The new limit of Principal Chiral Field and its relation to N=4 SYM

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Anisotropic Principal Chiral Field

The Lagrangian density

$$\mathcal{L} = \frac{1}{g_\perp}\Omega_\mu^+\Omega_\mu^- + \frac{1}{2g_{||}}\Omega_\mu^z\Omega_\mu^z + h\Omega_t^z$$

$$\Omega_\mu^a = \mathrm{iTr}(\sigma^a G^+ \partial_\mu G)$$

where G belongs to the SU(2) group and h is an external "magnetic" field acting in the U(1) sector.

The model is exactly solvable (Wiegmann 1984, Kirillov and Reshetikhin 1984).

New limit

However! There is a limit which has never been discussed:

$$h > M, \quad g_\perp / g_{||} \to 0$$

At $h = 0$ this limit leads to the O(3) sigma model (Wiegmann 1985):

$$\Omega^+_\mu \Omega^-_\mu = (\partial_\mu \mathbf{N})^2, \quad \mathbf{N}^2 = 1$$

Bethe ansatz

$$e^{iML\sinh\theta_a} = \prod_{b\neq a}^{n} \mathcal{S}_0(\theta_a - \theta_b) \times$$

$$\prod_{\alpha=1}^{m_1} \frac{\theta_a - \lambda_\alpha - i\pi/2}{\theta_a - \lambda_\alpha + i\pi/2} \prod_{\alpha=1}^{m_2} \frac{\sinh[\nu(\theta_a - \mu_\alpha - i\pi/2}{\sinh[\nu(\theta_a - \mu_\alpha + i\pi/2}$$

$$\prod_{a=1}^{n} \frac{\lambda_\alpha - \theta_a - i\pi/2}{\lambda_\alpha - \theta_a + i\pi/2} = \prod_{b\neq a}^{m_1} \frac{\lambda_\alpha - \lambda_\beta - i\pi}{\lambda_\alpha - \lambda_\beta + i\pi}$$

$$\prod_{a=1}^{n} \frac{\sinh[\nu(\mu_\alpha - \theta_a - i\pi/2)]}{\sinh[\nu(\mu_\alpha - \theta_a + i\pi/2)]} = \prod_{b\neq a}^{m_2} \frac{\sinh[\nu(\mu_\alpha - \mu_\beta - i\pi)]}{\sinh[\nu(\mu_\alpha - \mu_\beta + i\pi)]}$$

where M is the soliton mass.

ν is a function of $g_\perp/g_\parallel$.

$$E = M\sum_{a=1}^{n} \cosh\theta_a - h(n/2 - m_2)$$

At $\nu > 1$ the factor $S_0(\theta)$ has poles at $\theta = -i\pi(1 - 1/\nu)$ giving rise to the bound states with the masses

$$m_j = 2M \sin(\pi j/2\nu), \quad j = 1, 2, ...\nu$$

We are interested in the limit $\nu \to \infty, m_1 =$ const.

Bethe ansatz for low-lying excitations

$$e^{\mathrm{i}p_b(\theta_a)L} = \prod_{a=1}^{n_b} \frac{\theta_a - \theta_b + \mathrm{i}\pi}{\theta_a - \theta_b - \mathrm{i}\pi} \prod_{\alpha=1}^{m} \frac{\theta_a - \lambda_\alpha - \mathrm{i}\pi}{\theta_a - \lambda_\alpha + \mathrm{i}\pi}$$

$$e^{\mathrm{i}p_f(\lambda_\alpha)} \prod_{a=1}^{n_b} \frac{\lambda_\alpha - \theta_a - \mathrm{i}\pi}{\lambda_\alpha - \theta_a + \mathrm{i}\pi} = \prod_{\beta=1}^{m} \frac{\lambda_\alpha - \lambda_\beta - \mathrm{i}\pi}{\lambda_\alpha - \lambda_\beta + \mathrm{i}\pi}$$

th energy equal to

$$E = \sum_a \epsilon_b(\theta_a) + \sum_\alpha \epsilon_f(\lambda_\alpha)$$

$$p_f(\lambda) = 2\int_{-Q}^{Q} \mathrm{d}\theta \sigma(\theta) \tan^{-1}[2(\lambda - \theta)/\pi]$$

$$\epsilon_f(\lambda) = \int_{-Q}^{Q} \mathrm{d}\theta \frac{\sigma(\theta)}{4(\lambda - \theta)^2 + \pi^2}$$

B has a gap, f does not

$$\frac{\quad}{\mathrm{d}\theta} = m\cosh\theta + s * \sigma,$$

$$\epsilon_b(\theta) = m\cosh\theta + s * E$$

$$\int_{-Q}^{Q} \mathrm{d}\theta' \ln|\coth(\theta-\theta')|E(\theta') = m\pi\cosh\theta - \mu$$

$$\int_{-Q}^{Q} \mathrm{d}\theta' \ln|\coth(\theta-\theta')|\sigma(\theta') = \frac{m}{2}\cosh\theta$$

where m is the mass of the O(3) sigma model particle. In the small Q one can expand the hyperbolic functions. Then the solution of the second equation is

$$\sigma(\theta) = \frac{A}{\sqrt{Q^2-\theta^2}} + \sqrt{Q^2-\theta^2}$$

The values of A, Q is determined by two equations:

$$\int \mathrm{d}\theta\sigma(\theta) = n/L, \rightarrow A + Q^2/2 = n/\pi L$$

and

$$\pi A\ln Q + A\int_{-1}^{1}\mathrm{d}x\frac{\ln|x|}{\sqrt{1-x^2}}$$

$$+Q^2\int_{-1}^{1}\mathrm{d}x\ln|x|\sqrt{1-x^2} = -1$$

From the second equation it follows that

$$Q \sim \exp(-L/\pi n)$$

which means that at small densities one can neglect Q in comparison with A and treat

$$\sigma(\theta) \approx \frac{(n/L)}{\pi\sqrt{Q^2 - \theta^2}}, \quad \ln(1/Q) \sim mL/n$$

The spectrum

$$\epsilon_f(k) = \sqrt{1 + Q^2 \sin^2(k/2)} - 1$$