcal{O}(0)]\right\rangle = \left\langle \mathrm{Tr}\,[\mathcal{O}'^\dagger(t)\,\mathcal{O}(0)]\right\rangle .$$

This is a single trace over two operators separated in space.



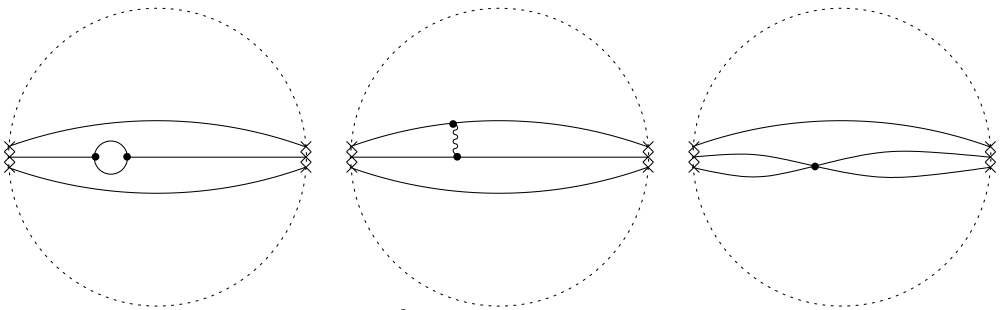
Spin chain

The planar 2-point function of local operators has an interpretation in terms of periodic spin-chains. At leading order there is a similar identification to the one in the last slide, only that it is periodic. In our case the same seem interpretation seems to hold, only that the spin-chain is open. Each word has a first letter and a last letter and there is only a single trace, so the order has to be kept

From this we find that the expectation value of the Wilson loop (with words of length K) is

$$\left\langle W[\mathcal{O}'^\dagger(t)\,\mathcal{O}(0)]\right\rangle \propto \left(\frac{\lambda}{8\pi^2 t^2}\right)^K I\,.$$

I is the identity matrix identifying the two insertions.



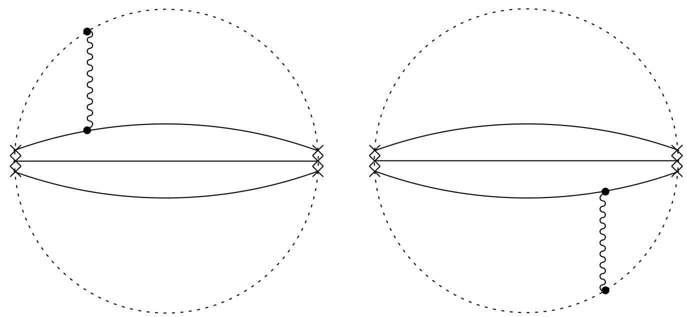
At one-loop there are three types of diagrams that do not involve the Wilson loop. They are the self energy graph (a), a gluon exchange "H-diagram" (b) and the scalar interaction vertex "X-diagram" (c). Each graph diverges and to renomalize it the Wilson loop has to be multiplied by the relevant Z-factors

$$Z_{\text{self-energy}} = I + \frac{\lambda}{8\pi^2}\ln\Lambda\, I\,, \qquad Z_{\text{H}} = I - \frac{\lambda}{16\pi^2}\ln\Lambda\, I\,,$$

$$Z_{\text{X}} = I + \frac{\lambda}{16\pi^2}(I - 2P)\ln\Lambda\,.$$

P is the permutation matrix replacing two adjacent letters.

There are also graphs the involve the Wilson loop



Those do not depend on the flavor index (Z or X) and give a renormalization factor

$$Z_{\text{boundary}} = I - \frac{\lambda}{8\pi^2}\ln\Lambda\, I\,.$$

Together K self-energy graphs, $K-1$ of the H and X ones and the two boundary contributions give

$$Z_{\text{total}} = I + \frac{\lambda}{8\pi^2}\ln\Lambda\sum_{k=1}^{K-1}(I - P_{k,k+1})\,.$$

Open and closed spin-chains

Consider the closed spin-chain of length $2K$. It has a one-loop Hamiltonian

$$Z_{\text{closed}} = I + \frac{\lambda}{16\pi^2} \ln \Lambda \sum_{k=1}^{2K} (I - P_{k,k+1})\,.$$

Now impose reflection invariance, so site k and $2K+1-k$ are equal. Then the spin-chain is effectively open with length K and the interaction terms between the K and $K+1$ spins as well as between the first and last spin cancel, so the Hamiltonian can be written as

$$Z_{\text{total}} = I + \frac{\lambda}{8\pi^2} \ln \Lambda \sum_{k=1}^{K-1} (I - P_{k,k+1})\,,$$

which is what we had before.

Using this orbifolding trick it's easy to solve this open spin-chain.

The Bethe ansatz

This Z-factor gives the matrix of anomalous dimensions which has to be diagonalized. In the case of local operators this matrix is the same as a Heisenberg spin-chain Hamiltonian that was solved by Bethe. In our case it's the open version of the spin-chain, instead of the periodic one.

Simplest way to solve it is by doubling the chain and solving for symmetric configurations of the resulting periodic chain.

Blah, blah, blah Magnon

$$|\psi\rangle = \sum_{k=1}^{K} \cos p(k - 1/2)\, |k\rangle\,.$$

Yada, yada, yada rapidities

$$u_n = \frac{1}{2} \cot \frac{p_n}{2}\,.$$

La, la, la Bethe equations

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^{2K} = \prod_{\substack{k=1\\k\neq j}}^{M} \frac{(u_j - u_k + i)(u_j + u_k + i)}{(u_j - u_k - i)(u_j + u_k - i)}\,.$$

badabeem, badabam anomalous dimensions

$$\gamma_n = \frac{\lambda}{2\pi^2}\sum_{k=1}^{M}\sin^2\frac{p_k}{2} = \frac{\lambda}{8\pi^2}\sum_{k=1}^{M}\frac{1}{u_k^2 + 1/4}\,.$$

For a single impurity

$$p_n = \frac{\pi n}{K}\,, \qquad u_n = \frac{1}{2}\cot\frac{\pi n}{2K}\,, \qquad \gamma_n = \frac{\lambda}{2\pi^2}\sin^2\frac{\pi n}{2K} \sim \frac{\lambda n^2}{8K^2}\,.$$

For many impurities can take the thermodynamic limit (hold on to your excitment, details to come...).

String theory description

In this case too we can find the minimal surfaces corresponding to the solution. They should be Wilson loops, so open strings reaching the boundary but they should also carry charges, so there should be some angular momentum around the sphere directions.

We are able to find the solution for one angular momentum, then we go to the BMN limit and quantize the small fluctuations around it. Finally we study the system with two angular momenta.

Working with global Lorentzian AdS_5 it is possible to map one insertion to past infinity and one to the future. Then the Wilson loop runs up and down two antipodal points on the boundary $S^3 \times \mathbb{R}$. The metric is

$$ds^2 = L^2 \left[-\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\Omega_3^2 + d\theta^2 + \sin^2 \theta \, d\phi^2 + \cdots \right] ,$$

One angular momentum

We take the ansatz

$$\rho = \rho(\sigma)\,, \qquad \theta = \theta(\sigma)\,, \qquad t = \omega\tau\,, \qquad \phi = w\tau\,.$$

The string Lagrangean in the conformal gauge reduces to

$$\mathcal{L} = \frac{L^2}{4\pi\alpha'}\left[\rho'^2 + \omega^2\cosh^2\rho + \theta'^2 - w^2\sin^2\theta\right]\,.$$

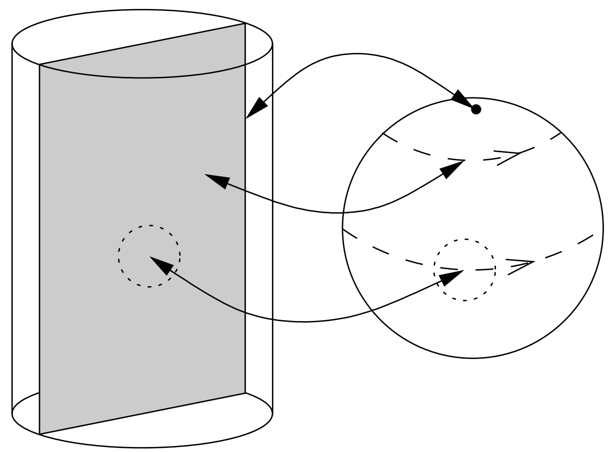
The equations of motion are quite simple

$$\rho'' - \omega^2\cosh\rho\sinh\rho = 0\,,$$

$$\theta'' + w^2\cos\theta\sin\theta = 0\,.$$

The relevant solution requires $w = \omega$ and is

$$\sinh\rho = \frac{1}{\sinh\omega\sigma}\,, \qquad \sin\theta = \tanh\omega\sigma\,.$$



This solution preserves $1/4$ of the supersymmetries, the intersection of those preserved by the basic Wilson loop and those preserved by the scalar operators $\mathrm{Tr}\, Z^J$ (the BMN ground state). This can be seen from both the gauge theory and from string theory.

The BMN limit

It is also possible to zoom in on the center of AdS_5 and the equator of S^2 and find this classical solution in the maximally supersymmetric pp-wave geometry studied by BMN. The Wilson loop will become an infinite solitonic string running through that space.

Quantizing the fluctuations of this string we find the modes that are analogous to the X insertions into the spin-chain. The spectrum of those turns out to be the same as found from the one-loop gauge theory calculation

$$\Delta - J = \omega_n = \sqrt{1 + \frac{\lambda n^2}{4J^2}} \sim 1 + \frac{\lambda n^2}{8J^2} \,.$$

Two angular momenta

We were able to go further and look for classical solutions carrying angular momentum in two planes. Those correspond to spin-chains with a large number of both Zs and Xs.

On the gauge theory side that is given by the spin-chain in the thermodynamic limit. That is quite tedious to review, but it involves solving for a large number of Bethe roots that condense into cuts in the plane. Luckily for us, the people studying closed spin-chains chose cuts that are symmetric around the imaginary axis, so they automatically are also solutions of our Bethe equations.

The only difference was that we had to take only one of the cuts into consideration, so that left us with the task of playing around with some factors of 2.

We arrived at the equation for the dimension of operator with M impurities in a chain of length K

$$\gamma_M = \frac{\lambda}{8\pi^2 K} K(k)\Big(2E(k) - (2-k^2)K(k)\Big)\,,$$

where k is given by the solution to

$$\frac{M}{K} = \frac{1}{2} - \frac{1}{2\sqrt{1-k^2}}\frac{E(k)}{K(k)}\,.$$

$K(k)$ and $E(k)$ are elliptic integrals.

On the string theory side we found equations similar to those of folded strings, but slightly more complicated.

The solution is as follows: Away from the center of AdS_5 the surface will look like the previous one but at the center of space it will move off the S^2 into another direction on S^5. The equations describing this part of the string are exactly the same as for folded strings, only that instead of backtracking they connect to the rest of the world-sheet and reach the boundary.

The quantum numbers carried by those operators are almost exactly half of the folded strings and this exactly agrees with the solution of the Bethe equations of the open spin-chain in the thermodynamic limit.

Discussion

- There are many many Wilson loop observables and tried to organize the calculation around simple cases:
  - SUSY.
  - small deformations of the circle and $SL(2,\mathbb{R})$.
  - Other ways?
- The circle and the line preserve an $SL(2,\mathbb{R})$ allowing to define a CFT living along the loop.
- For two insertions into the circle found an open spin-chain that gives the anomalous dimensions of the insertions. $\langle W \rangle = \frac{\#}{d^{2\Delta}}$.
- In other cases the overall numerical coefficient $\#$ was calculated using the matrix model.

- The Gaussian matrix model has a very rich structure and "knows" about strings, D3-branes and D5-branes.
- the $1/4$ BPS loop allows to define some kind of BMN limit.
- Much more to do
  - Full super-spin-chain.
  - Higher loops (integrability?)
  - higher point functions of insertions, or more general Wilson loops
  - ....

Thanks

The end