Serre Relation and Higher Grade Generators of the AdS/CFT Yangian Symmetry

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Collaborate with Sanefumi Moriyama, Based on arXiv:0902.3299[hep-th]

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Introduction & Results

AdS/CFT duality: Strong/Weak duality

Integrability ⇒ possibility to construct all-loop correspondence

su(2|2) Spin-Chain Model ['05 Beisert]

The common ∞ symmetries were discovered.

=Yangian symmetry

- Better understanding of Yangian symmetry

⇒ Better understanding of all-loop correspondence

Assumption Evaluation Rep. $J_n|\chi\rangle = u^n J|\chi\rangle$ of Yangian.

If the Assumption does not hold → ∞ sym. in AdS/CFT ≠ Yangian!?

Results

(I) We proved the Assumption.(Ev. Rep is valid!)
(II) We constructed higher grade generators.

Ev. Rep ⇒ Reminiscent of String Worldsheet (open problem)

(1) Current alg $J_n = z^n J \overset{?}{\leftrightarrow} J_n|\chi\rangle = u^n J|\chi\rangle$

(2) KZ eq $\kappa\frac{\partial}{\partial z_i}\Phi = \Sigma_{i\neq j}\frac{(J^A)_i\otimes(J_A)_j}{z_i - z_j}\Phi \overset{?}{\leftrightarrow} \mathcal{R}_{ij} = 1 + h\frac{(J^A)_i\otimes(J_A)_j}{u_i-u_j}+\cdots,$

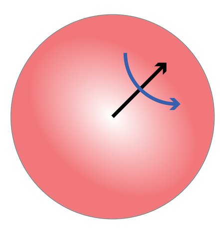
Classical r-matrix ['07 Beisert-Spill, Moriyama-Torrielli]

su(2|2) Spin-Chain Model (review)

Why 'su(2|2)'?

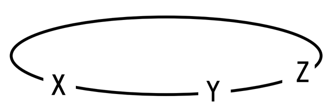
Vacua$|0\rangle = \mathrm{Tr}(\mathcal{Z}\mathcal{Z}\cdots\mathcal{Z})$fixed,

$\mathrm{psu}(2,2|4) \xrightarrow{\text{broken}} [\mathrm{psu}(2|2)]^2 \ltimes \mathbf{R}$



Why 'Spin-Chain'?

$\mathrm{Tr}(\cdots\mathcal{X}\cdots\mathcal{Y}\cdots\mathcal{Z}\cdots) \longleftrightarrow$



Spins : Fundamental Rep.**2**|**2** of su(2|2)

Remarkable fact

Algebra determines 2-particle $\mathcal{R}$(scattering) matrix up-to overall factor.

(∵) "Off-Shell" formalism (Central extension)

$\mathrm{Tr}(\cdots\mathcal{X}\cdots\mathcal{Y}\cdots\mathcal{Z}\cdots) \longrightarrow [\cdots\mathcal{X}\cdots\mathcal{Y}\cdots\mathcal{Z}\cdots]_\infty$

$\mathrm{psu}(2|2)\ltimes\mathbf{R} \longrightarrow \mathrm{psu}(2|2)\ltimes\langle \mathbf{C},\mathbf{P},\mathbf{K}\rangle$

Yangian Symmetry

$[\Delta\widehat{J}^A, \mathcal{R}_{12}]|\chi_1\chi_2\rangle = 0 \quad {}^\forall\widehat{J}^A \in \mathcal{Y}(\mathrm{psu}(2|2)\ltimes \mathrm{R}^3)$

Coproduct $\Delta\widehat{J}^A = \widehat{J}^A\otimes 1 + 1\otimes\widehat{J}^A + \underbrace{\tfrac{1}{2}J^B\otimes J^C f_{CB}{}^A}_{\text{Non-local}}$

if we assume a Representation (Evaluation Rep.)!

$\widehat{J}|\chi\rangle = uJ|\chi\rangle$ (u : spectral parameter)

Serre Relation (review)

Yangian algebra

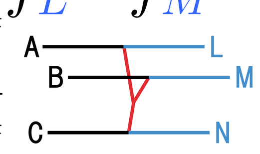
Generators Grade-0 J(Lie alg) & Grade-1 $\widehat{J}$

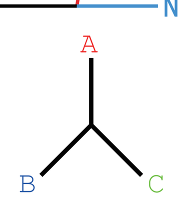
Commutation Relation & Jacobi id.

$[J^A, J^B] = J^C f_C{}^{AB}, \quad [J^A, \widehat{J}^B] = \widehat{J}^C f_C{}^{AB}, \quad [J^{(A},[J^B, J^{C)}]] = 0$

Serre Relation

$[\widehat{J}^{(A},[\widehat{J}^B, J^{C)}]] = -\frac{1}{24}f_L{}^{AI}f_M{}^{BJ}f_N{}^{CK}f_{IJK}\{J^L, J^M, J^N\}$

$= -\frac{1}{24}$  $\{J^L, J^M, J^N\}$

Notation $f_{ABC} \longleftrightarrow$ 

The role of the Serre relation

(1)Assurance of Homomorphism of Coproduct Δ

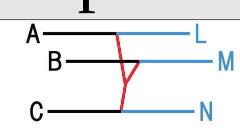
$[[\Delta J^{(A},\Delta\widehat{J}^B],\Delta\widehat{J}^{C)}] = \Delta[[J^{(A},\widehat{J}^B],\widehat{J}^{C)}]$

(2)Constraint on Grade-2 generators $\underbrace{\text{(LHS)}}_{\text{Grade-2}} = \underbrace{\text{(RHS)}}_{\text{Grade-0}}$

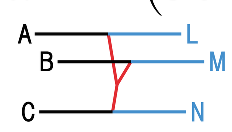
(I) Proof of the compatibility

Proof of Serre Relation

Ev. Rep. $\widehat{J}|\chi\rangle = uJ|\chi\rangle$ is compatible with Serre relation?

$[\widehat{J}^{(A},[\widehat{J}^B, J^{C)}]]|\chi\rangle =$  $\{J^L, J^M, J^N\}|\chi\rangle$

Since (LHS) $= u^2\times$ (Jacobi id)$|\chi\rangle = 0$,

We proved (RHS) $=$  $\{J^L, J^M, J^N\}|\chi\rangle = 0$ ∴ Yes! ∎

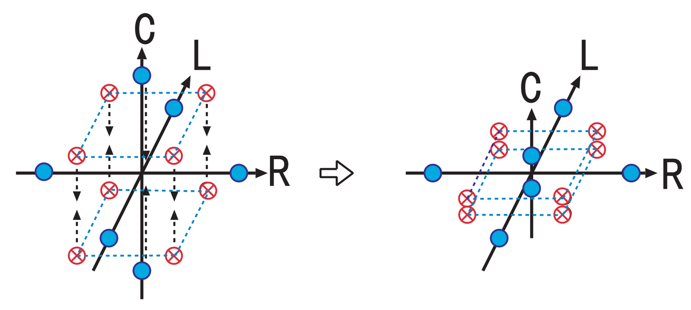
cf. ['08 Spill-Torrielli] with two evaluation parameters!?

Difficulty-(1)

Killing form g_{AB} of psu(2|2) ⋉ R^3 is degenerate.

[Solution] Exceptional Lie superalgebra $\mathrm{d}(2,1;\varepsilon)$

- $\mathrm{d}(2,1;\varepsilon) \xrightarrow{\varepsilon\to 0} \mathrm{psu}(2|2)\ltimes\mathrm{R}^3$



- Non-degenerate Killing form g_{AB}

Difficulty-(2)

The computations are complicated.

[Solution] 3-dimensional γ-matrix formalism

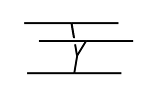
$\mathrm{d}(2,1;\varepsilon) \supset \mathrm{su}(2)_R\times\mathrm{su}(2)_L\times\mathrm{su}(2)_C \qquad \mathrm{su}(2) = \mathrm{so}(3)$

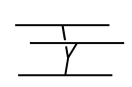
$(\gamma^{\boldsymbol{A}})^K{}_L = (\gamma^A_{A'})^K{}_L = -\sqrt{2}\left(\delta^K_{A'}\delta^A_L - \frac{1}{2}\delta^K_L\delta^A_{A'}\right)$

Clifford algebra $(\gamma^{\boldsymbol{A}})^K{}_L(\gamma^{\boldsymbol{B}})^L{}_M + (\gamma^{\boldsymbol{B}})^K{}_L(\gamma^{\boldsymbol{A}})^L{}_M = 2\delta^K_M g^{\boldsymbol{AB}}$

(II) Higher Grade Generators

[Def] Grade-2 generators

$[\widehat{J}^B, \widehat{J}^C] = \widehat{\widehat{J}}{}^A f_A{}^{BC} + X^{BC}$ with $X^{(A|D}f_D{}^{|BC)} =$  J^3-**(A)**

(∵) Serre relation $[\widehat{J}^{(A|},\widehat{J}^D]f_D{}^{|BC)} =$  J^3.

Gauge transformation : $X^{BC} \to X^{BC} - Y^A f_A{}^{BC}$

Canonical Gauge (... Too Strong)

$X^{BC}f_{CB}{}^A = 0 \Rightarrow \widehat{\widehat{J}}{}^A = \frac{1}{c_2}[\widehat{J}^B, \widehat{J}^C]f_{CB}{}^A \quad (f_A{}^{BC}f_{CBD} = c_2 g_{AD})$

However, $c_2 = 0$ for $\mathrm{d}(2,1;\varepsilon)$ ($\xrightarrow{\varepsilon\to 0}$ psu(2|2) ⋉ R^3).

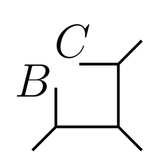
Alternative Suitable Gauge

$X^{BC}|\chi\rangle = 0$ -**(B)**

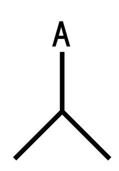
(∵) Consistent with Evaluation Representation.

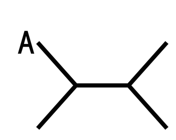
$[\widehat{J}^B, \widehat{J}^C]|\chi\rangle = \widehat{\widehat{J}}{}^A f_A{}^{BC}|\chi\rangle + X^{BC}|\chi\rangle$

Solution to **(A)**&**(B)**

$X^{BC} =$  $J^3 - 6J^A f_A{}^{BC}$

Symmetry of R-matrix (valid for $c_2 = 0$)

$\Delta\widehat{\widehat{J}}{}^A = \widehat{\widehat{J}}{}^A\otimes 1 + 1\otimes\widehat{\widehat{J}}{}^A + \frac{1}{2}$  $(\widehat{J}\otimes J + J\otimes\widehat{J})$

$+\frac{1}{24}$  $(J^2\otimes J + J\otimes J^2)$

$\Rightarrow [\Delta\widehat{\widehat{J}}, R_{12}]|\chi_1\chi_2\rangle = 0$

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