Transcedentality and Eden-Staudacher equation

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Content

1 Introduction

Pomeron contribution

$$\sigma_t \sim (s/m^2)^{\Delta}\,,\ j_P(t) = 1 + \Delta + j'_P\, t$$

Pomeranchuck theorem and Odderon

$$\frac{\sigma_t^{pp}}{\sigma_t^{p\overline{p}}} \to 1\,,\ \sigma_t^{pp} - \sigma_t^{p\overline{p}} \sim (s/m^2)^{\Delta_{Odd}}$$

Pomeron and Odderon as glueballs

$$P = gg\,,\ O = ggg$$

Parton distributions in hadrons

$$n_r(x) = \int d^2k_\perp n_r(x, \vec{k}_\perp)$$

DGLAP and BFKL equations

$$\frac{d\,n_r(x)}{d\ln\,Q^2} = -W_r\,n_r(x) + \sum_s \int_x^1 \frac{dy}{y}\,W_{s\to r}(\frac{x}{y})\,n_s(y)$$

$$\frac{d\,n(x,\vec{k})}{d\ln\frac{1}{x}} = 2\omega(-\vec{k}^2)n(x,\vec{k}) + \int d^2k'\,K(\vec{k},\vec{k}')\,n(x,\vec{k}')$$

2 Gluon reggeization

Leading logarithmic approximation (LLA)

$$M_{AB}^{A'B'}(s,t) = s\,\alpha_s \sum_{n=0}^{\infty} (\alpha_s\,\ln s)^n\; a_n(t)$$

Region of applicability

$$\alpha_s \ln s \sim 1,\; \alpha_s = \frac{g^2}{4\pi} \to 0$$

Born amplitude

$$M_{AB}^{A'B'}(s,t)|_{Born} = g\,T^c_{A'A}\,\delta_{\lambda_{A'}\lambda_A}\,\frac{2s}{t}\,g\,T^c_{B'B}\,\delta_{\lambda_{B'}\lambda_B}$$

Gluon reggeization

$$M_{AB}^{A'B'}(s,t) = s^{\omega(t)}\,M_{AB}^{A'B'}(s,t)|_{Born}$$

Gluon Regge trajectory in LLA

$$\omega(-|q|^2) = -\frac{\alpha_c}{4\pi^2}\,N_c\int d^2k\,\frac{|q|^2}{|k|^2|q-k|^2} \approx -\frac{\alpha_c}{2\pi}\,\ln\frac{|q^2|}{\lambda^2}$$

3 BFKL equation (1975)

Production amplitude in the MR kinematics

$$M_{2\to 1+n} \sim \frac{s_1^{\omega_1}}{|q_1|^2}\, gT^{d_1}_{c_2c_1} C(q_2,q_1)\, \frac{s_2^{\omega_2}}{|q_2|^2} ... C(q_n,q_{n-1})\, \frac{s_n^{\omega_n}}{|q_n|^2}$$

Reggeon-Reggeon-gluon vertex

$$C(q_2,q_1) = \frac{q_2\, q_1^*}{q_2 - q_1}$$

Impact parameter coordinates

$$\rho_k = x_k + iy_k\,,\ \rho_k^* = x_k - iy_k\,,\ p_k = i\frac{\partial}{\partial \rho_k}\,,\ p_k^* = i\frac{\partial}{\partial \rho_k^*}$$

Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation

$$E\,\Psi(\vec{\rho}_1,\vec{\rho}_2) = H_{12}\,\Psi(\vec{\rho}_1,\vec{\rho}_2)\ ,\ \Delta = -\frac{\alpha_s N_c}{2\pi}\,E$$

BFKL Hamiltonian

$$H_{12} = \ln\,|p_1p_2|^2 + \frac{1}{p_1p_2^*}\ln\,|\rho_{12}|^2 p_1p_2^*$$

$$+\frac{1}{p_1^*p_2}\ln\,|\rho_{12}|^2 p_1^*p_2 - 4\psi(1)\,,\ \rho_{12} = \rho_1 - \rho_2$$

4 Möbius invariance (1986)

Möbius transformations

$$\rho_k \to \frac{a\rho_k + b}{c\rho_k + d_k}$$

Möbius group generators

$$M_3^{(r)} = \rho_r \, \partial_r \, , \;\; M_+^{(r)} = \partial_r \, , \;\; M_-^{(r)} = -\rho_r^2 \, \partial_r$$

Casimir operator for two gluons

$$M^2 = \left(\sum_{r=1}^{2} \vec{M}^{(r)} \right)^2 = \rho_{12}^2 \, p_1 \, p_2 \, , \;\; \rho_{12} = \rho_1 - \rho_2$$

Eigenvalue equations

$$M^2 f_{m,\widetilde{m}} = m(m{-}1) f_{m,\widetilde{m}} \, , \;\; M^{*2} f_{m,\widetilde{m}} = \widetilde{m}(\widetilde{m}{-}1) f_{m,\widetilde{m}}$$

Principal series of unitary representations

$$m = 1/2 + i\nu + n/2 \, , \;\; \widetilde{m} = 1/2 + i\nu - n/2$$

5 Reggeized gluon interactions

Pomeron wave function (1986)

$$f_{m,\widetilde{m}}(\overrightarrow{\rho_1},\overrightarrow{\rho_2};\overrightarrow{\rho_0}) = \left(\frac{\rho_{12}}{\rho_{10}\,\rho_{20}}\right)^{m}\left(\frac{\rho^*_{12}}{\rho^*_{10}\,\rho^*_{20}}\right)^{\widetilde{m}}$$

Pomeron energy

$$E_{m,\widetilde{m}} = \varepsilon_m + \varepsilon_{\widetilde{m}} \quad , \quad \varepsilon_m = \psi(m) + \psi(1-m) - 2\psi(1)$$

BFKL Pomeron intercept

$$\Delta = 4\,\frac{\alpha_s}{\pi} N_c\,\ln 2$$

Violation of the Froissart bound

$$\sigma \sim s^{\Delta} > c\ln^2 s$$

Effective Lagrangian for reggeon fields $A_\pm$ (1995)

$$L = L_{QCD} + tr\,(j_- A_+ + j_+ A_-)\,,$$

$$j_\pm = \frac{1}{g}\,\partial_\pm\,P\,\exp\left(g\int_{-\infty}^{x} v_\pm(x')\,dx'^{\,\pm}\right)$$

6 Holomorphic separability

Bartels-Kwiecinski-Praszalowicz equation (1980)

$$E\,\Psi(\vec{\rho}_1, ..., \vec{\rho}_n) = H\,\Psi(\vec{\rho}_1, ..., \vec{\rho}_n)\ ,\ H = \sum_{k<l} \frac{\vec{T}_k \vec{T}_l}{-N_c}\, H_{kl}$$

Holomorphic separability at large N_c (1989)

$$H = \frac{h + h^*}{2}\ ,\ h = \sum_{k=1}^{n} h_{k,k+1}$$

Pair hamiltonian

$$h_{12} = \ln(p_1 p_2) + \frac{1}{p_1}\ \ln \rho_{12}\ p_1 + \frac{1}{p_2}\ \ln \rho_{12}\ p_2 - 2\Psi(1)$$

Holomorphic factorization (1989)

$$\Psi(\vec{\rho}_1, \vec{\rho}_2, ..., \vec{\rho}_n) = \sum_{r,s} a_{r,s}\,\Psi_r(\rho_1, ..., \rho_n)\,\Psi_s(\rho_1^*, ..., \rho_n^*)$$

Singularities

$$\Psi_r(\rho_1, ..., \rho_n) \sim \rho_{ij}^{\gamma}$$

7 Integrability (1993)

Two different normalization conditions

$$\|f\|_1^2 = \int \prod_{r=1}^{n} \frac{d^2\rho_r}{|\rho_{r,r+1}|^2} \, |f|^2 \, ,$$

$$\|f\|_2^2 = \int \prod_{r=1}^{n} d^2\rho_r \left| \prod_{r=1}^{n} p_r \;\, f \right|^2$$

First integral of motion

$$A = q_n = \rho_{12}\rho_{23}...\rho_{n1}\, p_1 p_2 ... p_n \, , \;\; [h, A] = 0$$

Transfer and monodromy matrices

$$T(u) = tr\, t(u)\, , \;\; t(u) = L_1 L_2 ... L_n = \sum_{r=0}^{n} u^{n-r}\, q_r \, ,$$

$$L_k = \left(\begin{array}{cc} u + \rho_k\, p_k & p_k \\ -\rho_k^2\, p_k & u - \rho_k\, p_k \, , \;\; \hat{l} = u\, \hat{1} + i\, \hat{P} \end{array} \right)$$

Yang-Baxter equation (1993)

$$t_{r_1'}^{s_1}(u)\, t_{r_2'}^{s_2}(v)\, l_{r_1 r_2}^{r_1' r_2'}(v-u) = l_{s_1' s_2'}^{s_1 s_2}(v-u)\, t_{r_2}^{s_2'}(v)\, t_{r_1}^{s_1'}(u)$$

8 Duality symmetry (1999)

Cyclic invariance at large N_c

$$\rho_i \to \rho_{i+1}\ ,\ \ p_i \to p_{i+1}$$

Two representations of the hamiltonian

$$h = \sum_{r=1}^{n} \left(2\ln p_r + p_r^{-1} \ln\left(\rho_{r,r+1}\rho_{r,r-1}\right) + 2\gamma\right) =$$

$$\sum_{r=1}^{n} \left(\rho_{r,r+1} \ln\left(p_r p_{r+1}\right)\, \rho_{r,r+1}^{-1} + 2\ln\, \rho_{r,r+1} + 2\gamma\right)$$

Duality transformation

$$\rho_{r,r+1} \to p_r \to \rho_{r-1,r}$$

Duality equation

$$\psi_{m,\widetilde{m}}(\overrightarrow{\rho_{12}}, ..., \overrightarrow{\rho_{n1}}) =$$

$$|\lambda_m| \int \prod_{k=1}^{n-1} \frac{d^2\rho'_{k-1,k}}{2\pi} \prod_{k=1}^{n} \frac{2e^{i\overrightarrow{\rho_{k,k+1}}\,\overrightarrow{\rho'_k}}}{\left|\rho'_{k,k+1}\right|^2}\, \psi^*_{\widetilde{m},m}(\overrightarrow{\rho'_{12}}, ..., \overrightarrow{\rho'_{n1}})$$

9 Heisenberg spin model

Local Hamiltonian with Möbius spins $\vec{M}_r$

$$h = \sum_{r=1}^{n} h(M^2_{r,r+1})\,,\ M^2_{r,r+1} = -\rho^2_{r,r+1}\partial_r\partial_{r+1}$$

Integrable pair interactions (L.(1994); F.,K.(1995))

$$h(M^2_{12}) = \Psi(\hat{m}_{12}) + \Psi(1-\hat{m}_{12}) - 2\Psi(1) = h_{12}$$

Casimir operator and conformal weights

$$M^2_{12} = -\hat{m}_{12}(1-\hat{m}_{12})$$

Monodromy matrix parametrization

$$t(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

Pseudovacuum state and the Bethe anzatz

$$C(u)\Omega_0 = 0\,,\ \Omega = \prod_k B(u_k)\Omega_0$$

10 Odderon wave function

Möbius symmetry

$$\psi(\vec{\rho}_1,\vec{\rho}_2,\vec{\rho}_3,\vec{\rho}_0)=\left(\frac{\rho_{23}}{\rho_{20}\rho_{30}}\right)^{m}\left(\frac{\rho_{23}^*}{\rho_{20}^*\rho_{30}^*}\right)^{\widetilde{m}}\phi_{m\widetilde{m}}(x,x^*)$$

Unharmonic ratio

$$x=\frac{\rho_{12}\,\rho_{30}}{\rho_{10}\,\rho_{32}}$$

Holomorphic factorization

$$\phi_{m,\widetilde{m}}(x,x^*)=\sum_{r,s=1}^{3}c_{rs}\phi_m^{(r)}(x)\,\phi_{\widetilde{m}}^{(s)}(x^*)$$

Duality symmetry operator

$$a_m=x(1-x)\,p^{1+m}\,,\;\; p=i\frac{d}{dx}$$

Eigenfunctions for integral of motion (1993)

$$a_m a_{1-m}\phi_m^{(r)}(x)=\lambda\phi_m^{(r)}(x)$$

11 Odderon energy

Hamiltonian in the normal form (1999)

$$\frac{h}{2} = -\log(x) - 3\psi(1) + \sum_{k=1}^{\infty} x^k f_k(P)\,,\;\; P = x\partial\,,$$

$$f_k(P) = -\frac{2}{k} + \frac{1}{2}\left(\frac{1}{P+k-m} + \frac{1}{P+k}\right) + \sum_{t=0}^{k} \frac{c_t(k)}{P+t}\,,$$

$$c_t(k) = \frac{(-1)^{k-t}\,\Gamma(m+t)\,((t-k)(m+t)+m\,k/2)}{k\,\Gamma(m-k+t+1)\,\Gamma(t+1)\,\Gamma(k-t+1)}$$

Wave function expansion

$$\Phi_m(P) = \sum_{n=0}^{\infty} \mu^{-n}\Phi_m^n(P)\,,\; \Phi_m^0(P) = 1\,,\;\; \mu = i\lambda$$

Holomorphic energy: $\varepsilon_m/2 =$

$$\log(\mu)+3\gamma+\left(\frac{3}{448}+\frac{13}{120}(m-1/2)^2-\frac{(m-1/2)^4}{12}\right)\frac{1}{\mu^2}$$

$$+\left(-\frac{4185}{2050048}-\frac{2151}{49280}(m-1/2)^2+\ldots\right)\frac{1}{\mu^4}+\ldots\,.$$

Odderon intercepts

$\Delta^{YW} \sim -0.1\Delta^{BFKL}$; $\Delta^{BLV} = 0\,,\; \sigma_{pp} - \sigma_{\bar{p}p} \to const$

12 Baxter-Sklyanin picture

Pseudo-vacuum state (F.,K.)

$$\Omega_0 = \prod_{r=1}^{n} \rho_r^{-2}\,, \;\; C^t(u)\Omega_0 = 0$$

Sklyanin wave function

$$\Omega = \prod_{r=1}^{n-1} Q(\hat{u}_r)\,\Omega_0\,, \;\; B^t(\hat{u}) = 0$$

Transfer matrix eigenvalue equation

$$(A(u) + B(u))\;\Omega = \Lambda(u)\,\Omega$$

Baxter equation

$$\Lambda(u)\,Q(u) = (u+i)^n Q(u+i) + (u-i)^n Q(u-i)$$

Eigenvalues of integrals of motion

$$\Lambda(u) = \sum_{r=0}^{n} \mu_r i^r u^{n-r}, \;\; \mu_0 = 1, \mu_1 = 0, \mu_2 = m(m-1)$$

13 Baxter function and energy

Solutions of the Baxter equation (H.,L. ; D.,M.,K.)

$$Q^{(t)}(u) = \sum_{r=0}^{\infty} \frac{P_{t-1}(u)}{(u-ir)^t} + \sum_{r=0}^{\infty} \frac{P_{n-t-2}(u)}{(u+ir)^{n-t-1}}$$

Linear relations (V.,L.)

$$(\delta_t + \pi \coth(\pi u))\, Q^{(t)}(u) = Q^{(t+1)}(u) + \alpha_t Q^{(t-1)}(u)$$

Holomorphic factorization

$$Q(\vec{u}) = \sum_{t,l=0}^{n-1} C_{t,l} Q^{(t)}(u) Q^{(l)}(u^*)$$

Energy quantization (V.,L.)

$$\epsilon = n + i\frac{d}{du} \ln(u+i)^t Q^{(t)}(u)|_{u=i}\,,\ t = 1, 2, ..., n-1$$

Intercepts and anomalous dimensions

$$\Delta_{Odd} = 0\,,\ \Delta_4\,;\ \gamma_3(\omega)\,,\ \gamma_4(\omega)$$

14 Pomeron in a thermostat

Invariants in the t-channel

$$t = 4E^2 \,, \;\; s = -2(\vec{p})^2 (1 - \cos\theta)$$

Periodicity of fields at a finite temperature

$$\phi(x_4) = \phi(x_4 + \frac{1}{T})$$

Regge kinematics in the s-channel

$$s >> T^2 \sim -t > 0$$

BFKL equation (V.,L.)

$$H_{12}\Psi = \Psi \,, \;\; H_{12} = h_{12} + h^*_{12} \,, \;\; \rho_r \to \frac{2\pi}{T}\,\rho_r$$

Holomorphic Hamiltonian

$$h_{12} = \sum_{r=1}^{2} \left[\Omega(p_r) + \frac{1}{p_r}\, G(\rho_{12})\, p_r \right]$$

$$\Omega(p) = \frac{\pi T}{2\lambda} + \frac{1}{2}\,[\psi(1+ip) + \psi(1-ip) - 2\psi(1)]$$

$$G(\rho_{12}) = -\frac{\pi\, T}{2\lambda} + \ln\left(2\sinh\frac{\rho_{12}}{2} \right)$$

15 Integrability at finite T

Conformal transformation

$$\rho_r = \ln\, \rho'_r$$

Integral of motion and Hamiltonian

$$A = -(\rho'_{12})^2 \frac{\partial}{\partial \rho'_1}\, \frac{\partial}{\partial \rho'_2}\,,$$

$$h_{12} = \ln(p'_1\, p'_2) + \frac{1}{p'_1}\, \log(\rho'_{12})\, p'_1 + \frac{1}{p'_2}\, \log(\rho'_{12})\, p'_2 - 2\psi(1)$$

Operator identity

$$\frac{1}{2}\left[\psi\left(1 + z\frac{\partial}{\partial z}\right) + \psi\left(-z\frac{\partial}{\partial z}\right)\right] = \ln\, z + \ln\, \frac{\partial}{\partial z}$$

Integrable Heisenberg model with spins

$$M_k = \partial_k \quad , \quad M_+ = e^{-\rho_k}\, \partial_k \quad , \quad M_- = -e^{\rho_k}\, \partial_k$$

Baxter function for the Pomeron state (V.,L.)

$$Q(p) = \psi(p, -p)$$

16 Pomeron in $N=4$ SUSY

BFKL kernel in the two-loop approximation (K., L.)

$$\omega = 4\,\hat{a}\,\chi(n,\gamma) + 4\,\hat{a}^2\,\Delta(n,\gamma)\,,\;\; \hat{a} = g^2 N_c/(16\pi^2)$$

Non-analytic terms in QCD

$$\Delta_{QCD}(n,\gamma) = c_0\delta_{n,0} + c_2\delta_{n,0} + ...$$

Hermitian separability in $N=4$ SUSY

$$\Delta(n,\gamma) = \phi(M) + \phi(M^*) - \frac{\rho(M)+\rho(M^*)}{2\hat{a}/\omega}\,,\; M = \gamma + \frac{|n|}{2}\,,$$

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2)\,,\; \beta'(z) = \frac{1}{4}\left[\Psi'\Big(\frac{z+1}{2}\Big) - \Psi'\Big(\frac{z}{2}\Big)\right]$$

Maximal transcedentality

$$\phi(M) = 3\zeta(3) + \Psi^{''}(M) - 2\Phi(M) + 2\beta^{'}(M)\Big(\Psi(1) - \Psi(M)\Big)$$

$$+\sum_{k=0}^{\infty}\frac{(-1)^k}{k+M}\left(\Psi'(k+1)\;-\;\frac{\Psi(k+1)-\Psi(1)}{k+M}\right)$$

17 Anomalous dimensions

Operator expansion

$$< \phi(\overrightarrow{\rho_1})\, \phi(\overrightarrow{\rho_2})\, \phi(\overrightarrow{\rho_{1'}})\, \phi(\overrightarrow{\rho_{2'}}) >\sim$$

$$< \phi(\overrightarrow{\rho_1})\, \phi(\overrightarrow{\rho_2}) \sum_n |\rho_{1'2'}|^{2\Gamma_\omega(n)} \left(\frac{\rho_{1'2'}}{\rho^*_{1'2'}}\right)^{\frac{|n|}{2}} O_{|n|,\nu_\omega}(\overrightarrow{\rho_{1'}}) >$$

Perturbative expansion

$$\Gamma_\omega = 1 + \frac{|n|}{2} - \gamma_n(\omega)\,,\ \gamma_n(\omega)|_{\omega\to 0} = \frac{g^2 N_c}{4\pi^2\omega} + O(g^4)$$

Hypothesis (K., L. (2000))

$$\lim_{j\to -r} \gamma(j) = \lim_{|n|\to -r-1} \gamma_n(j+r)$$

Prediction (A.K., L.L. (2003))

$$\gamma(j)|_{j\to -r} = \frac{g^2 N_c}{4\pi^2}\frac{1}{j+r} + \left(\frac{g^2 N_c}{4\pi^2}\right)^2 \frac{1+(-1)^r}{2(j+r)^3} + \ldots$$

Earlier result (L.L. (1997))

$$\gamma(j) = \frac{g^2 N_c}{16\pi^2}\gamma^{LLA}(j),\quad \gamma^{LLA}(j) = 4\Big(\Psi(1) - \Psi(j-1)\Big)$$

Integrable Heisenberg spin model (L.L. (1997))

One loop anomalous dimensions

Wilson twist-2 operators

$$\mathcal{O}^g_{\mu_1,\ldots,\mu_j} = \hat{S} G^a_{\rho\mu_1} D_{\mu_2} D_{\mu_3} \ldots D_{\mu_{j-1}} G^a_{\rho\mu_j} \, ,$$

$$\tilde{\mathcal{O}}^g_{\mu_1,\ldots,\mu_j} = \hat{S} G^a_{\rho\mu_1} D_{\mu_2} D_{\mu_3} \ldots D_{\mu_{j-1}} \tilde{G}^a_{\rho\mu_j} \, ,$$

$$\mathcal{O}^q_{\mu_1,\ldots,\mu_j} = \hat{S} \bar{\Psi}^a \gamma_{\mu_1} D_{\mu_2} \ldots D_{\mu_j} \Psi^a \, ,$$

$$\tilde{\mathcal{O}}^q_{\mu_1,\ldots,\mu_j} = \hat{S} \bar{\Psi}^a \gamma_5 \gamma_{\mu_1} D_{\mu_2} \ldots D_{\mu_j} \Psi^a \, ,$$

$$\mathcal{O}^\varphi_{\mu_1,\ldots,\mu_j} = \hat{S} \bar{\Phi}^a D_{\mu_1} D_{\mu_2} \ldots D_{\mu_j} \Phi^a$$

Anomalous dimension matrix (L.L. (1999))

$$\gamma_{gg} = -\frac{8}{j-1} + \frac{8}{j} - \frac{8}{j+1} + \frac{8}{j+2} + 8S_1(j) \, , \;\; \gamma_{q\varphi} = -\frac{16}{j} \, ,$$

$$\gamma_{gq} = -\frac{8}{j-1} + \frac{8}{j} - \frac{4}{j+1} \, , \;\; \gamma_{g\varphi} = -\frac{8}{j-1} + \frac{8}{j} \, ,$$

$$\gamma_{qg} = -\frac{16}{j} + \frac{32}{j+1} - \frac{32}{j+2} \, , \;\; \gamma_{qq} = -\frac{16}{j} + \frac{16}{j+1} + 8S_1(j),$$

$$\gamma_{\varphi g} = -\frac{24}{j+1} + \frac{24}{j+2} \, , \;\; \gamma_{\varphi q} = -\frac{12}{j+1} \, , \;\; \gamma_{\varphi\varphi} = 8S_1(j),$$

$$\widetilde{\gamma}_{gg} = -\frac{16}{j} + \frac{16}{j+1} + 8S_1(j) \, , \;\; \widetilde{\gamma}_{gq} = -\frac{8}{j} + \frac{4}{j+1} \, ,$$

$$\widetilde{\gamma}_{qg} = \frac{16}{j} - \frac{32}{j+1} \, , \;\; \widetilde{\gamma}_{qq} = \frac{8}{j} - \frac{8}{j+1} + 8S_1(j)$$

Two-loop universal anomalous dimension

Diagonalization in the Born Approximation

$$\begin{vmatrix} 8S_1(j-2) & 0 & 0 \\ 0 & 8S_1(j) & 0 \\ 0 & 0 & 8S_1(j+2) \end{vmatrix},$$

$$\begin{vmatrix} 8S_1(j-1) & 0 \\ & \\ 0 & 8S_1(j+1) \end{vmatrix}$$

Universal anomalous dimension

$$U\gamma U^{+} = \gamma_{uni}(j)\,,\ \gamma^{(0)}_{uni}(j) = -4S_1(j-2)\,,\ S_r(j) = \sum_{i=1}^{j} \frac{1}{i^r}$$

Most transcendental functions (A.K.,L.L. (2000))

$$\gamma_{uni}(j) = \hat{a}\gamma^{(0)}_{uni}(j) + \hat{a}^2\gamma^{(1)}_{uni}(j) + \hat{a}^3\gamma^{(2)}_{uni}(j) + \ldots$$

Two-loop dimension (A.K.,L.L.,V.V (2003))

$$\gamma^{(1)}_{uni}(j+2)/8 = 2S_1(j)\left(S_2(j)+S_{-2}(j)\right)-2S_{-2,1}(j)+S_3(j)$$

$$+S_{-3}(j),\ S_{-r}(j) = \sum_{i=1}^{j} \frac{(-1)^i}{i^r}, S_{-2,1} = \sum_{m=1}^{j} \frac{(-1)^m}{m^2} S_1(m)$$

Three-loop anomalous dimension

QCD dimension matrix (S.M., J.V., A. V. (2004)

$$\gamma^{ik}_{QCD}(j) = ..., \;\; i,k = g,q$$

N=4 universal dimension (A.K.,L.L.,A.O.,V.V.)

$$\gamma^{N=4}_{uni}(j) = \hat{a}\gamma^{(0)}_{uni}(j) + \hat{a}^2\gamma^{(1)}_{uni}(j) + \hat{a}^3\gamma^{(2)}_{uni}(j) + \ldots ,$$

$$\gamma^{(2)}_{uni}(j+2)/32 = -12\,\left(S_{-3,1,1} + S_{-2,1,2} + S_{-2,2,1}\right)$$

$$+6\,\left(S_{-4,1} + S_{-3,2} + S_{-2,3}\right) - 3\,S_{-5} - 2\,S_3\,S_{-2} - S_5$$

$$-2\,S_1^2\left(3\,S_{-3} + S_3 - 2\,S_{-2,1}\right) - S_2\left(S_{-3} + S_3 - 2\,S_{-2,1}\right)$$

$$+24\,S_{-2,1,1,1} - S_1\left(8S_{-4} + S^2_{-2} + 4S_2S_{-2} + 2S_2^2\right)$$

$$-S_1\left(3S_4 - 12S_{-3,1} - 10S_{-2,2} + 16S_{-2,1,1}\right)$$

Harmonic sums

$$S_a(j) = \sum_{m=1}^{j} \frac{1}{m^a}\,, \;\; S_{a,b,c,\cdots}(j) = \sum_{m=1}^{j} \frac{1}{m^a}\, S_{b,c,\cdots}(m)\,,$$

$$S_{-a}(j) = \sum_{m=1}^{j} \frac{(-1)^m}{m^a}, \; S_{-a,b,\cdots}(j) = \sum_{m=1}^{j} \frac{(-1)^m}{m^a}\, S_{b,\cdots}(m),$$

$$\overline{S}_{-a,b,c\cdots}(j) = (-1)^j S_{-a,b,\ldots}(j) + S_{-a,b,\cdots}(\infty)\left(1-(-1)^j\right)$$

Comparison with other approaches (A.K.,L.L.,A.O.,V.V.)

Singularities at $j = 1 + \omega \to 0$

$$\gamma_{uni}^{N=4}(j) = \hat{a}\frac{4}{\omega} - 32\zeta_3\,\hat{a}^2 + 32\zeta_3\,\hat{a}^3\frac{1}{\omega} + \ldots$$

DL resummation at $j + 2r = \omega \to 0$

$$\gamma_{uni} = 4\,\frac{\hat{a}}{\omega} + \frac{\gamma_{uni}^2}{\omega}$$

Anomalous dimensions at large j

$$\gamma_{uni} = a(z)\,\ln j\,,\;\; z = \frac{\alpha N_c}{\pi} = 4\hat{a}$$

Perturbative results

$$a = -z + \frac{\pi^2}{12}\,z^2 - \frac{11}{720}\pi^4 z^3 + \ldots$$

Polyakov AdS/CFT prediction

$$\lim_{z\to\infty} a = -z^{1/2} + \frac{3\ln 2}{4\pi} + \ldots$$

Resummation

$$\widetilde{a} = -z + \frac{\pi^2}{12}\,\widetilde{a}^2 = -z + \frac{\pi^2}{12}\,z^2 - \frac{1}{72}\pi^4 z^3 + \ldots$$

Eden-Staudacher approach

Anomalous dimension at large j

$$\gamma = 8\,g^2\,\sigma(0)\,\ln j = 4\,g\,\sqrt{2}\,f(0)\,\ln j$$

$$\sigma(t) = \epsilon\,f(x)\;,\;\; t = \epsilon\,x\;,\;\; \epsilon = \frac{1}{g\,\sqrt{2}}\,,$$

Eden-Staudacher equation

$$\epsilon\,f(x) = \frac{t}{e^t - 1}\left(\frac{J_1(x)}{x} - \int_0^\infty dx'\,K(x,x')\,f(x')\right)\;,$$

$$K(x,y) = \frac{J_1(x)\,J_0(y) - J_1(y)\,J_0(x)}{x-y}\,.$$

Mellin transformation

$$f(x) = \int_{-i\infty}^{i\infty} \frac{d\,j}{2\pi\,i}\,e^{x\,j}\,\phi(j)\,.$$

Anzatz for the solution

$$\phi(j) = \sum_{n=1}^{\infty} \phi_{n,\epsilon}(j)\,\left(\delta_{n,1} - a_{n,\epsilon}\right)\;,$$

$$\phi_{n,\epsilon}(j) = \sum_{s=1}^{\infty} \frac{\left(\sqrt{(j+s\,\epsilon)^2+1} + j + s\,\epsilon\right)^{-n}}{\sqrt{(j+s\,\epsilon)^2+1}}$$

Analytic properties of the kernel

Linear set of equations

$$a_{n,\epsilon} = \sum_{n'=1}^{\infty} K_{n,n'}(\epsilon)\ (\delta_{n',1} - a_{n',\epsilon})\ ,$$

Kernel

$$K_{n,n'}(\epsilon) = 2n \sum_{R=0}^{\infty} (-1)^R \frac{2^{-2R-n-n'}}{\epsilon^{2R+n+n'}}$$

$$\zeta(2R+n+n')\,\frac{(2R+n+n'-1)!\,(2R+n+n')!}{R!\,(R+n)!\,(R+n')!\,(R+n+n')!}$$

Transcedentality with integer coefficients

$$\gamma(\epsilon) = 8 \sum_{k=1}^{\infty} \left(-\frac{1}{4\epsilon^2}\right)^k \sum_{[s_t]} c_{[s_t]} \prod_r \zeta(s_r)\ , \ \sum_t s_t = 2k-2$$

Another representation for $K_{n,n'}(\epsilon)$

$$\frac{\Gamma^2(\frac{n+n'+1}{2})\,\Gamma(\frac{n+n'}{2}+1)\,\Gamma(\frac{n+n'}{2})}{\pi\,\Gamma(n)\,\Gamma(n'+1)\,\Gamma(n+n'+1)} \sum_{k=1}^{\infty} \left(\frac{2}{k\,\epsilon}\right)^{n+n'} F\left(\frac{-4}{k^2\epsilon^2}\right)$$

Finite distance singularity of the kernel

$$K_{n,n'} \approx \frac{n}{\pi} \left(\frac{2}{\epsilon}\right)^{n+n'} (1+\frac{4}{\epsilon^2})\ \ln(1+\frac{4}{\epsilon^2})$$

Analytic properties of the solution

New variable

$$z = j + \sqrt{j^2+1}$$

Dispersion representation

$$\xi(z) = \int_L \frac{dz'}{2\,\pi\, i}\, \frac{\xi(z') - \xi(-1/z')}{z - z'}$$

Linearized "unitarity"

$$\frac{\xi(z) - \xi(-1/z)}{2\sqrt{j^2+1}} = 1 - \sum_{s=1}^{\infty} \frac{\xi\left(j + s\,\epsilon + \sqrt{(j+s\,\epsilon)^2+1}\right)}{\sqrt{(j+s\,\epsilon)^2+1}}$$

ES equation at strong couplings

$$-\epsilon\, z\, \frac{\partial}{\partial z}\, \chi(z) = \frac{1}{z} - \int_L \frac{dz'}{2\,\pi\, i}\, \frac{z'^2+1}{z'}\, \frac{\chi(\widetilde{z}')}{z - z'}$$

Singularity at $z \to 0$

$$\chi_{sing}(z) = -\frac{1}{J_0(2\,\epsilon^{-1})} \int_L \frac{d\,z'}{2\pi i}\, \frac{\exp \frac{z'^2-1}{\epsilon\, z'}}{z - z'}$$

Singular part of the anomalous dimension

$$\gamma_{sing} = \frac{2}{\epsilon}\, \frac{J_1(2\epsilon^{-1})}{J_0(2\epsilon^{-1})} \approx \frac{2\, g}{\sqrt{2}}\, \tan\left(\frac{2}{\epsilon} - \frac{\pi}{4}\right)$$

Pomeron and graviton at $N=4$ SUSY (OKLV)

BFKL Pomeron in a diffusion approximation

$$j = 2 - \Delta - D\,\nu^2$$

Anomalous dimension of twist-2 operators

$$\gamma = 1 + \frac{j-2}{2} + i\nu$$

Constraint from the conservation of $T_{\mu\nu}$

$$\gamma = (j-2)\left(\frac{1}{2} - \frac{1/\Delta}{1+\sqrt{1+(j-2)/\Delta}}\right)$$

AdS/CFT for the graviton Regge trajectry

$$j = 2 + \frac{\alpha'}{2}\,t\,,\;\; t = E^2/R^2\,,\;\; \alpha' = \frac{R^2}{2}\,\Delta$$

Large coupling asymptotics

$$\gamma_{|z\to\infty} = -\sqrt{j-2}\,\Delta^{-1/2}_{|j\to\infty} = \sqrt{\pi j}\,z^{1/4}$$

Pomeron intercept at large α and its perturbative estimate

$$j = 2 - \Delta\,,\;\; \Delta = \frac{1}{\pi}\,z^{-1/2} \approx \frac{\sqrt{3}}{2\pi} z^{-1/2}$$

Resummation of the DGLAP and BFKL equations

Slope of the anomalous dimension

$$\gamma'(2) = \frac{1}{2} - \frac{1}{2\Delta} = -\frac{\pi^2}{6} z + \frac{\pi^4}{72} z^2 - \frac{\pi^6}{540} z^3 + \ldots$$

Resummation equation

$$\frac{\pi^2}{6} z = -\widetilde{b} + \frac{1}{2}\widetilde{b}^2 \,, \;\; b = \gamma'(2)$$

Weak and strong coupling asymptotics

$$\widetilde{b} = -\frac{\pi^2}{6} z + \frac{\pi^4}{72} z^2 - \frac{\pi^6}{432} z^3 + \ldots \,, \;\; \widetilde{\Delta} = \frac{\sqrt{3}}{2\pi} z^{-1/2}$$

Two-loop BFKL results

$$\omega_0 = 1 - \Delta = 4 \ln 2 \, z - a_1 \, z^2 + \ldots$$

$$D = 14 \, \zeta(3) \, z - b_1 z^2 + \ldots \,, \;\; \zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

Resummation for large couplings

$$4 \ln 2 \, z = \frac{(1-\Delta) - \left(2 - a_1 \, (4 \, \ln 2)^{-2}\right) (1-\Delta)^2}{\Delta^2}$$

$$14 \, \zeta(3) \, z = \frac{1 - \sqrt{1 - 2 \, D^3 + c D^4}}{D^2}$$

18 Discussion

1. High energies and the reggeized gluon calculus.
2. Pomeron and Odderon as gluon composite states.
3. Möbius invariance of the reggeon interactions.
4. Holomorphic separability and factorization.
5. Duality symmetry and signature degeneracy.
6. Integrability of the reggeon dynamics.
7. Heisenberg model with the Möbius spins.
8. Baxter-Sklyanin representation.
9. Integrability at finite T.
10. BFKL Pomeron in N=4 SUSY.
11. Transcedentality for anomalous dimensions.
12. AdS/CFT correspondence and resummation.
13. Diffusion and the graviton Regge trajectory.
14. Pomeron intercept at large coupling constants.