Quantum Symmetries and Integrable Sigma Models

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Abstract

We report upon the first steps of the development of a programme for the systematic solution of certain classes of quantised integrable sigma models. The aim is to enable *ab initio* calculations of the spectra of models relevant to the AdS/CFT correspondence. We identify algorithmically the underlying quantum symmetry algebra of the sigma model and use this as the input to the quantum inverse scattering method for a lattice discretisation. Here we illustrate the progress of this programme with non-trivial examples of integrable sigma models including the sausage model and the *SS*-model.

Background

It is well known that conformal field theories possess a "hidden" quantum symmetry algebra which can be seen manifestly in free field realisations. More precisely, the *screening operators* of the free field theory generate a representation of this quantum symmetry algebra [1, Ch. 11]. Recall that these operators appear in the correlators of the (deformed) theory (*cf.* the Coulomb gas picture). Analogously, perturbing a theory by quite general fields leads to similar insertions in the correlators. It is therefore not unreasonable to regard such perturbations as *generalised screening operators*. This analogy becomes even more apt when one considers the integrals of motion of the perturbed system [2]. But do these generalised screening operators describe an interesting quantum symmetry algebra?

Our thesis is that when the perturbed system is *integrable*, the generalised screening operators precisely describe the quantum symmetry algebra responsible for this integrability. Recall that the key insight behind the quantum inverse scattering method is that the integrable structure can be traced back to the quasitriangularity of the quantum symmetry algebra $\mathscr{U}$ (a Hopf algebra). Specifically, there exists a universal R-matrix $\mathscr{R} \in \mathscr{U} \otimes \mathscr{U}$ satisfying an abstract Yang-Baxter equation [3] from which the building blocks of integrability are obtained by choosing representations of $\mathscr{U}$. In particular, the *transfer matrices* are recovered from an "evaluation representation" $\pi_a(\lambda)$ (λ is the spectral parameter) on a finite-dimensional auxiliary space and an infinite-dimensional representation π_q on the physical quantum space:

$$T_a(\lambda) = \mathrm{Tr}_{\pi_a}\left(\pi_a(\lambda) \otimes \pi_q\right)(\mathscr{R}). \tag{1}$$

Knowing the quantum symmetry algebra is therefore the fundamental requirement for implementing the quantum inverse scattering method. Our strategy is to use the generalised screening operators to make this identification for integrable sigma models (in their dual description). While these screening operators may be used for π_q in some cases [4], in general this choice leads to ultraviolet divergences. We construct instead a *lattice regularisation* of such theories. This amounts to taking π_q to be a tensor product of "discretised screening operator" representations, one for each lattice site.

There is insufficient space to describe our programme in detail here; such a description will appear in our upcoming article [5]. Instead, we content ourselves with showing how this programme works for three important examples, each of which is viewed as a dual sigma model describing an integrable deformation of a system of free bosons.

Results

We will consider here three different integrable sigma models: The sine-Gordon model, the sausage model [6] and the *SS*-model [7]. They each admit dual formulations with respective lagrangians

$$\begin{aligned}
\mathscr{L}_{\mathrm{sG}} &= \frac{1}{2}\left(\partial_\mu \phi\right)^2 + g\cos\left(\beta\phi\right), && (2)\\
\mathscr{L}_{\mathrm{sausage}} &= \frac{1}{2}\left[\left(\partial_\mu \phi^{(1)}\right)^2 + \left(\partial_\mu \phi^{(2)}\right)^2\right] + g\cos\left(\alpha_1\phi^{(1)}\right)\cos\left(\alpha_2\phi^{(2)}\right), && (3)\\
\mathscr{L}_{SS} &= \frac{1}{2}\left[\left(\partial_\mu \phi^{(1)}\right)^2 + \left(\partial_\mu \phi^{(2)}\right)^2 + \left(\partial_\mu \phi^{(3)}\right)^2\right] \\
&\qquad + g\left[\cos\left(\alpha_1\phi^{(1)} + \alpha_2\phi^{(2)}\right)e^{\mathrm{i}\alpha_3\phi^{(3)}} + \cos\left(\alpha_1\phi^{(1)} - \alpha_2\phi^{(2)}\right)e^{-\mathrm{i}\alpha_3\phi^{(3)}}\right], && (4)
\end{aligned}$$

where $\sum_i \alpha_i^2 = 1$. It is clear that these models are one or two-parameter deformations of one, two or three free bosons. Expanding the cosines, we obtain 2, 4 and 4 perturbing terms, respectively, each of which is a product of exponentials of classical bosons. Quantising, these become products of the usual vertex operators $\mathscr{V}_a^{(i)}(z)$. Our task is now to determine the algebra generated by the corresponding generalised screening operators. This follows from the standard exchange relations of the vertex operators:

$$\mathscr{V}_a^{(i)}(z)\,\mathscr{V}_b^{(j)}(w) = e^{\mathrm{i}\pi ab\delta_{i,j}}\,\mathscr{V}_b^{(j)}(w)\,\mathscr{V}_a^{(i)}(z) \qquad (\arg z > \arg w). \tag{5}$$

sine-Gordon

For this model, we have one boson and two perturbing vertex operators, $V_0(z) = \mathscr{V}_{\mathrm{i}\beta}(z)$ and $V_1(z) = \mathscr{V}_{-\mathrm{i}\beta}(z)$. The exchange relations take the form

$$V_i(z)\,V_j(w) = q^{A_{i,j}}\,V_j(w)\,V_i(z), \tag{6}$$

where $q = e^{-\mathrm{i}\pi\beta^2}$ and $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ is the Cartan matrix of $\widehat{\mathfrak{sl}}(2)$. The generalised screening operators $Q_i = \oint V_i(z)\,\frac{\mathrm{d}z}{2\pi\mathrm{i}}$ satisfy the fourth-order q-Serre relations of $\mathscr{U}_q\left(\widehat{\mathfrak{sl}}(2)\right)$:

$$Q_{0001} - [3]_q Q_{0010} + [3]_q Q_{0100} - Q_{1000} = 0 \quad \text{and} \quad Q_{1110} - [3]_q Q_{1101} + [3]_q Q_{1011} - Q_{0111} = 0. \tag{7}$$

Here, $Q_{ab\cdots c}$ is shorthand for $Q_a Q_b \cdots Q_c$. The Q_i satisfy no other independent relations (to order 7).

The generalised screening operators therefore form a representation of the quantised enveloping algebra of the nilpotent subalgebra $\mathfrak{n}_-$ of $\widehat{\mathfrak{sl}}(2)$. This can be extended to the corresponding Borel subalgebra $\mathfrak{b}_-$ by using the zero-mode (momentum operator) P of the free boson: $K_0 = e^{-2\pi\beta P}$ and $K_1 = e^{2\pi\beta P}$. We therefore conclude that the quantum symmetry algebra of the sine-Gordon model is $\mathscr{U}_q\left(\mathfrak{b}_-\left(\widehat{\mathfrak{sl}}(2)\right)\right)$. We have no realisation of $\mathfrak{n}_+$, nor do we expect one — for genuine screening operators, such a realisation makes use of the conformal symmetry [1].

The quantum symmetry algebra becomes a Hopf algebra upon defining the standard coproduct

$$\Delta(Q_i) = Q_i \otimes \mathbf{1} + K_i^{-1} \otimes Q_i \quad \text{and} \quad \Delta(K_i) = K_i \otimes K_i, \tag{8}$$

counit $\varepsilon(Q_i) = 0$ and $\varepsilon(K_i) = 1$, and antipode $S(Q_i) = -K_i Q_i$ and $S(K_i) = K_i^{-1}$. Taking the two-dimensional evaluation representations of $\widehat{\mathfrak{sl}}(2)$ with generic spectral parameters λ and μ, and extending them to $\mathscr{U}_q\left(\widehat{\mathfrak{sl}}(2)\right)$, we recover the well-known form of the R-matrix.

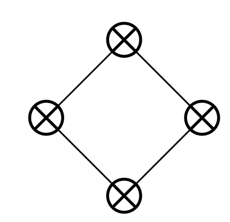
Sausage

The sausage model has two bosons and four perturbing vertex operators, $V_0 = \mathscr{V}_{\mathrm{i}\alpha_1}^{(1)}\mathscr{V}_{\mathrm{i}\alpha_2}^{(2)}$, $V_1 = \mathscr{V}_{\mathrm{i}\alpha_1}^{(1)}\mathscr{V}_{-\mathrm{i}\alpha_2}^{(2)}$, $V_2 = \mathscr{V}_{-\mathrm{i}\alpha_1}^{(1)}\mathscr{V}_{-\mathrm{i}\alpha_2}^{(2)}$ and $V_3 = \mathscr{V}_{-\mathrm{i}\alpha_1}^{(1)}\mathscr{V}_{\mathrm{i}\alpha_2}^{(2)}$. The exchange relations then take the form

$$V_i(z)\,V_j(w) = -q^{A_{i,j}}\,V_j(w)\,V_i(z), \tag{9}$$

where $q = e^{-2\pi\mathrm{i}\alpha_1^2}$ and A is a Cartan matrix for $\widehat{\mathfrak{psl}}(2|2)$:

$$A = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}.$$



This suggests that the generalised screening operators should satisfy $Q_i^2 = 0$, $Q_0Q_2 + Q_2Q_0 = Q_1Q_3 + Q_3Q_1 = 0$, and the q-Serre relations (listed in [8]) of $\mathscr{U}_q\left(\widehat{\mathfrak{psl}}(2|2)\right)$ — indeed we find that they do. However, we also find an additional relation,

$$Q_{1234} + Q_{2341} + Q_{3412} + Q_{4123} + [2]_q\left(Q_{1324} - Q_{2413}\right) - Q_{3214} - Q_{2143} - Q_{1432} - Q_{4321} = 0, \tag{10}$$

which is not listed in [8]. We have checked that this is a genuine q-Serre relation of $\mathscr{U}_q\left(\widehat{\mathfrak{psl}}(2|2)\right)$.

Setting $K_0 = K_2^{-1} = e^{\pi\left(\alpha_1 P^{(1)} + \alpha_2 P^{(2)}\right)}$ and $K_1 = K_3^{-1} = e^{\pi\left(\alpha_1 P^{(1)} - \alpha_2 P^{(2)}\right)}$, we obtain a representation of the corresponding Borel subalgebra (specified by the above Dynkin diagram). The quantum symmetry algebra of the sausage model is thus $\mathscr{U}_q\left(\mathfrak{b}_-\left(\widehat{\mathfrak{psl}}(2|2)\right)\right)$. However, we may relax this conclusion slightly and consider instead the quantised enveloping algebra of the central extension $\widehat{\mathfrak{sl}}(2|2)$. This is convenient for R-matrix considerations as it possesses four-dimensional evaluation representations. Note however that (10) is not a relation of this extended algebra. The R-matrix is

$$R = \left(\begin{array}{cccccccccccccccc}
\lambda q^{-1}-q\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda-\mu & 0 & 0 & f(\mu) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda-\mu & 0 & 0 & 0 & 0 & 0 & f(\mu) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda-\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f(\mu) & 0 & 0 & 0 \\
0 & f(\lambda) & 0 & 0 & \lambda-\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu q^{-1}-\lambda q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda-\mu & 0 & 0 & f(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu-\lambda & 0 & 0 & 0 & 0 & 0 & f(\lambda) & 0 & 0 \\
0 & 0 & f(\lambda) & 0 & 0 & 0 & 0 & 0 & \lambda-\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & f(\mu) & 0 & 0 & \lambda-\mu & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda q^{-1}-q\mu & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda-\mu & 0 & 0 & f(\lambda) & 0 \\
0 & 0 & 0 & f(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda-\mu & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & f(\mu) & 0 & 0 & 0 & 0 & 0 & \mu-\lambda & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f(\mu) & 0 & 0 & \lambda-\mu & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu q^{-1}-\lambda q
\end{array}\right), \tag{11}$$

where $f(a) = a\left(q - q^{-1}\right)$.

SS-Model

The *SS*-model is a *two-parameter* family of integrable sigma models built by perturbing three free bosons with four vertex operators, $V_0 = \mathscr{V}_{\mathrm{i}\alpha_1}^{(1)}\mathscr{V}_{\mathrm{i}\alpha_2}^{(2)}\mathscr{V}_{\mathrm{i}\alpha_3}^{(3)}$, $V_1 = \mathscr{V}_{\mathrm{i}\alpha_1}^{(1)}\mathscr{V}_{-\mathrm{i}\alpha_2}^{(2)}\mathscr{V}_{-\mathrm{i}\alpha_3}^{(3)}$, $V_2 = \mathscr{V}_{-\mathrm{i}\alpha_1}^{(1)}\mathscr{V}_{\mathrm{i}\alpha_2}^{(2)}\mathscr{V}_{-\mathrm{i}\alpha_3}^{(3)}$ and $V_3 = \mathscr{V}_{-\mathrm{i}\alpha_1}^{(1)}\mathscr{V}_{-\mathrm{i}\alpha_2}^{(2)}\mathscr{V}_{\mathrm{i}\alpha_3}^{(3)}$. The exchange relations then take the form

$$V_i(z)\,V_j(w) = -q_1^{A_{i,j}^{(1)}}\,q_2^{A_{i,j}^{(2)}}\,q_3^{A_{i,j}^{(3)}}\,V_j(w)\,V_i(z), \tag{12}$$

where $q_i = e^{-2\pi\mathrm{i}\alpha_i^2}$ (so $q_1q_2q_3 = 1$) and the $A^{(i)}$ are Cartan matrices for $\mathfrak{sl}(2|1) \oplus \mathfrak{sl}(2|1)$:

$$A^{(1)} = \begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \end{pmatrix} \quad \text{and} \quad A^{(3)} = \begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{pmatrix}. \tag{13}$$

However, the generalised screening operators satisfy $Q_i^2 = 0$ and four cubic "q-Serre relations":

$$\begin{aligned}
\left(q_1 - q_1^{-1}\right)\left(Q_{120} - Q_{021}\right) + \left(q_2 - q_2^{-1}\right)\left(Q_{012} - Q_{210}\right) + \left(q_3 - q_3^{-1}\right)\left(Q_{201} - Q_{102}\right) &= 0, \\
\left(q_1 - q_1^{-1}\right)\left(Q_{130} - Q_{031}\right) + \left(q_2 - q_2^{-1}\right)\left(Q_{301} - Q_{103}\right) + \left(q_3 - q_3^{-1}\right)\left(Q_{013} - Q_{310}\right) &= 0, \\
\left(q_1 - q_1^{-1}\right)\left(Q_{302} - Q_{203}\right) + \left(q_2 - q_2^{-1}\right)\left(Q_{230} - Q_{032}\right) + \left(q_3 - q_3^{-1}\right)\left(Q_{023} - Q_{320}\right) &= 0, \\
\left(q_1 - q_1^{-1}\right)\left(Q_{312} - Q_{213}\right) + \left(q_2 - q_2^{-1}\right)\left(Q_{123} - Q_{321}\right) + \left(q_3 - q_3^{-1}\right)\left(Q_{231} - Q_{132}\right) &= 0.
\end{aligned} \tag{14}$$

Note that these relations have no classical counterpart when the $q_i \to 1$. We have found no other (independent) relations up to order 7.

This demonstrates that the quantum symmetry algebra of the *SS*-model is not related to any quantum affine superalgebra, but rather forms the first example of a new, previously undiscovered class of algebras underlying integrability. Defining Cartan elements K_i in the obvious way, we have verified that the standard coproduct, counit and antipode define a Hopf algebra structure on this new quantum symmetry algebra. It is almost surely quasi-triangular, and work is currently in progress to verify this.

Finally, we mention that the *SS*-model formally reduces to the sausage model if we set $q_2 = 1$. At the level of the symmetry algebras, we see that (14) then implies that $Q_0Q_2 + Q_2Q_0$ and $Q_1Q_3 + Q_3Q_1$ are *central* in the specialised algebra. This accords well with the fact that these combinations both vanish in the symmetry algebra of the sausage. It would be very interesting to see if the other Serre relations of the sausage can be derived from (14) in this way.

Future Work

Our current goal is to use the results presented here to construct Lax connections describing lattice discretisations of these sigma models. This has been completed for sine-Gordon, and is underway for the sausage (results will appear in [5]). In a parallel effort, these Lax connections will be used to solve for the sigma model spectrum by combining the method of separation of variables with Q-operator technology. For sine-Gordon, this will be reported on in [9].

Moreover, we are still investigating the structure of the new quantum symmetry algebra of the *SS*-model with the aim of generalising the concept of R and L-matrices to this case. We are also using our formalism to construct new families of integrable sigma models with the aim of producing other examples of symmetry algebras "beyond" quantum affine superalgebras. We believe that a detailed understanding of integrability in this context will require a solid mathematical study of these more general Hopf algebras.

References

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