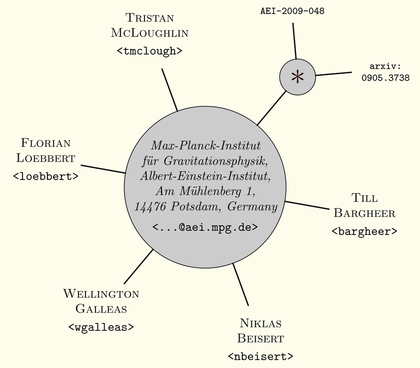
# Exacting $\mathcal{N}=4$ Superconformal Symmetry*

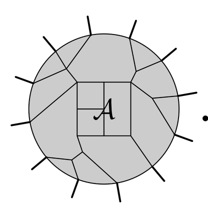

MAX-PLANCK-GESELLSCHAFT

Till Bargheer, Niklas Beisert, Wellington Galleas, Florian Loebbert, Tristan McLoughlin

*Max–Planck–Institut für Gravitationsphysik, Albert–Einstein–Institut, Am Mühlenberg 1, 14476 Potsdam*


## Scattering Amplitudes

Scattering amplitudes in $\mathcal{N}=4$ SYM are conveniently expressed in the spinor helicity superspace: The light-like momentum $p$ of each external particle is converted to a bi-spinor $p^{a\dot a}=\lambda^a\bar\lambda^{\dot a}$, where $\lambda^a$ and $\bar\lambda^{\dot a}$ are conjugate bosonic Lorentz spinors with $a,\dot a=1,2$. It is advantageous to compute scattering amplitudes for the superfield $\Phi(\lambda,\bar\lambda,\eta)$ with $\eta^A$, $A=1,\dots,4$ being fermionic spinors. The scattering amplitude for $n$ external particles is thus a superspace function

$$A_n(\lambda_1,\bar\lambda_1,\eta_1,\dots,\lambda_n,\bar\lambda_n,\eta_n)=$$  .

Amplitudes can be classified through their helicity measured by the number of $\eta$'s ranging between 8 for MHV and $4n-8$ for $\overline{\text{MHV}}$ amplitudes. Tree level MHV amplitudes of $\mathcal{N}=4$ SYM take the simple form [1]

$$A_n^{\text{MHV}}=\frac{\delta^4(P)\,\delta^8(Q)}{\langle 12\rangle\langle 23\rangle\dots\langle n1\rangle},$$

where $\langle j,k\rangle=\varepsilon_{ab}\lambda_j^a\lambda_k^b$ and

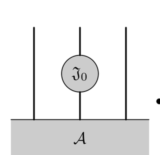
$$P^{a\dot b}=\sum_{k=1}^n\lambda_k^a\bar\lambda_k^{\dot b},\qquad Q^{aB}=\sum_{k=1}^n\lambda_k^a\eta_k^B.$$

## Free Representation

The free representation of the superconformal algebra on tree-level scattering amplitudes in $\mathcal{N}=4$ SYM can be written in a compact fashion. Most relevantly

$$\begin{aligned}(\mathfrak{Q}_0)^{aB}&=\lambda^a\eta^B, & (\mathfrak{S}_0)_{aB}&=\partial_a\partial_B,\\ (\bar{\mathfrak{Q}}_0)^{\dot a}_B&=\bar\lambda^{\dot a}\partial_B, & (\bar{\mathfrak{S}}_0)^B_{\dot a}&=\eta^B\bar\partial_{\dot a},\\ (\mathfrak{P}_0)^{a\dot b}&=\lambda^a\bar\lambda^{\dot b}, & (\mathfrak{K}_0)_{a\dot b}&=\partial_a\bar\partial_{\dot b},\end{aligned}\tag{1}$$

where $\partial_a=\partial/\partial\lambda^a$, $\bar\partial_{\dot a}=\partial/\partial\bar\lambda_{\dot a}$ and $\partial_A=\partial/\partial\eta^A$.
The action of the generators $\mathfrak{J}_0$ takes the standard tensor product form

$$\mathfrak{J}_0=\sum_{k=1}^n\mathfrak{J}_{0,k}=$$  .

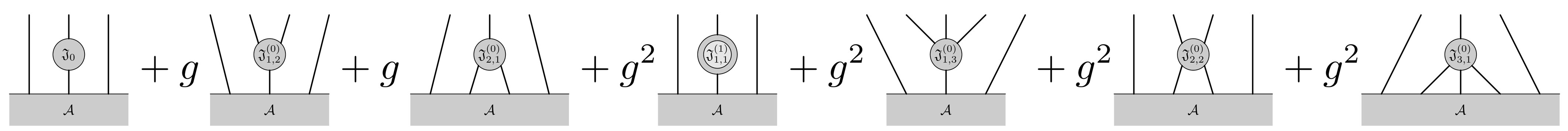
Here $\mathfrak{J}_{0,k}$ is the representation of the conformal symmetry generator $\mathfrak{J}_0$ on the $k$-th leg $(\lambda_k,\bar\lambda_k,\eta_k)$ of $A_n$ as specified in (1). Invariance of $A_n$ corresponds to the statement

$$\mathfrak{J}_0A_n=0.\tag{2}$$

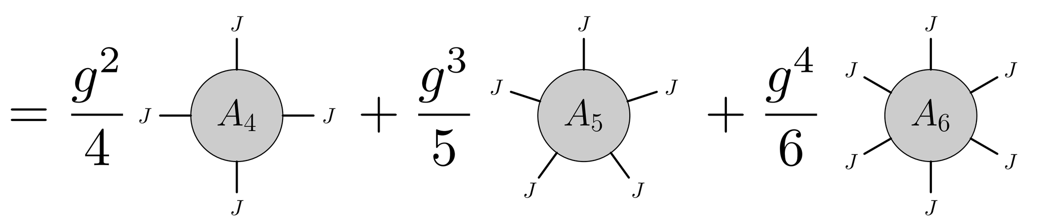
## Higher Loop Representation

Inspired by the representation of the symmetry generators on local operators [3], assume the following perturbative representation $\mathfrak{J}(g)$ for some symmetry generator $\mathfrak{J}$ around the free representation $\mathfrak{J}_0=\mathfrak{J}^{(0)}_{1,1}$:

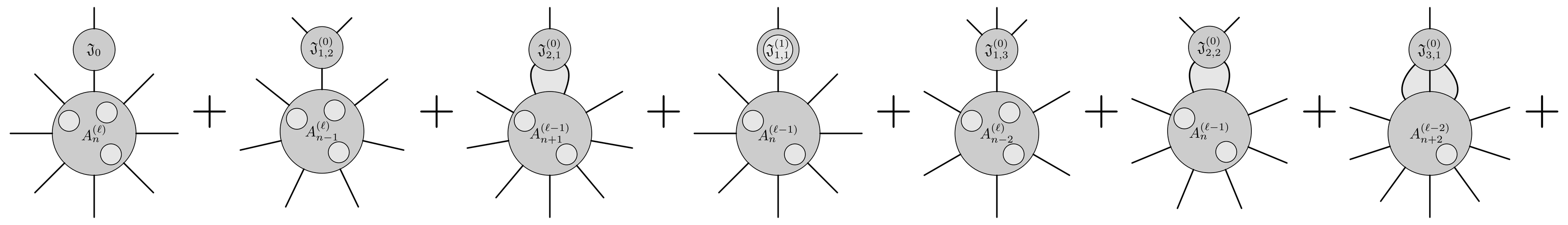
$$\mathfrak{J}(g)=\sum_{m,n=1}^\infty\sum_{\ell=0}^\infty g^{2\ell+m+n-2}\mathfrak{J}^{(\ell)}_{m,n}=\mathfrak{J}_0+g\,\mathfrak{J}^{(0)}_{1,2}+g\,\mathfrak{J}^{(0)}_{2,1}+g^2\,\mathfrak{J}^{(1)}_{1,1}+g^2\,\mathfrak{J}^{(0)}_{1,3}+g^2\,\mathfrak{J}^{(0)}_{2,2}+g^2\,\mathfrak{J}^{(0)}_{3,1}+\dots\tag{3}$$



At order $g$, the generators relate $A_n$ to $A_{n-1}$. The first term in (3) is the free generator $\mathfrak{J}_0=\mathfrak{J}^{(0)}_{1,1}$. The contributions $\mathfrak{J}^{(0)}_{1,2},\mathfrak{J}^{(0)}_{1,3}$ increase the number of legs by one or two, respectively, while $\mathfrak{J}^{(0)}_{2,1},\mathfrak{J}^{(0)}_{3,1}$ decrease it. In order to construct an invariant of the symmetry algebra, write all amplitudes in terms of a generating functional $\mathcal{A}[J]$ with sources $J(\Lambda_k)$:

$$\mathcal{A}(g)[J]=\sum_{n=4}^\infty\frac{g^{n-2}}{n}\int d\Lambda\,A_n(\Lambda)\,\mathrm{Tr}\big(J(\Lambda_1)\dots J(\Lambda_k)\big)=$$ 

Here, $\Lambda_k=(\lambda_k,\bar\lambda_k,\eta_k)$ and $\Lambda=(\Lambda_1,\dots,\Lambda_k)$.
Collecting contributions with equal numbers of legs and powers in $g$, the invariance equation (2) takes the form:

$$\mathfrak{J}(g)\mathcal{A}(g)=0\quad\Rightarrow\quad 0=$$ 

## Holomorphic Anomaly

Indeed, corrections to the free representation of superconformal symmetry are necessary: In Lorentz signature, tree level scattering amplitudes are not exactly invariant under the free representation. This is due to the holomorphic anomaly [2] contributing when two external momenta become collinear,

$$\frac{\partial}{\partial\bar\lambda^{\dot a}}\frac{1}{\langle\lambda,\mu\rangle}=\pi\delta^2(\langle\lambda,\mu\rangle)\,\varepsilon_{\dot a\dot b}\bar\mu^{\dot b}.$$

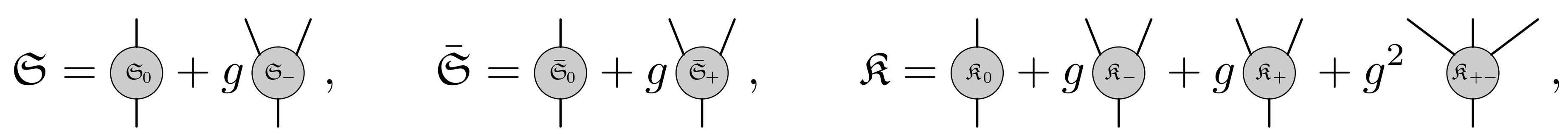
For instance, the action of the free generator $\bar{\mathfrak{S}}_0$ on MHV amplitudes reduces to an expression which does not vanish in the collinear limit of two external momenta

$$(\bar{\mathfrak{S}}_0)^B_{\dot a}A_n^{\text{MHV}}=-\pi\sum_{k=1}^n\varepsilon_{\dot a\dot b}\big(\bar\lambda^{\dot b}_{k-1}\eta^B_k-\bar\lambda^{\dot b}_k\eta^B_{k-1}\big)\frac{\delta^2(\langle\lambda_{k-1},\lambda_k\rangle)\,\delta^4(P)\,\delta^8(Q)}{\langle 12\rangle\dots\langle k-1,k\rangle^0\dots\langle n1\rangle}.\tag{4}$$

## Tree Level Correction

At tree level, only the supersymmetry generators $\mathfrak{S}$, $\bar{\mathfrak{S}}$ and the generator of special conformal transformations $\mathfrak{K}$ receive corrections (the subscript refers to the helicity):

$$\mathfrak{S}=\dots,\qquad\bar{\mathfrak{S}}=\dots,\qquad\mathfrak{K}=\dots,$$



where e.g. $\bar{\mathfrak{S}}_+$ cancels the contribution (4) of the free generator $\mathfrak{S}_0$ and is given by [4]

$$(\bar{\mathfrak{S}}_+)^B_{\dot a}J(\Lambda)=\pi\int d^4\bar\eta'\,d\alpha\,d\varphi\,e^{3i\varphi}\varepsilon_{\dot a\dot\gamma}\bar\lambda_1^{\dot\gamma}\bar\eta_2^B[J(\Lambda_1),J(\Lambda_2)],\qquad\begin{aligned}\lambda_1&=\lambda e^{i\varphi}\sin\alpha, & \bar\eta_1&=(\bar\eta\sin\alpha+\bar\eta'\cos\alpha)e^{-i\varphi},\\ \lambda_2&=\lambda\cos\alpha, & \bar\eta_2&=\bar\eta\cos\alpha-\bar\eta'\sin\alpha.\end{aligned}$$

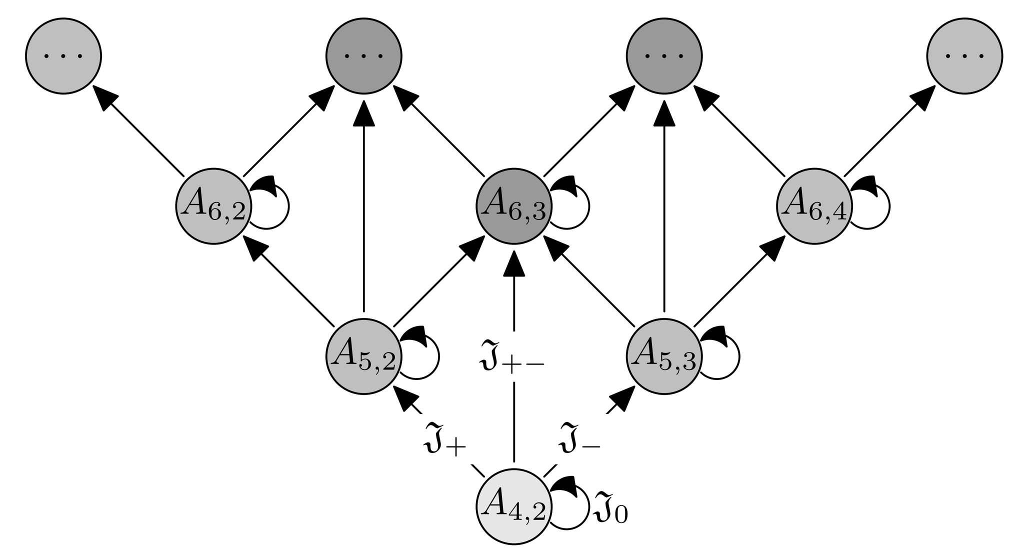
With similar corrections to the generators $\mathfrak{S}_0$, $\mathfrak{K}_0$, the superconformal algebra closes up to gauge transformations,

$$[\mathfrak{J}_a,\mathfrak{J}'_b]\sim\mathfrak{G}_{ab},\qquad\mathfrak{G}_{ab}J(\Lambda)\sim[\partial_a\partial_bJ(0),J(\Lambda)].$$

## All Amplitudes

Employing the BCFW recursion relations [5], generic amplitudes show a specific scaling behaviour when two external momenta become collinear. Acting with the generators $\mathfrak{S}_0$, $\bar{\mathfrak{S}}_0$, $\mathfrak{K}_0$ shows the corrections from the MHV case to render the set of all amplitudes invariant. At tree level, the invariance equation generalises to [4]:

$$\mathfrak{J}_0A_{n,k}+\mathfrak{J}_+A_{n-1,k}+\mathfrak{J}_-A_{n-1,k-1}+\mathfrak{J}_{+-}A_{n-2,k-1}=0.$$



Here, $A_{n,k}$ is the $n$-point amplitude with $k$ negative-helicity particles; $A_n^{\text{MHV}}=A_{n,2}$ and $A_n^{\overline{\text{MHV}}}=A_{n,n-2}$. Dual superconformal symmetry is free of holomorphic anomalies and thus needs not be corrected. Hence it can be argued that the general tree-level amplitude is invariant under the Yangian symmetry of [6]; we are confident that the Yangian uniquely determines the amplitude.

## Open Problems

- Show explicitly that tree amplitudes are uniquely determined by Yangian symmetry.
- Promote corrections to loop level. Does the exact algebra determine loop-level amplitudes?
- What about conformal inversions?
- Apply to $\mathcal{N}<4$ gauge theories with matter?
- Similar effects for $E_{7(7)}$ in supergravity?

## References


[1] V. P. Nair, *"A current algebra for some gauge theory amplitudes"*, Phys. Lett. B214, 215 (1988).

[2] F. Cachazo, P. Svrcek and E. Witten, *"Gauge theory amplitudes in twistor space and holomorphic anomaly"*, JHEP 0410, 077 (2004), hep-th/0409245.

[3] N. Beisert, *"The SU(2/3) Dynamic Spin Chain"*, Nucl. Phys. B682, 487 (2004), hep-th/0310252.

[4] T. Bargheer, N. Beisert, W. Galleas, F. Loebbert and T. McLoughlin, *"Exacting N=4 Superconformal Symmetry"*, arxiv:0905.3738.

[5] R. Britto, F. Cachazo and B. Feng, *"New recursion relations for tree amplitudes of gluons"*, Nucl. Phys. B715, 499 (2005), hep-th/0412308.

[6] J. M. Drummond, J. M. Henn and J. Plefka, *"Yangian symmetry of scattering amplitudes in $\mathcal{N}=4$ super Yang-Mills theory"*, JHEP 0905, 046 (2009), arxiv:0902.2987.