Correlation Functions of Heisenberg Spin Chains: Algebraic Bethe Ansatz Approach

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Quantum integrable models

Interests

- Exacts results not accessible by usual techniques
- Direct applications : condensed matter, solid state physics,..., strings?
- Mathematics : quantum groups, knot theory,...

What can we compute?

Hamiltonian spectrum, scattering matrix , partition function, critical exponents,....
(Bethe, Onsager, Yang, Baxter, McCoy, Faddeev, Zamolodchikov,...)

Correlation functions?

The spin-1/2 XXZ Heisenberg chain

$$H = \sum_{m=1}^{M} \left(\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta \left(\sigma_m^z \sigma_{m+1}^z - 1 \right) \right) - \frac{h}{2} \sum_{m=1}^{M} \sigma_m^z$$

- Spectrum :
  Bethe ansatz : Bethe, Orbach, Walker, Yang and Yang,...
  Algebraic Bethe ansatz : Faddeev, Sklyanin, Taktadjan,...

- Correlation functions :
  Free fermion point $\Delta = 0$: Lieb, Shultz, Mattis, Wu, McCoy, Sato, Jimbo, Miwa
  Starting 1985 Izergin, Korepin : first attempts using Bethe ansatz for general Δ
  General Δ : multiple integral representations
  ⋆ 1996 Jimbo and Miwa $\longrightarrow$ from qKZ equation
  ⋆ 1999 Kitanine, M, Terras $\longrightarrow$ from Algebraic Bethe Ansatz
  ⋆ Several new developments (Jimbo, Miwa, Smirnov,..., Gohmann, Klumper,...)

Correlation functions

At zero temperature only the ground state $|\omega\rangle$ contributes :

$$g_{12} = \langle\omega|\theta_1\theta_2|\omega\rangle$$

Two main strategies to evaluate such a function:

(i) compute the action of local operators on the ground state $\theta_1\theta_2|\omega\rangle = |\tilde{\omega}\rangle$ and then calculate the resulting scalar product:

$$g_{12} = \langle\omega|\tilde{\omega}\rangle$$

(ii) insert a sum over a complete set of eigenstates $|\omega_i\rangle$ to obtain a sum over one-point matrix elements (form factor type expansion) :

$$g_{12} = \sum_i \langle\omega|\theta_1|\omega_i\rangle \cdot \langle\omega_i|\theta_2|\omega\rangle$$

Algebraic Bethe ansatz and correlation functions

- Hamiltonian eigenstates

  Algebraic Bethe ansatz : $\sigma_m^\alpha \longrightarrow T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$

  $$T(\lambda) \equiv T_{a,1\ldots N}(\lambda) = L_{aN}(\lambda - \xi_N) \ldots L_{a1}(\lambda - \xi_1)$$

  $L_{an}(\lambda)$ being 2×2 matrices with entries function of $\sigma_n^{x,y,z}$ operators in site n.

  Yang-Baxter algebra : $R_{12}(\lambda_1, \lambda_2)T_1(\lambda_1)T_2(\lambda_2) = T_2(\lambda_2)T_1(\lambda_1)R_{12}(\lambda_1, \lambda_2)$

  Commuting conserved charges : $t(\lambda) = A(\lambda) + D(\lambda), \qquad [t(\lambda), t(\mu)] = 0$

  Hamiltonian : $H = 2 \sinh \eta \, \frac{\partial}{\partial \lambda} \log t(\lambda)\Big|_{\lambda = \frac{\eta}{2}} + c$ for all $\xi_j = 0$.

  Eigenstates of $t(\mu)$: $|\psi\rangle = \prod_k B(\lambda_k)|0\rangle$ with $\{\lambda_k\}$ solution of the Bethe equations.

- Action of local operators on eigenstates

Resolution of the quantum inverse scattering problem : $\sigma_m^\alpha \longleftarrow T(\lambda)$

$$\begin{aligned} \sigma_j^- &= \prod_{k=1}^{j-1} t(\xi_k) \cdot B(\xi_j) \cdot \prod_{k=1}^{j} t^{-1}(\xi_k), \\ \sigma_j^+ &= \prod_{k=1}^{j-1} t(\xi_k) \cdot C(\xi_j) \cdot \prod_{k=1}^{j} t^{-1}(\xi_k), \\ \sigma_j^z &= \prod_{k=1}^{j-1} t(\xi_k) \cdot (A-D)(\xi_j) \cdot \prod_{k=1}^{j} t^{-1}(\xi_k), \end{aligned} \tag{1}$$

$+$ Yang-Baxter algebra for A, B, C, D to get the action on arbitrary states, for example

$$\langle 0| \prod_{k=1}^{N} C(\lambda_k)\, A(\lambda_{N+1}) = \sum_{a'=1}^{N+1} \Lambda_{a'}\ \langle 0| \prod_{\substack{k=1\\ k\neq a'}}^{N+1} C(\lambda_k)$$

- Scalar products :

$$\langle 0| \prod_{j=1}^{N} C(\mu_j) \prod_{k=1}^{N} B(\lambda_k) |0\rangle = \frac{\det U(\{\mu_j\},\{\lambda_k\})}{\det V(\{\mu_j\},\{\lambda_k\})} \tag{2}$$

for $\{\lambda_k\}$ a solution of Bethe equations and $\{\mu_j\}$ an arbitrary set of parameters, :

$$U_{ab} = \partial_{\lambda_a}\tau(\mu_b,\{\lambda_k\}), \qquad V_{ab} = \frac{1}{\sinh(\mu_b - \lambda_a)}, \qquad 1 \leqslant a, b \leqslant N, \tag{3}$$

where $\tau(\mu_b,\{\lambda_k\})$ is the eigenvalue of the transfer matrix $t(\mu_b)$
(Slavnov formula, proofs from F-matrix by Kitanine, M, Terras)

Matrix elements of local operators

For example :

$$\langle 0| \prod_{j=1}^{N} C(\mu_j)\, \sigma_n^z \prod_{k=1}^{N} B(\lambda_k)\, |0\rangle =$$

$$= \langle 0| \prod_{j=1}^{N} C(\mu_j) \prod_{k=1}^{n-1} t(\xi_k) \cdot \big(A - D\big)(\xi_j) \cdot \prod_{k=1}^{n} t^{-1}(\xi_k) \prod_{k=1}^{N} B(\lambda_k)\, |0\rangle$$

Here the sets $\{\lambda_k\}$ and $\{\mu_j\}$ are both solutions of Bethe equations $\longrightarrow$

$$\langle 0| \prod_{j=1}^{N} C(\mu_j)\, \sigma_n^z \prod_{k=1}^{N} B(\lambda_k)\, |0\rangle = \Phi_n \, \langle 0| \prod_{j=1}^{N} C(\mu_j)\, \big(A - D\big)(\xi_j) \prod_{k=1}^{N} B(\lambda_k)\, |0\rangle$$

Hence it leads to determinant representations of these matrix elements (using the scalar product formula)

Correlation functions : elementary blocks

$$F_m(\{\epsilon_j,\epsilon'_j\}) = \frac{\langle\psi_g|\prod_{j=1}^{m} E_j^{\epsilon'_j,\epsilon_j}|\psi_g\rangle}{\langle\psi_g|\psi_g\rangle} \qquad\qquad E_{lk}^{\epsilon',\epsilon} = \delta_{l,\epsilon'}\delta_{k,\epsilon} \tag{4}$$

Solution of the quantum inverse scattering problem + Yang-Baxter algebra of operators $T(\lambda)$ $\longrightarrow$ Multiple integral formula for the correlation functions

$$F_m(\{\epsilon_j,\epsilon'_j\}) = (\prod_{k=1}^{m}\int_{C_k^h} d\lambda_k)\ \ \Omega_m(\{\lambda_k\},\ \{\epsilon_j,\epsilon'_j\})\ S_h(\{\lambda_k\}) \tag{5}$$

where $\Omega_m(\{\lambda_k\},\ \{\epsilon_j,\epsilon'_j\})$ is purely algebraic and $S_h(\{\lambda_k\})$, C_k^h are depending on the regime and the magnetic field h.

$\longrightarrow$ Proof of the results and conjectures of Jimbo, Miwa et al. and extension to the non zero magnetic field h (a case where the quantum affine symmetry is broken)

What about this result ?

$\longrightarrow$ A priori, the problem is solved:

- expression of all elementary blocks $\langle\psi_g|\, E_1^{\epsilon'_1,\epsilon_1} \ldots E_m^{\epsilon'_m,\epsilon_m}\,|\psi_g\rangle$
- any correlation function $= \sum$ (elementary blocks)

$\longrightarrow$ From a practical point of view, there are **two main problems:**

(1) physical correlation function = huge sum of elementary blocks at large distances
Example: two-point function

$$\langle\psi_g|\sigma_1^z\,\sigma_m^z|\psi_g\rangle \equiv \langle\psi_g|(E_1^{11}-E_1^{22})\underbrace{\prod_{j=2}^{m-1}(E_j^{11}+E_j^{22})}_{\text{propagator}}(E_m^{11}-E_m^{22})|\psi_g\rangle$$

$$= \sum_{2^m \text{ terms}} \text{(elementary blocks)} \underset{m\to\infty}{\sim} ?$$

$\rightsquigarrow$ re-summation

(2) each block has a complicated expression

Example: emptiness formation probability for $h = 0$ in the massless regime ($-1 < \Delta = \cosh\zeta < 1$)

$$\tau(m) \equiv \langle\psi_g| \prod_{k=1}^{m} \frac{1-\sigma_k^z}{2} |\psi_g\rangle$$

$$= (-1)^m \Big(-\frac{\pi}{\zeta}\Big)^{\frac{m(m-1)}{2}} \int_{-\infty}^{\infty} \frac{d^m\lambda}{2\pi} \prod_{a>b}^{m} \frac{\sinh\frac{\pi}{\zeta}(\lambda_a-\lambda_b)}{\sinh(\lambda_a-\lambda_b-i\zeta)}$$

$$\times \prod_{j=1}^{m} \frac{\sinh^{j-1}(\lambda_j-i\zeta/2)\,\sinh^{m-j}(\lambda_j+i\zeta/2)}{\cosh^m\frac{\pi}{\zeta}\lambda_j}$$

$\rightsquigarrow$ dependence on m ?

(1)+(2) $\Rightarrow$ difficult to analyse! $\Rightarrow$ new tools needed!

Emptiness formation probability

Integral representation as a single elementary block but previous expression not symmetric

$\longrightarrow$ **symmetrisation** of the integrand:

$$\tau(m) = \lim_{\xi_1,\ldots\xi_m \to -\frac{i\zeta}{2}} \frac{1}{m!} \int_{-\infty}^{\infty} d^m\lambda \prod_{a,b=1}^{m} \frac{1}{\sinh(\lambda_a - \lambda_b - i\zeta)}$$

$$\times \prod_{a<b}^{m} \frac{\sinh(\lambda_a - \lambda_b)}{\sinh(\xi_a - \xi_b)} \cdot Z_m(\{\lambda\},\{\xi\}) \cdot \det{}_m[\rho(\lambda_j,\xi_k)]$$

where $Z_m(\{\lambda\},\{\xi\})$ is the partition function of the 6-vertex model with domain wall boundary conditions (Izergin) and $\rho(\lambda,\xi) = [-2i\zeta \sinh \frac{\pi}{\zeta}(\lambda_j - \xi_k)]^{-1}$ is the inhomogeneous version of the density for the ground state (massless regime $\Delta = \cos\zeta$, $h = 0$).

$$Z_m(\{\lambda\},\{\xi\}) = \prod_{a=1}^{m}\prod_{b=1}^{m} \frac{\sinh(\lambda_a-\xi_b)\sinh(\lambda_a-\xi_b-i\zeta)}{\sinh(\lambda_a-\lambda_b-i\zeta)} \cdot \frac{\det_m\left(\frac{-i\sin\zeta}{\sinh(\lambda_j-\xi_k)\sinh(\lambda_j-\xi_k-i\zeta)}\right)}{\prod\limits_{a>b}^{m}\sinh(\xi_a-\xi_b)}$$

Exact computation for $\Delta = 1/2$

The determinant structure combined with the periodicity properties at $\Delta = 1/2$ enable us to separate the multiple integral and to compute them :

$$\tau_{inh}(m,\{\xi_j\}) = \frac{(-1)^{\frac{m^2-m}{2}}}{2^{m^2}} \prod_{a>b}^{m} \frac{\sinh 3(\xi_b-\xi_a)}{\sinh(\xi_b-\xi_a)} \prod_{\substack{a,b=1\\a\neq b}}^{m} \frac{1}{\sinh(\xi_a-\xi_b)} \cdot \det_m\left(\frac{3\sinh\frac{\xi_j-\xi_k}{2}}{\sinh\frac{3(\xi_j-\xi_k)}{2}}\right)$$

$$\tau(m) = \left(\frac{1}{2}\right)^{m^2} \prod_{k=0}^{m-1} \frac{(3k+1)!}{(m+k)!} = \left(\frac{1}{2}\right)^{m^2} A_m$$

$\longrightarrow$ A_m - number of alternating sign matrices

$\longrightarrow$ first exact result for $\Delta \neq 0$

Asymptotic Results:

⋆ massless case $(-1 < \Delta = \cos\zeta \leqslant 1)$

$$\lim_{m\to\infty} \frac{\log\tau(m)}{m^2} = \log\frac{\pi}{\zeta} + \frac{1}{2}\int_{\mathbb{R}-i0} \frac{d\omega}{\omega} \frac{\sinh\frac{\omega}{2}(\pi-\zeta)\cosh^2\frac{\omega\zeta}{2}}{\sinh\frac{\pi\omega}{2}\sinh\frac{\omega\zeta}{2}\cosh\omega\zeta}$$

$$= \begin{cases} -\frac{1}{2}\log 2 & \text{for } \Delta = 0 \\ \frac{3}{2}\log 3 - 3\log 2 & \text{for } \Delta = \frac{1}{2} \\ \log\left[\frac{\Gamma(\frac{3}{4})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4})}\right] & \text{for } \Delta = 1 \text{ (XXX chain)} \end{cases}$$

⋆ massive case $(\Delta = \cosh\zeta > 1)$

$$\lim_{m\to\infty} \frac{\log\tau(m)}{m^2} = -\frac{\zeta}{2} - \sum_{n=1}^{\infty} \frac{e^{-n\zeta}}{n} \frac{\sinh(n\zeta)}{\cosh(2n\zeta)} \xrightarrow[\zeta\to 0]{} \log\left[\frac{\Gamma(\frac{3}{4})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4})}\right] \quad \text{(XXX)}$$

$$\xrightarrow[\zeta\to +\infty]{} -\infty \qquad \text{(Ising)}$$

Generating function for σ^z correlation functions

$$Q^{\kappa}_{1,m} = \prod_{n=1}^{m} \left(\frac{1+\kappa}{2} + \frac{1-\kappa}{2} \cdot \sigma^z_n \right) = \prod_{a=1}^{m} (A + \kappa D)(\xi_a) \prod_{b=1}^{m} (A + D)^{-1}(\xi_b)$$

Generating function (polynomial in κ):

$$\langle Q^{\kappa}_{1,m} \rangle = \frac{\langle \psi(\{\lambda\}) | Q^{\kappa}_{1,m} | \psi(\{\lambda\}) \rangle}{\langle \psi(\{\lambda\}) | \psi(\{\lambda\}) \rangle},$$

where $|\psi(\{\lambda\})\rangle$ is an eigenstate of $\mathcal{T}(\mu)$ depending on the N parameters λ_j satisfying Bethe equations. We have,

$$\frac{1}{2}\langle (1-\sigma^z_1)(1-\sigma^z_{m+1}) \rangle = \frac{\partial^2}{\partial \kappa^2} \langle \left(Q^{\kappa}_{1,m+1} - Q^{\kappa}_{1,m} - Q^{\kappa}_{2,m+1} + Q^{\kappa}_{2,m} \right) \rangle \bigg|_{\kappa=1}$$

$$Q_m(\kappa) = \sum_{s=0}^{m} \kappa^s G_s(m).$$

$$G_s(m) = \frac{1}{s!(m-s)! \sin^m \zeta} \prod_{j<k} \frac{1}{\sinh(\xi_j - \xi_k)} \int_{-\infty}^{\infty} d\lambda_1 \ldots \int_{-\infty}^{\infty} d\lambda_m \times$$
$$\times\ Z_m(\{\lambda\}, \{\xi\}) \ \det_m(\rho(\{\lambda\}, \{\xi\})) \times \Theta_s(\{\lambda\}).$$

$$\Theta_m^s(\lambda_1, \ldots, \lambda_m) = \prod_{m \geqslant j > k > s} \frac{\sinh(\lambda_j - \lambda_k)}{\sinh(\lambda_j - \lambda_k + i\zeta) \sinh(\lambda_j - \lambda_k - i\zeta)}$$
$$\times \prod_{k=1}^{s} \prod_{j=s+1}^{m} \frac{1}{\sinh(\lambda_j - \lambda_k)} \prod_{s \geqslant j > k \geqslant 1} \frac{\sinh(\lambda_j - \lambda_k)}{\sinh(\lambda_j - \lambda_k + i\zeta) \sinh(\lambda_j - \lambda_k - i\zeta)}$$

Re-summation for the generating function

In the thermodynamic limit :

$$Q^{\kappa}_{1,m} = \sum_{n=0}^{m} \frac{1}{(n!)^2} \oint_{\Gamma} \prod_{j=1}^{n} \frac{dz_j}{2\pi i} \int_C d^n\lambda \prod_{b=1}^{n} \prod_{a=1}^{m} \frac{f(z_b - \xi_a)}{f(\lambda_b - \xi_a)}$$

$$W_n(\{\lambda\},\{z\}) \cdot \det_n\Big[\tilde{M}_{jk}(\{\lambda\}|\{z\})\Big] \cdot \det_n\Big[\rho(\lambda_j, z_k)\Big],$$

where

$$\tilde{M}_{jk} = t(z_k - \lambda_j) + \kappa t(\lambda_j - z_k) \prod_{a=1}^{n} \frac{\sinh(\lambda_a - \lambda_j - i\zeta)}{\sinh(\lambda_j - \lambda_a - i\zeta)} \prod_{a=1}^{n} \frac{\sinh(\lambda_j - z_a - i\zeta)}{\sinh(z_a - \lambda_j - i\zeta)}.$$

with $t(x) = \frac{\sin\zeta}{\sinh(x)\sinh(x-i\zeta)}$ and $f(x) = \frac{\sinh(x+\eta)}{\sinh(x)}$.

Note: for $\kappa = 1$ and $n > 0$ we obtain $\det_n \tilde{M}_{jk} = 0$

Master equation for σ^z correlation functions

Let the inhomogeneities $\{\xi\}$ be generic and the set $\{\lambda\}$ be an admissible off-diagonal solution of the Bethe equations (cf.Tarasov - Varchenko). Then there exists $\kappa_0 > 0$ such, that for $|\kappa| < \kappa_0$ the expectation value of the operator $Q^{\kappa}_{1,m}$:

$$\langle Q^{\kappa}_{1,m} \rangle = \frac{1}{N!} \oint\limits_{\Gamma\{\xi\}\cup\Gamma\{\lambda\}} \prod_{j=1}^{N} \frac{dz_j}{2\pi i} \cdot \prod_{a,b=1}^{N} \sinh^2(\lambda_a - z_b) \cdot \prod_{a=1}^{m} \frac{\tau_\kappa(\xi_a|\{z\})}{\tau(\xi_a|\{\lambda\})}$$

$$\times \frac{\det_N\left(\frac{\partial \tau_\kappa(\lambda_j|\{z\})}{\partial z_k}\right) \cdot \det_N\left(\frac{\partial \tau(z_k|\{\lambda\})}{\partial \lambda_j}\right)}{\prod\limits_{a=1}^{N} \mathcal{Y}_\kappa(z_a|\{z\}) \cdot \det_N\left(\frac{\partial \mathcal{Y}(\lambda_k|\{\lambda\})}{\partial \lambda_j}\right)}.$$

The integration contour is such that the only singularities of the integrand within the contour $\Gamma\{\xi\} \cup \Gamma\{\lambda\}$ which contribute to the integral are the points $\{\xi\}$ and $\{\lambda\}$ (hep-th/0406190).

Twisted transfer matrix : $\mathcal{T}_\kappa(\lambda) = A(\lambda) + \kappa D(\lambda)$, $\quad [\mathcal{T}_\kappa(\lambda), \mathcal{T}_\kappa(\mu)] = 0$

Eigenstates of $\mathcal{T}_\kappa(\mu)$ and H obtained from : $|\psi\rangle = \prod_k B(\lambda_k)|0\rangle$, $\{\lambda_k\}$ solution of the (twisted) Bethe equations :

$$\mathcal{Y}_\kappa(\lambda_j|\{\lambda\}) = 0, \qquad j = 1, \ldots, N.$$

$$\mathcal{Y}_\kappa(\mu|\{\lambda\}) = a(\mu)\prod_{k=1}^{N}\sinh(\lambda_k - \mu + \eta) + \kappa\, d(\mu)\prod_{k=1}^{N}\sinh(\lambda_k - \mu - \eta)$$

Eigenvalue $\tau_\kappa(\mu|\{\lambda\})$ of the operator $\mathcal{T}_\kappa(\mu)$:

$$\tau_\kappa(\mu|\{\lambda\}) = a(\mu)\prod_{k=1}^{N}\frac{\sinh(\lambda_k - \mu + \eta)}{\sinh(\lambda_k - \mu)} + \kappa\, d(\mu)\prod_{k=1}^{N}\frac{\sinh(\mu - \lambda_k + \eta)}{\sinh(\mu - \lambda_k)}$$

Time-dependent master equation

$$\langle Q_\kappa(m,t)\rangle = \frac{1}{N!}\oint_{\Gamma\{\pm\frac{\eta}{2}\}\cup\Gamma\{\lambda\}} \prod_{j=1}^{N}\frac{dz_j}{2\pi i}\cdot\prod_{b=1}^{N} e^{it\big(E(z_b)-E(\lambda_b)\big)+im\big(p(z_b)-p(\lambda_b)\big)}$$

$$\times\prod_{a,b=1}^{N}\sinh^2(\lambda_a - z_b)\cdot\frac{\det_N\left(\frac{\partial\tau_\kappa(\lambda_j|\{z\})}{\partial z_k}\right)\cdot\det_N\left(\frac{\partial\tau(z_k|\{\lambda\})}{\partial\lambda_j}\right)}{\prod_{a=1}^{N}\mathcal{Y}_\kappa(z_a|\{z\})\cdot\det_N\left(\frac{\partial\mathcal{Y}(\lambda_k|\{\lambda\})}{\partial\lambda_j}\right)}$$

$$E(z) = \frac{2\sinh^2\eta}{\sinh(z-\frac{\eta}{2})\sinh(z+\frac{\eta}{2})}$$

$$p(\lambda) = i\log\left(\frac{\sinh(\lambda-\frac{\eta}{2})}{\sinh(\lambda+\frac{\eta}{2})}\right)$$

Dynamical correlation functions (infinite lattice)

$$\langle Q_\kappa(m,t)\rangle = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \int_{-\Lambda_h}^{\Lambda_h} d^n\lambda \oint_{\Gamma\{\pm\frac{\eta}{2}\}} \prod_{j=1}^{n} \frac{dz_j}{2\pi i} \cdot G_n(\kappa, \{\lambda\}, \{z\})$$

$$\times \prod_{b=1}^{n} e^{it(E(z_b)-E(\lambda_b))+im(p(z_b)-p(\lambda_b))} \cdot \det_n[\mathcal{R}_n^\kappa(\lambda_j, z_k)].$$

$$\mathcal{R}_n^\kappa(\lambda, z|\{\lambda\},\{z\}) = \begin{cases} \rho(\lambda, z), & z \sim \eta/2; \\ -\kappa^{-1}\rho(\lambda, z+\eta) \prod_{b=1}^{n} \frac{\sinh(z-\lambda_b+\eta)\sinh(z_b-z+\eta)}{\sinh(\lambda_b-z+\eta)\sinh(z-z_b+\eta)}, & z \sim -\eta/2. \end{cases}$$

From master equation to form factor expansion

Evaluating the master equation by the residues outside the integration contour we arrive at the expansion over form factors for $\langle Q^{\kappa}_{1,m}\rangle$. Recall that the only poles outside the contour $\Gamma\{\xi\} \cup \Gamma\{\lambda\}$ which contribute to the integral, are the admissible off-diagonal solutions. Hence

$$\langle Q^{\kappa}_{1,m}\rangle = (-1)^N \sum_{\{\mu\}} \prod_{a,b=1}^{N} \sinh^2(\lambda_a - \mu_b) \cdot \prod_{a=1}^{m} \frac{\tau_\kappa(\xi_a|\{\mu\})}{\tau(\xi_a|\{\lambda\})}$$

$$\times \frac{\det_N\left(\frac{\partial \tau_\kappa(\lambda_j|\{\mu\})}{\partial \mu_k}\right)}{\det_N\left(\frac{\partial \mathcal{Y}_\kappa(\mu_k|\{\mu\})}{\partial \mu_j}\right)} \cdot \frac{\det_N\left(\frac{\partial \tau(\mu_k|\{\lambda\})}{\partial \lambda_j}\right)}{\det_N\left(\frac{\partial \mathcal{Y}(\lambda_k|\{\lambda\})}{\partial \lambda_j}\right)}$$

the sum is taken on all admissible off-diagonal solutions μ_j of the twisted Bethe equations.

$$\langle Q^{\kappa}_{1,m}\rangle = \sum_{\{\mu\}} \prod_{a=1}^{m} \frac{\tau_\kappa(\xi_a|\{\mu\})}{\tau(\xi_a|\{\lambda\})} \cdot \frac{\langle \psi(\{\lambda\})|\psi_\kappa(\{\mu\})\rangle}{\langle \psi_\kappa(\{\mu\})|\psi_\kappa(\{\mu\})\rangle} \cdot \frac{\langle \psi_\kappa(\{\mu\})|\psi(\{\lambda\})\rangle}{\langle \psi(\{\lambda\})|\psi(\{\lambda\})\rangle}$$

It remains to use that the state $|\psi_\kappa(\{\mu\})\rangle$ is the eigenstate of $\mathcal{T}_\kappa(\xi)$ with the eigenvalue $\tau_\kappa(\xi|\{\mu\})$ and the state $|\psi(\{\lambda\})\rangle$ is the eigenstate of $\mathcal{T}(\xi)$ with the eigenvalue $\tau(\xi|\{\lambda\})$:

$$\langle Q^\kappa_{1,m}\rangle = \sum_{\{\mu\}} \frac{\langle\psi(\{\lambda\})|\prod_{b=1}^{m}\mathcal{T}_\kappa(\xi_b)|\psi_\kappa(\{\mu\})\rangle \cdot \langle\psi_\kappa(\{\mu\})|\prod_{b=1}^{m}\mathcal{T}^{-1}(\xi_b)|\psi(\{\lambda\})\rangle}{\langle\psi_\kappa(\{\mu\})|\psi_\kappa(\{\mu\})\rangle \cdot \langle\psi(\{\lambda\})|\psi(\{\lambda\})\rangle}$$

Observe that we did not use the completeness of the set $|\psi_\kappa(\{\mu\})\rangle$. The sum over eigenstates of $\mathcal{T}_\kappa$ appears automatically as the result of the evaluation of the multiple integral by the residues outside the integration contour.

Taking the second lattice derivative and then differentiating twice with respect to κ at $\kappa = 1$,

$$\langle\sigma^z_1\sigma^z_{m+1}\rangle = \langle\sigma^z_1\rangle\cdot\langle\sigma^z_{m+1}\rangle + \sum_{\{\mu\}\neq\{\lambda\}} \frac{\langle\psi(\{\lambda\})|\sigma^z_1|\psi(\{\mu\})\rangle \cdot \langle\psi(\{\mu\})|\sigma^z_{m+1}|\psi(\{\lambda\})\rangle}{\langle\psi(\{\mu\})|\psi(\{\mu\})\rangle \cdot \langle\psi(\{\lambda\})|\psi(\{\lambda\})\rangle}$$

Generating function at $\Delta = \frac{1}{2}$

Inhomogeneous case (multiple integrals can be separated):

$$\langle Q_\kappa(m)\rangle = \frac{3^m}{2^{m^2}} \prod_{a>b}^{m} \frac{\sinh 3(\xi_a - \xi_b)}{\sinh^3(\xi_a - \xi_b)} \sum_{n=0}^{m} \kappa^{m-n} \sum_{\substack{\{\xi\}=\{\xi_{\gamma_+}\}\cup\{\xi_{\gamma_-}\} \\ |\gamma_+|=n}} \det_m \hat{\Phi}^{(n)}$$

$$\times \prod_{a\in\gamma_+} \prod_{b\in\gamma_-} \frac{\sinh(\xi_b - \xi_a - \frac{i\pi}{3}) \sinh(\xi_a - \xi_b)}{\sinh^2(\xi_b - \xi_a + \frac{i\pi}{3})},$$

$$\hat{\Phi}^{(n)}(\{\xi_{\gamma_+}\}, \{\xi_{\gamma_-}\}) = \left(\begin{array}{c|c} \Phi(\xi_j - \xi_k) & \Phi(\xi_j - \xi_k - \frac{i\pi}{3}) \\ \hline \Phi(\xi_j - \xi_k + \frac{i\pi}{3}) & \Phi(\xi_j - \xi_k) \end{array}\right), \quad \Phi(x) = \frac{\sinh \frac{x}{2}}{\sinh \frac{3x}{2}}.$$

Homogeneous limit

$$\langle Q_\kappa(m)\rangle = \frac{(-1)^{\frac{m^2-m}{2}} 3^m}{2^{m^2} m!} \prod_{a>b}^{m} \frac{\sinh 3(\xi_a - \xi_b)}{\sinh(\xi_a - \xi_b)} \sum_{n=0}^{m} \kappa^{m-n} C_m^n \oint_{\Gamma\{\xi - \frac{i\pi}{6}\}} \frac{d^n z}{(2\pi i)^n}$$

$$\oint_{\Gamma\{\xi + \frac{i\pi}{6}\}} \frac{d^{m-n} z}{(2\pi i)^{m-n}} \times \prod_{b=1}^{m} \left\{ \prod_{j=1}^{n} \frac{1}{\sinh(z_j - \xi_b + \frac{i\pi}{6})} \prod_{j=n+1}^{m} \frac{1}{\sinh(z_j - \xi_b - \frac{i\pi}{6})} \right\}$$

$$\times \prod_{a=1}^{n} \prod_{b=n+1}^{m} \frac{\sinh(z_a - z_b - \frac{i\pi}{3}) \sinh(z_a - z_b + \frac{i\pi}{3})}{\sinh^2(z_a - z_b)} \cdot \det_m \Phi(z_j - z_k). \quad (6)$$

Here the integration contours $\Gamma\{\xi \mp \frac{i\pi}{6}\}$ surround the points $\{\xi - \frac{i\pi}{6}\}$ for $z_1, \ldots, z_n$ and $\{\xi + \frac{i\pi}{6}\}$ for $z_{n+1}, \ldots, z_m$ respectively.

Generating function : exact results

If the lattice distance m is not too large, the representations can be successfully used to compute $\langle Q_\kappa(m)\rangle$ explicitely.
First results for $P_m(\kappa) = 2^{m^2}\langle Q_\kappa(m)\rangle$ up to $m = 9$:

$$P_1(\kappa) = 1 + \kappa,$$

$$P_2(\kappa) = 2 + 12\kappa + 2\kappa^2,$$

$$P_3(\kappa) = 7 + 249\kappa + 249\kappa^2 + 7\kappa^3,$$

$$P_4(\kappa) = 42 + 10004\kappa + 45444\kappa^2 + 10004\kappa^3 + 42\kappa^4$$

$$P_5(\kappa) = 429 + 738174\kappa + 16038613\kappa^2 + 16038613\kappa^3 + 738174\kappa^4 + 429\kappa^5,$$

$$P_6(\kappa) = 7436 + 96289380\kappa + 11424474588\kappa^2 + 45677933928\kappa^3 + 11424474588\kappa^4 + 96289380\kappa^5 + 7436\kappa^6.$$

Two-point functions at small distances

$$
\begin{aligned}
\langle\sigma_1^z\sigma_2^z\rangle &= -\,2^{-1},\\
\langle\sigma_1^z\sigma_3^z\rangle &= 7\cdot 2^{-6},\\
\langle\sigma_1^z\sigma_4^z\rangle &= -\,401\cdot 2^{-12},\\
\langle\sigma_1^z\sigma_5^z\rangle &= 184453\cdot 2^{-22},\\
\langle\sigma_1^z\sigma_6^z\rangle &= -\,95214949\cdot 2^{-31},\\
\langle\sigma_1^z\sigma_7^z\rangle &= 1758750082939\cdot 2^{-46},\\
\langle\sigma_1^z\sigma_8^z\rangle &= -\,30283610739677093\cdot 2^{-60},\\
\langle\sigma_1^z\sigma_9^z\rangle &= 5020218849740515343761\cdot 2^{-78}.
\end{aligned}
$$

Exact vs asymptotic results

m	$\langle \sigma_1^z \sigma_{m+1}^z \rangle$ Exact	$\langle \sigma_1^z \sigma_{m+1}^z \rangle$ Asymptotics
1	-0.5000000000	-0.5805187860
2	0.1093750000	0.1135152692
3	-0.0979003906	-0.0993588501
4	0.0439770222	0.0440682654
5	-0.0443379157	-0.0444087865
6	0.0249933420	0.0249365346
7	-0.0262668452	-0.0262404925
8	0.0166105110	0.0165641239

Numerical methods for dynamical correlation functions in a field

(Work in progress with J. S. Caux, R. Hagemans)
Subspaces : fixed number of reversed spins M, number of sites N even, and $2M \leq N$, the other sector being accessible through a change in the reference state.

Eigenstates in each subspace are completely characterized for $2M \leq N$ by a set of rapidities $\{\lambda_j\}$, $j = 1, \ldots, M$, solution to the Bethe equations

$$\left[\frac{\sinh(\lambda_j + i\zeta/2)}{\sinh(\lambda_j - i\zeta/2)}\right]^N = \prod_{k \neq j}^{M} \frac{\sinh(\lambda_j - \lambda_k + i\zeta)}{\sinh(\lambda_j - \lambda_k - i\zeta)}, \qquad j = 1, \ldots, M$$

where $\Delta = \cos\zeta$.

Strip for rapidities : $-\pi/2 < \mathrm{Im}\lambda \leq \pi/2$

$$\text{atan}\left[\frac{\tanh(\lambda_j)}{\tan(\zeta/2)}\right] - \frac{1}{N}\sum_{k=1}^{M} \text{atan}\left[\frac{\tanh(\lambda_j - \lambda_k)}{\tan\zeta}\right] = \pi\frac{I_j}{N}.$$

Here, I_j are distinct half-integers which can be viewed as quantum numbers: each choice of a set $\{I_j\}$, $j = 1, \ldots, M$ (with I_j defined mod(N)) uniquely specifies a set of rapidities, and therefore an eigenstate.

$$E = J\sum_{j=1}^{M} \frac{-\sin^2\zeta}{\cosh 2\lambda_j - \cos\zeta} - h(\frac{N}{2} - M),$$

$$q = \sum_{j=1}^{M} i\ln\left[\frac{\sinh(\lambda_j + i\zeta/2)}{\sinh(\lambda_j - i\zeta/2)}\right] = \pi M + \frac{2\pi}{N}\sum_{j=1}^{M} I_j \mod 2\pi.$$

The ground state is given by $I_j^0 = -\frac{M+1}{2} + j$, $j = 1, \ldots, M$, and all excited states are in principle obtained from the different choices of sets $\{I_j\}$.

Form factors for the Fourier-transformed spin operators $S_q^a = \frac{1}{\sqrt{N}} \sum_{j=1}^N e^{iqj} S_j^a$.

$$|\langle\{\mu\}|S_q^z|\{\lambda\}\rangle|^2 = \frac{N}{4} \prod_{j=1}^M \left|\frac{\sinh(\mu_j - i\zeta/2)}{\sinh(\lambda_j - i\zeta/2)}\right|^2 \prod_{j>k=1}^M \left|\sinh^2(\mu_j - \mu_k) + \sin^2\zeta\right|^{-1} \times$$

$$\times \prod_{j>k=1}^M \left|\sinh^2(\lambda_j - \lambda_k) + \sin^2\zeta\right|^{-1} \frac{|\det[\mathbf{H}(\{\mu\},\{\lambda\}) - 2\mathbf{P}(\{\mu\},\{\lambda\})]|^2}{|\det \mathbf{\Phi}(\{\mu\}) \det \mathbf{\Phi}(\{\lambda\})|}$$

$$\mathbf{H}_{ab}(\{\mu\},\{\lambda\}) = \frac{1}{\sinh(\mu_a - \lambda_b)} \left[\prod_{j \neq a}^{M} \sinh(\mu_j - \lambda_b - i\zeta) - \right.$$

$$\left. \left[\frac{\sinh(\lambda_b + i\zeta/2)}{\sinh(\lambda_b - i\zeta/2)} \right]^N \prod_{j \neq a}^{M} \sinh(\mu_j - \lambda_b + i\zeta) \right]$$

$$\mathbf{P}_{ab}(\{\mu\},\{\lambda\}) = \frac{1}{\sinh^2 \mu_a + \sin^2 \zeta/2} \prod_{k=1}^{M} \sinh(\lambda_k - \lambda_b - i\zeta),$$

Structure factor :

$$S^{a\bar{a}}(q,\omega) = 2\pi \sum_{\alpha \neq GS} |\langle GS|S_q^a|\alpha\rangle|^2 \delta(\omega - \omega_\alpha)$$

$$S^{a\bar{a}}(q,\omega) = \frac{1}{N}\sum_{j,j'=1}^{N} e^{iq(j-j')}\int_{-\infty}^{\infty} dt e^{i\omega t}\langle S_j^a(t) S_{j'}^{\bar{a}}(0)\rangle_c$$

Sum rule :

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{1}{N}\sum_q S^{zz}(q,\omega) = \frac{1}{4} - \langle S^z\rangle^2 = \frac{1}{4}\left[1-(1-\frac{2M}{N})^2\right]$$

Table 1: Comparison of equal-time correlation functions $\langle S_j^a S_{j+l}^{\bar{a}} \rangle$ at zero field for $\Delta = 0.25$ for small distances $l = 1, 2, 3$. Subscript p refers to the exact polynomial representation, whereas ff refers to our results obtained by summing form factors for all states up to three holes, thereby achieving saturation of the sum rule to 99.88 % (S^{zz}) and 95.61 % (S^{-+}).

l	S_p^{zz}	S_{ff}^{zz}	S_p^{-+}	S_{ff}^{-+}
1	-0.113489	-0.113337	-0.316807	-0.311455
2	0.0129789	0.0129605	0.180965	0.180967
3	-0.0163964	-0.0163965	-0.152364	-0.152466

Table 2: Comparison of equal-time correlation functions $\langle S_j^a S_{j+l}^{\bar{a}} \rangle$ at zero field for $\Delta = 0.75$ for small distances $l = 1, 2, 3$. Subscript p refers to the exact polynomial representation, whereas ff refers to our results obtained by summing form factors for all states up to three holes, thereby achieving saturation of the sum rule to 98.73 % (S^{zz}) and 98.89 % (S^{-+}).

l	S_p^{zz}	S_{ff}^{zz}	S_p^{-+}	S_{ff}^{-+}
1	-0.136265	-0.136093	-0.305461	-0.304906
2	0.043132	0.043427	0.140331	0.140553
3	-0.035605	-0.035911	-0.117158	-0.117201

Table 3: $\langle S_j^a S_{j+d}^{\bar{a}} \rangle$ at zero field for $\Delta = 0.25$ for large distances d. Subscript ff refers to our results obtained by summing form factors for $N = 200$, for intermediate states including up to three holes, thereby achieving saturation of the sum rule to 99.88 % (S^{zz}) and 95.61 % (S^{-+}).

d	S_{CFT}^{zz}	S_{ff}^{zz}	S_{CFT}^{-+}	S_{ff}^{-+}
10	9.89624e-4	9.89375e-4	7.31376e-2	7.39883e-2
25	-3.77195e-4	-3.77096e-4	-4.41637e-2	-4.40758e-2
40	1.13604e-4	1.13539e-4	3.42939e-2	3.43211e-2
55	-1.11476e-4	-1.11424e-4	-2.95934e-2	-2.95775e-2
70	5.70852e-5	5.70336e-5	2.69545e-2	2.69692e-2
85	-7.21563e-5	-7.21058e-5	-2.56504e-2	-2.56405e-2
100	4.71343e-5	4.70918e-5	2.52109e-2	2.52193e-2

Table 4: Comparison of equal-time correlation functions $\langle S_j^a S_{j+d}^{\bar{a}} \rangle$ at zero field for $\Delta = 0.75$ for large distances d. Subscript CFT refers to the scaling prediction, whereas ff refers to our results obtained by summing form factors for $N = 200$, for intermediate states including up to three holes, thereby achieving saturation of the sum rule to 98.73 % (S^{zz}) and 98.89 % (S^{-+}).

d	S_{CFT}^{zz}	S_{ff}^{zz}	S_{CFT}^{-+}	S_{ff}^{-+}
10	7.10534e-3	7.26298e-3	4.22187e-2	4.36235e-2
25	-2.49782e-3	-2.51931e-3	-2.24156e-2	-2.21879e-2
40	1.39631e-3	1.39725e-3	1.57658e-2	1.58537e-2
55	-1.03764e-3	-1.03205e-3	-1.31010e-2	-1.30427e-2
70	8.26181e-4	8.18235e-4	1.14797e-2	1.15143e-2
85	-7.56746e-4	-7.47886e-4	-1.08240e-2	-1.07873e-2
100	7.13747e-4	7.04615e-4	1.05099e-2	1.05363e-2

Conclusions and Perspectives

New method to obtain correlation functions of quantum integrable models

- Generic tools for a large class of models
- Explicit results for the Heisenberg spin chains

Open new perspectives

- Asymptotic behavior of correlation functions (under study)
- Dynamical correlation functions and depending on temperature
- Applications to many different models (models with impurities, with boundaries, field theories,...)
- Numerical evaluations via form factor expansion