Discretized Minimal Surface and Gluon Scattering Amplitudes in N=4 SYM at Strong Coupling

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S. Dobashi, K.I. and K. Iwasaki, arXiv:0805.3594, JHEP 07 (2008)088

S. Dobashi and K.I., arXiv:0901.3046, Nucl. Phys. B819 (2009) 18

Gluon Scattering Amplitudes in $\mathcal{N}=4$ SYM

Planar L-loop, n-point amplitude

$$A_n^{(L)}(k_1,\cdots,k_n)=A_n^{(0)}(k_1,\cdots,k_n)\mathcal{M}_n^{(L)}(\epsilon)$$

the BDS conjecture Bern-Dixon-Smirnov, Anastasiou-Bern-Dixon-Kosower

$$\begin{aligned}\ln\mathcal{M}_n(\epsilon) &= \frac{A_2}{\epsilon^2}+\frac{A_1}{\epsilon}\\ &-\frac{1}{16}f(\lambda)\sum_{i=1}^{n}\left(\ln\left(\frac{\mu^2}{-s_{i,i+1}}\right)\right)^2-\frac{g(\lambda)}{4}\sum_{i=1}^{n}\ln\left(\frac{\mu^2}{-s_{i,i+1}}\right)+\frac{f(\lambda)}{4}F_n^{(BDS)}(0)+C\end{aligned}$$

For $n=4$

$$F_4^{BDS}=\frac{1}{2}\log^2\left(\frac{s}{t}\right)+\frac{2\pi^2}{3}$$

For $n\geq 5$, $F_n^{BDS}(0)=\frac{1}{2}\sum_{i=1}^{n}g_{n,i}$ (Mandelstam variables: $t_i^{[r]}\equiv(k_i+\ldots+k_{i+r-1})^2$)

$$g_{n,i}=-\sum_{r=2}^{[n/2]-1}\ln\left(\frac{-t_i^{[r]}}{-t_i^{[r+1]}}\right)\ln\left(\frac{-t_{i+1}^{[r]}}{-t_i^{[r+1]}}\right)+D_{n,i}+L_{n,i}+\frac{3}{2}\zeta_2$$

Test of the BDS conjecture (Weak Coupling)

- explicit loop calculation
  - 4-point up to 3-loops [BDS]
  - 5-point up to 2-loops [Cachazo et. al. , Bern et. al.]
  - $n(\geq 6)$-point 1-loop [Bern et. al.]
- Discrepancy in 6-point 2-loop amplitude[Bern et. al. 0803.1465]
  Gluon amplitudes=Wilson loop [Drummond et. al. 0803.1466]

$$\ln M_6^{MHV} = \ln W(C_6) + const., \quad F_6^{WL} = F_6^{BDS} + R_6(u), \quad R_6 \neq 0$$

Non-trivial dependence comes from the function of the cross-ratio

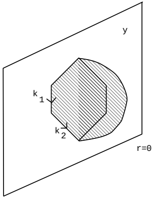
$$u_{ij,kl} = \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}, \quad (x_{ij}^2 = t_i^{[j-i]})$$

For $n = 6$, $u_{13,46}$, $u_{24,15}$, $u_{35,26}$ are independent cross ratios.

Gluon amplitudes = Wilson loop with light-like boundaries

- ends at $r = \frac{R^2}{z_{IR}} \to 0 \ (z_{IR} \to \infty)$
- y^μ: surrounded by the light-like segments

$$\mathcal{M}_n \sim \exp(-S_{NG})$$



S_{NG}: the value of the Nambu-Goto action for the surface surrounded light-like segments = Area of the minimal surface
static gauge: surface $y_0(y_1, y_2)$, $r(y_1, y_2)$ parametrized by (y_1, y_2)

$$S_{NG} = \frac{R^2}{2\pi} \int dy_1 dy_2 \frac{\sqrt{1 + (\partial_i r)^2 - (\partial_i y_0)^2 - (\partial_1 r \partial_2 y_0 - \partial_2 r \partial_1 y_0)^2}}{r^2}$$

Euler-Lagrange equations

$$\partial_i \left(\frac{\partial L}{\partial(\partial_i y_0)} \right) = 0, \quad \partial_i \left(\frac{\partial L}{\partial(\partial_i r)} \right) - \frac{\partial L}{\partial r} = 0,$$

non-linear differential equations, difficult to solve

Test of the BDS conjecture at Strong Coupling

Alday-Maldacena's Solution

4-point amplitude ($s=t$): $s=-(k_1+k_2)^2$, $t=-(k_1+k_4)^2$

boundary condition:

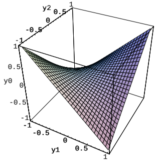
$r(\pm 1, y_2) = r(y_1, \pm 1) = 0$,

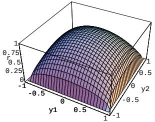
$y_0(\pm 1, y_2) = \pm y_2$,

$y_0(y_1, \pm 1) = \pm y_1$

solution: $\boxed{y_0 = y_1 y_2,}$

$$\boxed{r = \sqrt{(1-y_1^2)(1-y_2^2)}}$$





general (s,t)-solution ($SO(2,4)$ transformation)

conformal boost (b)+scale transformation (a)

$$r' = \frac{ar}{1+by_0}, \quad y_0' = \frac{a\sqrt{1+b^2}y_0}{1+by_0}, \quad y_1' = \frac{ay_1}{1+by_0}, \quad y_2' = \frac{ay_2}{1+by_0}$$

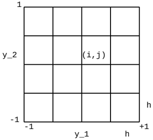
$-s(2\pi)^2 = \frac{8a^2}{(1-b)^2}$, $-t(2\pi)^2 = \frac{8a^2}{(1+b)^2}$

The area agrees with the BDS formula. Higher-point amplitudes?

Numerical Solutions of Minimal Surfaces in AdS

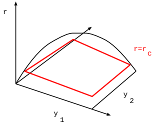
discretization

- square lattice with spacing $h = \frac{2}{M}$
- (i,j) $(i,j = 0, \cdots, M)$
  $y_0[i,j] = y_0(-1+hi, -1+hj)$
  $r[i,j] = r(-1+hi, -1+hj)$.



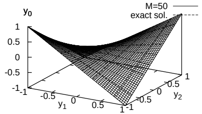
E-L Equations→ $2(M-1)^2$ nonlinear simultaneous equations for $y_0[i,j]$ and $r[i,j]$
Use the same momentum configurations as in Astefanesei-Dobashi-Ito-Nastase
Solve these equations numerically. (Newton's method)
Evaluate the action $S = \sum L[i,j]h^2$ using **the radial cut-off regularization**

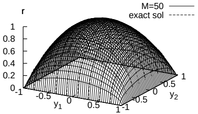
$$S[r_c] = \int_{r(y_1,y_2)\geq r_c} dy_1 dy_2 L$$
$$S^{dis}[r_c] = \sum_{r[i,j]\geq r_c} L[i,j]h^2$$

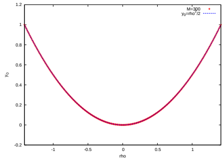


- quantitative check of the gluon amplitude/Wilson loop duality and the BDS conjecture at strong coupling
- some hints to obtain exact solutions

Minimal surface:4-point amplitude





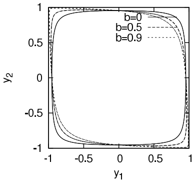


$y_1 = y_2$ section $(M = 300)$

$$S_4[r_c, b] = \int_S dy_1 dy_2 L, \quad L = \frac{1}{(1-y_1^2)(1-y_2^2)}$$

S: region surrounded by the cut-off curve C

$$r_c^2 = (1-y_1^2)(1-y_2^2)\frac{1}{(1+by_1y_2)^2}$$

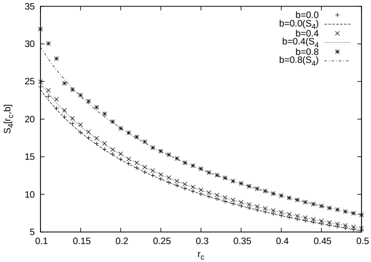


$$S_4[r_c, b] = \frac{1}{4}\log^2\left(\frac{r_c^2}{-8\pi^2 s}\right) + \frac{1}{4}\log^2\left(\frac{r_c^2}{-8\pi^2 t}\right) - \frac{1}{4}\log^2(\frac{s}{t}) - 3.289\ldots + O(r_c^2 \log r_c^2).$$

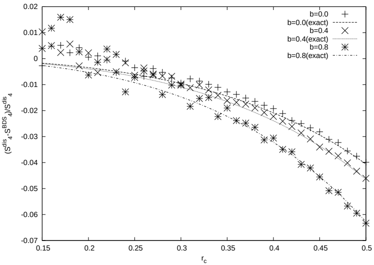
$$F_4^{BDS} = -\frac{1}{4}\log^2(\frac{s}{t}) - \frac{\pi^2}{3} = -\frac{1}{4}\log^2(\frac{s}{t}) - 3.28987\ldots$$

Numerical check of the BDS formula: 4-pt amplitude

$M = 520$



$S[r_c, b]$ vs $S^{dis}[r_c, b]$



$(S_4^{dis} - S_4^{BDS})/S_4^{dis}$

- Finite r_c correction $\leq 6\%$
- numerical error becomes large $r_c \leq 0.2$ (and large b)

n-point amplitude (conjecture)

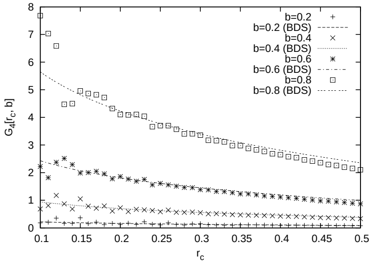
$$\tilde{S}_n[r_c] = \frac{1}{8}\sum_{i=1}^{n}\left(\log\frac{r_c^2}{-8\pi^2 s_{i,i+1}}\right)^2 + F_n(p_1,\cdots,p_n) + O(r_c^2\log^2 r_c),$$

$$F_n = -\frac{1}{2}F_n^{BDS} + R_n(u_{ij,kl})$$

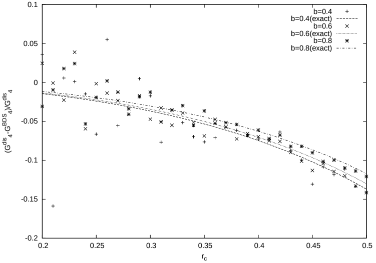
- remainder function: R_n
- It is difficult to distinguish finite r_c correction and remainder function, numerically.
- $\boxed{G_n^{dis}[r_c,b] = S_n^{dis}[r_c,b] - S_n^{dis}[r_c,0]}$: smaller r_c correction

Difference of areas with different boost parameters

4-point amplitude:



$G^{dis}[r_c, b]$ vs $G^{BDS}[r_c, b]$

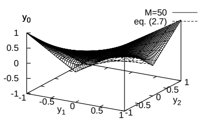


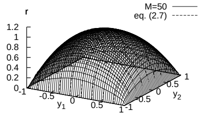
$(G^{dis} - G^{BDS})/G^{dis}$

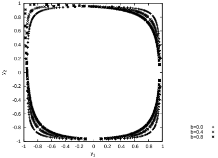
- small r_c: large fluctuation, large r_c: large r_c correction($\sim$10%)
- difference 5% at $r_c = 0.3$
- If we find deviation larger than finite r_c correction, this suggests the existence of the remainder function R_n.

Highre-point amplitudes

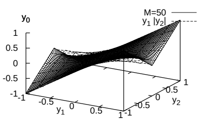
6-point solution1

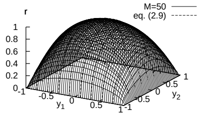


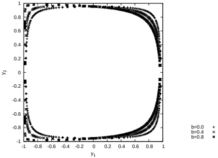




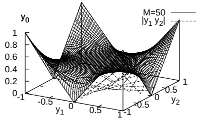
6-point solution2

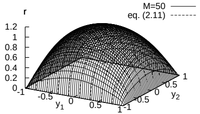


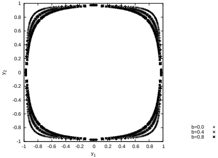




8-point

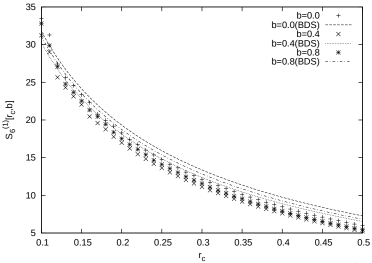




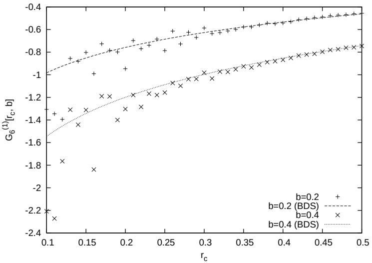


6-point amplitude solution 1

$S^{dis}[r_c, b]$ vs S^{BDS}



$G^{dis}[r_c, b]$ vs $G^{BDS}[r_c, b]$



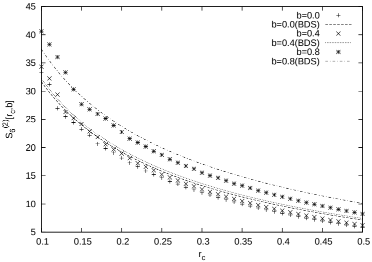
cross ratios are independent of b: $u_1 = u_2 = u_3 = 1$

$$
\begin{aligned}
S_6^{BDS(1)}[r_c, b] &= \frac{1}{8}\Big\{2\log^2\Big(\frac{r_c^2(1-b)}{8}\Big) + 2\log^2\Big(\frac{r_c^2(1+b)^2}{8}\Big) + \log^2\frac{r_c^2}{4} + \log^2\Big(\frac{r_c^2(1+b)^2}{16}\Big)\Big\} \\
&\quad -\frac{1}{2}\Big\{\log 2\log(1-b) - 2\log 2\log(1+b) - 2\log(1-b)\log(1+b) \\
&\quad +\frac{1}{2}\log^2(1-b) + 3\log^2(1+b)\Big\} - \frac{3\pi^2}{16}
\end{aligned}
$$

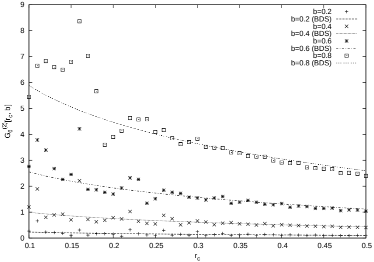
$\Longrightarrow R_6^{dis}$ does not depend on b and is non-zero constant.

6-point amplitude solution 2

$S^{dis}[r_c, b]$ vs $S^{BDS}[r_c, b]$



$G^{dis}[r_c, b]$ vs $G^{BDS}[r_c, b]$

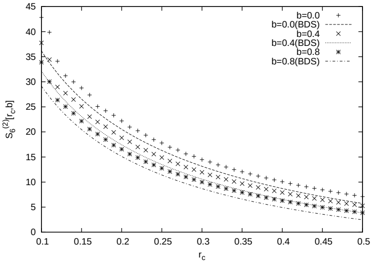


cross ratios are independent of b: $u_1 = u_2 = u_3 = 1$

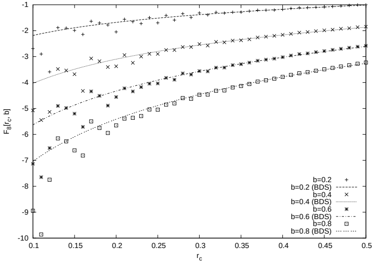
$$S_6^{BDS(2)}[r_c, b] = \frac{1}{8}\Big\{\log^2\big(\frac{r_c^2(1-b)^2}{8}\big) + \log^2\big(\frac{r_c^2(1+b)^2}{8}\big) + 2\log^2\big(\frac{r_c^2(1+b)}{8}\big) + 2\log^2\big(\frac{r_c^2(1-b)}{8}\big)\Big\}$$
$$-\frac{1}{2}\Big\{\frac{3}{2}\log^2(1-b) + \frac{3}{2}\log^2(1+b) - 2\log(1-b)\log(1+b)\Big\} - \frac{3\pi^2}{16}$$

8-point amplitude

$S^{dis}[r_c, b]$ $S^{BDS}[r_c, b]$



$G^{dis}[r_c, b]$ vs $G^{BDS}[r_c, b]$



cross ratios are independent of b: $u_{ijkl} = 1$

$$S_8^{BDS}[r_c, b] = \frac{1}{8}\left\{4\log^2\left(\frac{r_c^2(1+b)^2}{8}\right) + 4\log^2\left(\frac{r_c^2}{4}\right)\right\} - \frac{1}{2}\left\{4\log^2(1+b) - 4\log 2\log(1+b) - \frac{\pi^2}{6}\right\} - \frac{\pi^2}{2}$$

Conclusions and Outlook

- For the 4-point amplitude, $M = 520$ data is numerically consistent with the BDS formula.
- For the 6 and 8-point amplitudes, the present numerical solutions suggest non-zero constant remainder functions R_n.
- Non-trivial momentum configurations $\Longrightarrow R_n(u)$.
  Compare the deviation with the results from weak coupling analysis
  Anastasiou et al., 0902.2245
- Improve numerical solutions (larger M)
  - AdS_3 constraints: $r^2 - y_0^2 + y_1^2 + y_2^2 = 1$
    Jevicki-Jin-Kalousios 0712.1193, Alday-Maldacena 0903.4701; 0904.0663
  - Newton method→contragradient method
- Hints to obtain exact solution (without AdS_3 constraints)
  Numerical check of underlying integrable structure at strong coupling (dual conformal symmetry, fermionic T-duality, Yangian)
- non-AdS geometry (finite temperature) Ito-Nastase-Iwasaki