Symmetries of scattering amplitudes in $\mathcal{N}=4$ SYM

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Based on work in collaboration with

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Scattering amplitudes in $\mathcal{N}=4$ super Yang-Mills theory

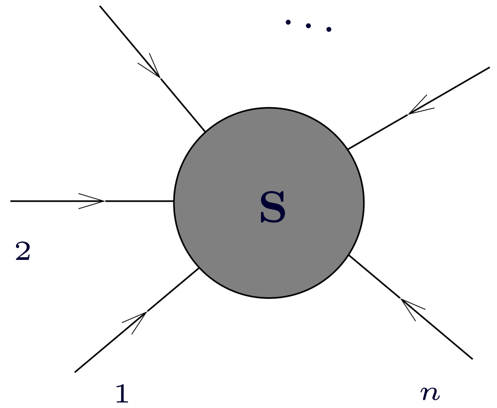
✔ Extended spectrum of asymptotic on-shell states

2 gluons with helicity $\pm 1\,,$ *6 scalars with helicity* $0\,,$ *8 gaugino with helicity* $\pm\frac{1}{2}$

all in the adjoint of the $SU(N_c)$ gauge group

✔ On-shell matrix elements of $S-$matrix:

$\mathcal{A}_n \;=$



- Nontrivial functions of Mandelstam variables $s_{i\ldots j} = (p_i + \ldots + p_j)^2$ and 't Hooft coupling $a = g^2_{\rm YM} N_c$
- Are independent on gauge choice
- Probe (hidden) symmetries of gauge theory

✔ Simpler than QCD amplitudes but they share many of the same properties

✔ In planar $\mathcal{N}=4$ SYM theory they have a remarkable iterative structure [Anastasiou,Bern,Dixon,Kosower]

✔ All-order conjectures [Bern,Dixon,Smirnov] and a proposal for strong coupling via AdS/CFT [Alday,Maldacena]

✔ Final goal: *compute the scattering amplitudes in planar* $\mathcal{N}=4$ *SYM theory*
for arbitrary 't Hooft coupling a

General properties of amplitudes in gauge theories

Tree amplitudes:

- ✔ Are well-defined in $D = 4$ dimensions (free from UV and IR divergences)
- ✔ Respect classical (Lagrangian) symmetries of gauge theory
- ✔ Gluon tree amplitudes are the same in all gauge theories

All-loop amplitudes:

- ✔ Loop corrections are not universal (gauge theory dependent)
- ✔ Free from UV divergences (when expressed in terms of renormalized coupling)
- ✔ Suffer from IR divergences $\to$ are not well-defined in $D = 4$ dimensions
- ✔ Some of the classical symmetries (dilatations, conformal boosts,...) are broken

Three questions in this talk:

- ✔ Do tree amplitudes in $\mathcal{N} = 4$ SYM have hidden symmetries?
- ✔ How powerful are these symmetries to completely determine the scattering amplitudes?
- ✔ What happens to these symmetries at loop level?

Color-ordered planar MHV, NMHV,... amplitudes

✔ Color-ordered planar partial amplitudes:

$$A_n = \mathrm{tr}\left[T^{a_1}T^{a_2}\dots T^{a_n}\right] A_n^{h_1,h_2,\dots,h_n}(p_1,p_2,\dots,p_n) + \text{[Bose symmetry]}$$

✗ Quantum numbers: light-like momenta ($p_i^2 = 0$), helicity $(h_i = 0, \pm\frac{1}{2}, \pm 1)$, color ($a_i$)

✗ Amplitudes suffer from IR divergences $\mapsto$ require regularization (dim.reg. with $D = 4 - 2\epsilon$)

✔ The amplitudes are classified according to their total helicity

$$h_{\rm tot} = h_1 + \dots + h_n = \{n, n-2, n-4, \dots, -(n-2), -n\}$$

✗ $h_{\rm tot} = \pm n, \pm(n-2)$: $\mapsto$ amplitudes vanish to all loops due to supersymmetry

✗ $h_{\rm tot} = n - 4$: $\mapsto$ MHV amplitudes $A^{--+\dots+}$, $A^{-+-\dots+}$

$$A_n^{\rm MHV} = A_n^{\rm MHV(tree)}(p_i, h_i) M_n^{\rm MHV}(\{s_{ij}\}; a)$$

All-loop corrections are described by a single scalar function! [Parke,Taylor]

✗ $h_{\rm tot} = n - 4 - 2p$: $\mapsto$ $\mathrm{N^p}$MHV amplitudes $A^{---+\dots+}$, $A^{--+-\dots+}$

$$A_n^{\rm N^pMHV} = \textit{much more complicated structure compared with MHV amplitudes}$$

Use supersymmetry to combine amplitudes into superamplitudes

From MHV amplitudes to MHV superamplitude in $\mathcal{N}=4$ SYM

- On-shell helicity states in $\mathcal{N}=4$ SYM:

$$G^{\pm}\ \text{(gluons } h=\pm 1\text{)},\qquad \Gamma_A,\ \bar{\Gamma}^A\ \text{(gluinos } h=\pm\tfrac{1}{2}\text{)},\qquad S_{AB}\ \text{(scalars } h=0\text{)}$$

- Can be combined into a single on-shell superstate [Mandelstam],[Brink et el]

$$\Phi(p,\eta)=G^{+}(p)+\eta^{A}\Gamma_{A}(p)+\frac{1}{2}\eta^{A}\eta^{B}S_{AB}(p)$$
$$+\frac{1}{3!}\eta^{A}\eta^{B}\eta^{C}\epsilon_{ABCD}\bar{\Gamma}^{D}(p)+\frac{1}{4!}\eta^{A}\eta^{B}\eta^{C}\eta^{D}\epsilon_{ABCD}G^{-}(p)$$

- Combine all MHV amplitudes into a single MHV superamplitude [Nair]

$$\mathcal{A}_n^{\mathrm{MHV}}=(\eta_1)^4(\eta_2)^4\times A\left(G_1^{-}G_2^{-}G_3^{+}\dots G_n^{+}\right)$$
$$+(\eta_1)^4(\eta_2)^2(\eta_3)^2\times A\left(G_1^{-}\bar{S}_2\,S_3\dots G_n^{+}\right)+\dots$$

- Spinor helicity formalism: [Xu,Zhang,Chang'87]

$$p_i^2=0\qquad\Leftrightarrow\qquad p_i^{\alpha\dot\alpha}\equiv p_i^{\mu}(\sigma_\mu)^{\alpha\dot\alpha}=\lambda_i^{\alpha}\,\tilde{\lambda}_i^{\dot\alpha}\equiv|i\rangle[i|$$

- Superamplitudes are functions of $\{\lambda_i,\tilde{\lambda}_i,\eta_i\}$

$$\mathcal{A}_n(\Phi_1,\Phi_2,\dots,\Phi_n)=\mathcal{A}_n(\lambda_1,\tilde{\lambda}_1,\eta_1;\dots\lambda_n,\tilde{\lambda}_n,\eta_n)$$

Tree MHV superamplitude

- ✔ All MHV amplitudes are combined into a single superamplitude (spinor notations $\langle ij \rangle = \lambda_i^\alpha \lambda_{j\alpha}$)

$$\mathcal{A}_n^{\mathrm{MHV}} = i \frac{\delta^{(4)}\left(\sum_{i=1}^n p_i\right)\, \delta^{(8)}\left(\sum_{i=1}^n \lambda_i^\alpha \eta_i^A\right)}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle}$$

- ✔ On-shell $\mathcal{N} = 4$ supersymmetry: [Nair]

$$q_\alpha^A = \sum_i \lambda_{i,\alpha} \eta_i^A \,, \quad \bar{q}_{A\,\dot{\alpha}} = \sum_i \tilde{\lambda}_{i,\dot{\alpha}} \frac{\partial}{\partial \eta_i^A} \qquad \Longrightarrow \qquad q_\alpha^A \mathcal{A}_n^{\mathrm{MHV}} = \bar{q}_{A\,\dot{\alpha}} \mathcal{A}_n^{\mathrm{MHV}} = 0$$

- ✔ (Super)conformal invariance [Witten]

$$k_{\alpha\dot{\alpha}} = \sum_i \frac{\partial^2}{\partial \lambda_i^\alpha \partial \tilde{\lambda}_i^{\dot{\alpha}}} \qquad \Longrightarrow \qquad k_{\alpha\dot{\alpha}} \mathcal{A}_n^{\mathrm{MHV}} = 0$$

  Much less trivial to verify for NMHV amplitudes

- ✔ In fact, (super)conformal symmetry is almost exact (due to holomorphic anomaly)

$$\bar{s}\, \mathcal{A}_n^{\mathrm{MHV}} \sim \sum_i \left(\eta_i \tilde{\lambda}_{i+1} - \eta_{i+1} \tilde{\lambda}_i \right) \delta^{(2)}(\lambda_i, \lambda_{i+1})\, \mathcal{A}_{n-1}^{\mathrm{MHV}}$$

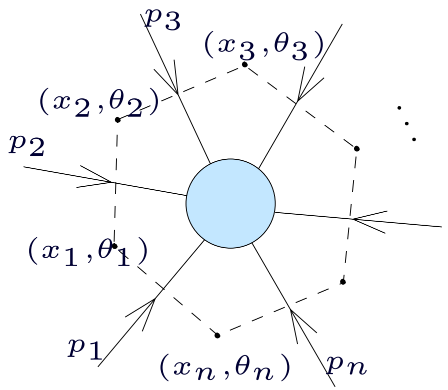
  $\bar{s}\, \mathcal{A}_n^{\mathrm{MHV}}$ is localized at collinear configurations $p_i || p_{i+1}$ [Bargheer,Beisert,Galleas,Loebbert,McLoughlin]

Dual $\mathcal{N}=4$ superconformal symmetry I

- The $\mathcal{N}=4$ superamplitudes possess a much bigger, dual superconformal symmetry

[Drummond, Henn, GK, Sokatchev]

- Chiral dual superspace $(x_{\alpha\dot\alpha}\,,\,\theta_\alpha^A\,,\,\lambda_\alpha)$:



✗ $p=\sum_{i=1}^{n} p_i = 0 \quad \longrightarrow \quad p_i = x_i - x_{i+1}$

✗ $q=\sum_{i=1}^{n} \lambda_i \eta_i = 0 \quad \longrightarrow \quad \lambda_{i\,\alpha}\,\eta_i^A = (\theta_i - \theta_{i+1})_\alpha^A$

- The MHV superamplitude in the dual superspace

$$\mathcal{A}_n^{\rm MHV} = i(2\pi)^4 \frac{\delta^{(4)}\left(x_1 - x_{n+1}\right)\,\delta^{(8)}\left(\theta_1 - \theta_{n+1}\right)}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}$$

- $\mathcal{N}=4$ supersymmetry in the dual superspace:

$$Q_{A\,\alpha} = \sum_{i=1}^{n}\frac{\partial}{\partial\theta_i^{A\,\alpha}}\,, \qquad \bar{Q}_{\dot\alpha}^{A} = \sum_{i=1}^{n}\theta_i^{A\,\alpha}\frac{\partial}{\partial x_i^{\dot\alpha\alpha}}\,, \qquad P_{\alpha\dot\alpha} = \sum_{i=1}^{n}\frac{\partial}{\partial x_i^{\dot\alpha\alpha}}$$

- Dual supersymmetry

$$Q_{A\,\alpha}\mathcal{A}_n^{\rm MHV} = \bar{Q}_{\dot\alpha}^{A}\mathcal{A}_n^{\rm MHV} = P_{\alpha\dot\alpha}\mathcal{A}_n^{\rm MHV} = 0$$

Dual $\mathcal{N} = 4$ superconformal symmetry II

- ✔ Super-Poincaré $+$ inversion $=$ conformal supersymmetry:
  - ✗ Inversions in the dual superspace

$$I[\lambda_i^{\alpha}] = (x_i^{-1})^{\dot\alpha\beta}\lambda_{i\beta}\,, \qquad I[\theta_i^{\alpha\, A}] = (x_i^{-1})^{\dot\alpha\beta}\theta_{i\beta}^{A}$$

  - ✗ Neighbouring contractions are dual conformal covariant

$$I[\langle i\, i+1\rangle] = (x_i^2)^{-1}\langle i\, i+1\rangle$$

  - ✗ Impose cyclicity, $x_{n+1} = x_1$, $\theta_{n+1} = \theta_1$, through delta functions. Then, only in $\mathcal{N} = 4$,

$$I[\delta^{(4)}(x_1 - x_{n+1})] = x_1^8\,\delta^{(4)}(x_1 - x_{n+1})$$
$$I[\delta^{(8)}(\theta_1 - \theta_{n+1})] = x_1^{-8}\,\delta^{(8)}(\theta_1 - \theta_{n+1})$$

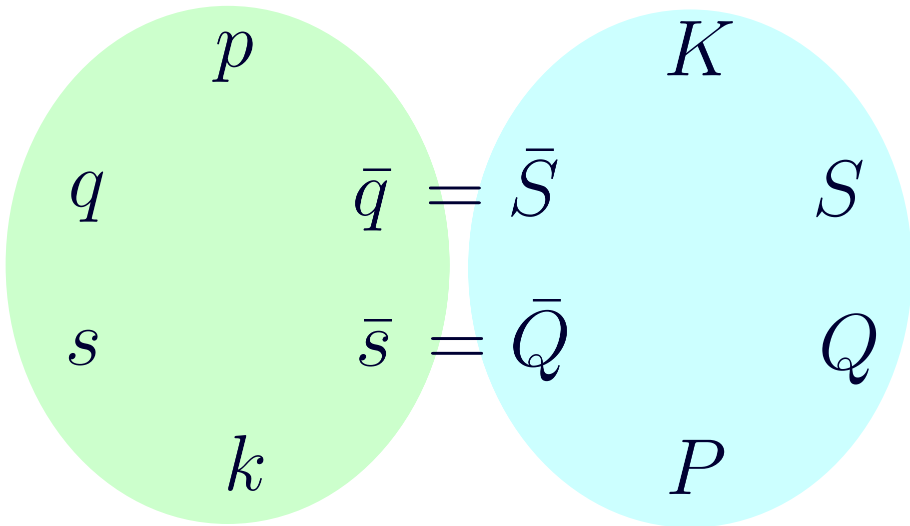
- ✔ The tree-level MHV superamplitude is covariant under dual conformal inversions

$$I\left[\mathcal{A}_n^{\rm MHV}\right] = \left(x_1^2 x_2^2 \dots x_n^2\right) \times \mathcal{A}_n^{\rm MHV}$$

- ✔ ***Dual superconformal covariance is a property of all tree-level superamplitudes (MHV, NMHV, N^{2}MHV,...) in $\mathcal{N} = 4$ SYM theory*** [Drummond,Henn,GK,Sokatchev],[Brandhuber,Heslop,Travaglini]

Symmetries at tree amplitudes

- ✔ The relationship between conventional and dual superconformal $su(2,2|4)$ symmetries: [Drummond,Henn,GK,Sokatchev]



- ✔ The same symmetries appear at strong coupling from invariance of $\mathrm{AdS}_5 \times \mathrm{S}^5$ sigma model under bosonic [Kallosh,Tseytlin] + fermionic T-duality [Berkovits,Maldacena],[Beisert,Ricci,Tseytlin,Wolf]

- ✔ (Infinite-dimensional) closure of two symmetries has Yangian structure [Drummond,Henn,Plefka]

- ✔ All tree-level amplitudes are known [Drummond,Henn] from the supersymmetric generalization of the BCFW recursion relations [Brandhuber,Heslop,Travaglini],[Bianchi,Elvang,Freeman],[Arkani-Hamed,Cachazo,Kaplan]

Are tree level amplitudes completely determined by the symmetries?

Invariants of both symmetries

- ✔ The ‘ratio’ of two tree superamplitudes

$$\mathcal{A}_n = \mathcal{A}_n^{\mathrm{MHV}} R_n = \mathcal{A}_n^{\mathrm{MHV}} \left[R_n^{\mathrm{MHV}} + R_n^{\mathrm{NMHV}} + \ldots \right]$$

- ✔ The ratio R_n–functions are invariants of ***both*** conventional (g) and dual (G) symmetries:

$$g \cdot R_n^{\mathrm{N^pMHV}} = G \cdot R_n^{\mathrm{N^pMHV}} = 0 \,,$$

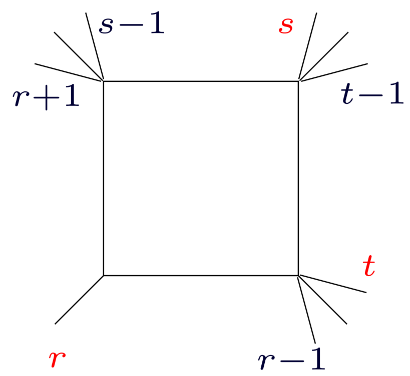
$$R_n^{\mathrm{N^pMHV}} = \text{ Polynomial in } \theta\text{'s of degree } 4p$$

Classification of $\mathrm{N^pMHV}$ superinvariants remains to be done

- ✗ MHV superinvariants $(p = 0)$ are trivial: $\qquad R_n^{\mathrm{MHV}} = \text{const}$
- ✗ NMHV superinvariants $(p = 1)$ are nontrivial:

[Drummond,Henn,GK,Sokatchev]

$$R_{rst}(x, \lambda, \theta) = \frac{\langle s-1 s\rangle\langle t-1 t\rangle \delta^{(4)}(\langle r|x_{rs}x_{st}|\theta_{tr}\rangle + \langle r|x_{rt}x_{ts}|\theta_{sr}\rangle)}{x_{st}^2 \langle r|x_{rs}x_{st}|t-1\rangle\langle r|x_{rs}x_{st}|t\rangle\langle r|x_{rt}x_{ts}|s-1\rangle\langle r|x_{rt}x_{ts}|s\rangle} =$$



Supersymmetric extenstion of three-mass box coefficients

[Bern,Dixon,Kosower]

How powerful are the symmetries?

General expression for the NMHV ratio function dictated by the symmetries

$$g \cdot R_n^{\rm NMHV} = G \cdot R_n^{\rm NMHV} = 0 \qquad \mapsto \qquad R_n^{\rm NMHV} = \sum_{r,s,t} c_{rst} R_{rst} \quad \text{(with } c_{rst} \text{ arbitrary!)}$$

- ✔ The combined action of conventional and dual superconformal symmetries *is not sufficient* to fix all the freedom in the tree-level amplitudes

- ✔ The additional information needed comes from the study of the analytic properties of the amplitudes: [GK, Sokatchev]

  - ✗ Tree amplitudes only have 'physical' poles in multi-particle invariant masses $(p_s + \ldots + p_{t-1})^2 = x_{st}^2 = 0$

  - ✗ Tree amplitudes should be free from spurious singularities

  Analytical properties of R_{rst}−invariants:

$$R_{rst} \sim \Big(\underbrace{x_{st}^2}_{\text{physical pole}} \times \underbrace{\langle r|x_{rs}x_{st}|t-1\rangle\langle r|x_{rs}x_{st}|t\rangle\langle r|x_{rt}x_{ts}|s-1\rangle\langle r|x_{rt}x_{ts}|s\rangle}_{\text{spurious poles}} \Big)^{-1}$$

  Kinematical configuration corresponding to spurious pole at $\langle r|x_{rs}x_{st}|t\rangle = 0$:

$$-x_{rt}^2 x_{r+1,s}^2 x_{t+1,s}^2 + x_{r+1,t}^2 x_{rs}^2 x_{t+1,s}^2 - x_{r+1,t+1}^2 x_{rs}^2 x_{ts}^2 + x_{r,t+1}^2 x_{r+1,s}^2 x_{ts}^2 = 0$$

  Spurious poles should cancel inside $R_n^{\rm NMHV}$*!*

Cancellation of spurious poles

- ✔ Each invariant R_{rst} has four sets of spurious poles

$$\langle r|x_{rs}x_{st}|t-1\rangle = \langle r|x_{rs}x_{st}|t\rangle = \langle r|x_{rt}x_{ts}|s-1\rangle = \langle r|x_{rt}x_{ts}|s\rangle = 0$$

- ✔ 'Master identity': all spurious poles of R_{rst} cancel in the linear combination of invariants:

$$R_{rst} + (R_{str} + R_{trs} - R_{s-1\,tr} - R_{t-1\,rs})$$

  - ✗ $n = 8$ NMHV: general expression consistent with all symmetries

$$R_8^{\mathrm{NMHV};0} = \alpha R_{147} + \beta R_{148} + \gamma R_{157} + \delta R_{158} + \varepsilon R_{168} + \text{cyclic}$$

    Cancellation of spurious poles leads to

$$\alpha - \beta = \alpha + \gamma - \delta = 2\alpha - \gamma = \delta - \varepsilon = \beta + \gamma - \delta = \beta + \gamma - \varepsilon = 0$$

    This system is overdetermined but it has a unique solution

$$\beta = \alpha\,, \qquad \gamma = 2\alpha\,, \qquad \delta = \epsilon = 3\alpha$$

- ✔ The same relations (with $\alpha = \frac{1}{8}$) ensure the correct behavior in the collinear limit $p_i || p_{i+1}$

$$R_n(\ldots, i, i+1, \ldots) \stackrel{i||i+1}{\longrightarrow} R_{n-1}(\ldots, \ell, \ldots)$$

Tree amplitudes are uniquely fixed by symmetries + analyticity condition [GK,Sokatchev]

Do symmetries survive loop corrections?

- ✔ Loop corrections to the amplitudes necessarily induce infrared divergences
- ✔ The scattering amplitudes are well-defined in $D = 4 - 2\epsilon_{\rm IR}$ dimensions only
- ✔ *All-loop planar* (super)amplitudes can be split into a IR divergent and a finite part

$$\mathcal{A}_n^{(\rm all-loop)} = \text{Div}(1/\epsilon_{\rm IR})\ [\text{Fin} + O(\epsilon_{\rm IR})]$$

- ✗ IR divergences (poles in $\epsilon_{\rm IR}$) exponentiate (in any gauge theory!) [Mueller],[Sen],[Collins],[Sterman],...

$$\text{Div}(1/\epsilon_{\rm IR}) = \exp\left\{-\frac{1}{2}\sum_{l=1}^{\infty} a^l \left(\frac{\Gamma_{\rm cusp}^{(l)}}{(l\epsilon_{\rm IR})^2} + \frac{G^{(l)}}{l\epsilon_{\rm IR}}\right)\sum_{i=1}^{n}(-s_{i,i+1})^{l\epsilon_{\rm IR}}\right\}$$

- ✗ *IR divergences* are in the one-to-one correspondence with *UV divergences* of Wilson loops [Ivanov,GK,Radyushkin]

$\Gamma_{\rm cusp}(a) = \sum_l a^l \Gamma_{\rm cusp}^{(l)} =$ *cusp anomalous dimension of Wilson loops*

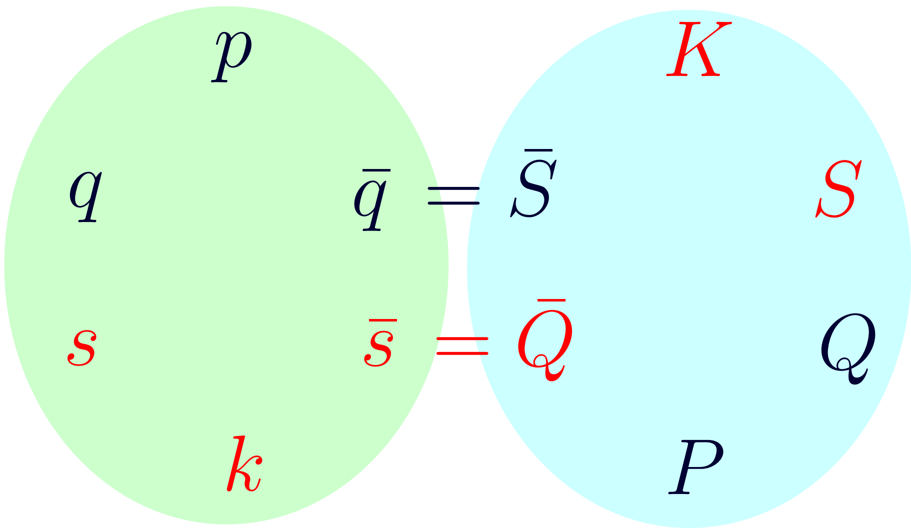
$G(a) = \sum_l a^l G_{\rm cusp}^{(l)} =$ *collinear anomalous dimension*

- ✔ *IR divergences preserve Poincaré supersymmetry but break conformal + dual conformal symmetry*

  IR divergences come from small momenta (=large distances) and, therefore, the conformal anomaly is not ‘localized’ (= difficult to control)

Anomalous symmetries at loop level

- ✔ Some symmetries ($p, q, \bar{q}, P, Q, \bar{S}, \ldots$) survive loop corrections while other ($s, \bar{s}, k, K, S, \bar{Q}, \ldots$) are broken



- ✔ But the anomalies are not independent: $[K, \bar{Q}] = S\,, \qquad [s, \bar{s}] = k$

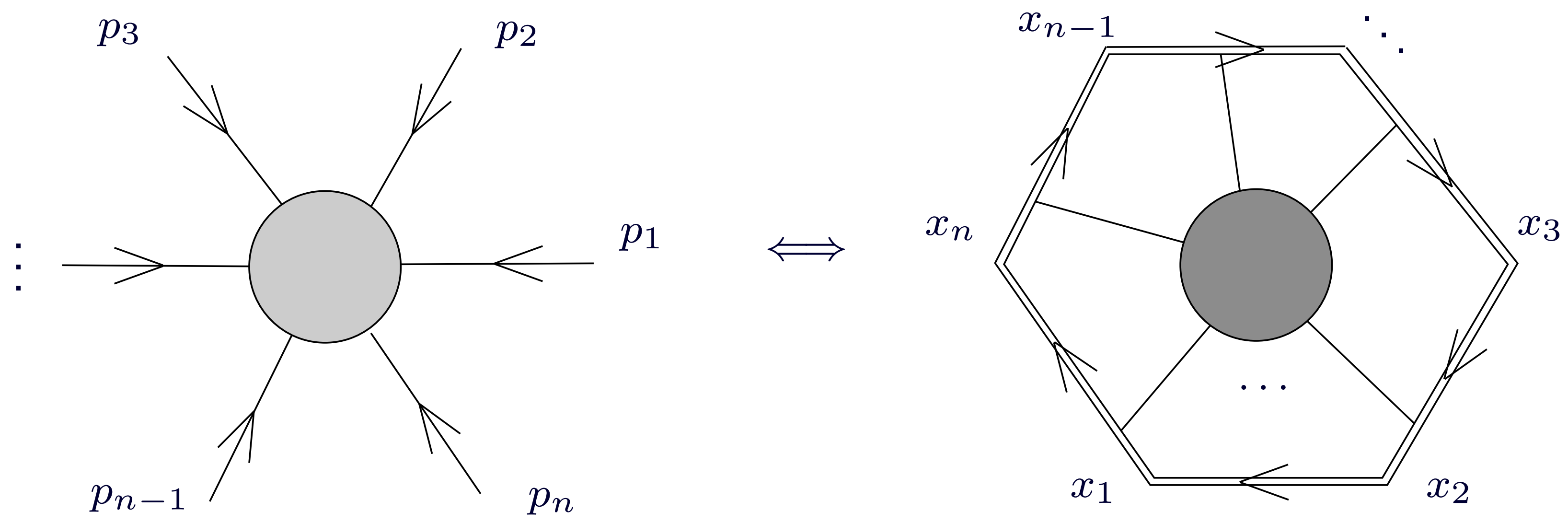
- ✔ Three independent anomalies are

$$s_{\alpha A} = \sum_i \frac{\partial^2}{\partial \lambda_i^\alpha \partial \eta_i^A}\,, \qquad \bar{Q}^A_{\dot\alpha} = \sum_i \eta_i^A \frac{\partial}{\partial \tilde{\lambda}_i^{\dot\alpha}}\,, \qquad K^{\alpha\dot\alpha} = \sum_i \Big[x_i^{\alpha\dot\beta} x_i^{\beta\dot\alpha} \frac{\partial}{\partial x_i^{\beta\dot\beta}} + x_i^{\beta\dot\alpha} \theta_i^{\alpha B} \frac{\partial}{\partial \theta_i^{\beta B}} + \ldots \Big]$$

- ✔ Dual conformal $K-$anomaly is *universal* for all superamplitudes (MHV, NMHV,...)

- ✔ $K-$anomaly can be determined to all loops from Wilson loop/MHV amplitude duality, whereas the $s-$ and $\bar{Q}-$anomalies are hard to control

MHV scattering amplitudes/Wilson loop duality



MHV amplitudes are dual to light-like Wilson loops

$$\ln \mathcal{A}_n^{(\mathrm{MHV})} \sim \ln W(C_n) + O(1/N_c^2)\,, \qquad C_n = \text{light-like } n\text{–(poly)gon}$$

✔ At strong coupling, agrees with the BDS ansatz [Alday,Maldacena]

✔ At weak coupling, the duality was verified against BDS ansatz to two loops for $n \geq 4$

[Drummond,Henn,GK,Sokatchev], [Anastasiou,Brandhuber,Heslop,Khoze,Spence,Travaglini]

Wilson loops match the BDS ansatz for $n = 4, 5$ but not for $n \geq 6$

✔ Scattering amplitude/Wilson loop duality also holds in QCD but in the Regge limit only [GK]

Dual conformal $K-$anomaly

Dual conformal symmetry of the amplitudes $\Leftrightarrow$ Conformal symmetry of Wilson loops

Dual conformal anomaly $\Leftrightarrow$ Conformal anomaly of Wilson loops

✔ How could Wilson loops have conformal anomaly in $\mathcal{N} = 4$ SYM?

✗ Were the Wilson loop well-defined (=finite) in $D = 4$ dimensions it would be conformal invariant

$$W(C_n) = W(C'_n)$$

✗ ... but $W(C_n)$ has cusp UV singularities $\mapsto$ dim.reg. breaks conformal invariance

$$W(C_n) = W(C'_n) \times \text{[cusp anomaly]}$$

✔ *All-loop* anomalous conformal Ward identities for the *finite part* of the Wilson loop

$$\ln W(C_n) = F_n^{(WL)} + \text{[UV divergencies]} + O(\epsilon)$$

Under special conformal transformations (boosts), to all orders, [Drummond,Henn,GK,Sokatchev]

$$K^\mu\, F_n \equiv \sum_{i=1}^{n} \left[2x_i^\mu (x_i \cdot \partial_{x_i}) - x_i^2 \partial_{x_i}^\mu\right] F_n = \frac{1}{2}\Gamma_{\text{cusp}}(a) \sum_{i=1}^{n} x^\mu_{i,i+1} \ln\left(\frac{x^2_{i,i+2}}{x^2_{i-1,i+1}}\right)$$

The same relations also hold at strong coupling [Alday,Maldacena],[Komargodski]

Dual conformal anomaly at work

Consequences of the conformal Ward identity for the finite part of the Wilson loop W_n:

✔ $n = 4, 5$ are special: there are no conformal invariants (too few distances due to $x^2_{i,i+1} = 0$)

$\Longrightarrow$ the Ward identity has a *unique all-loop solution* (up to an additive constant)

$$F_4 = \frac{1}{4}\Gamma_{\text{cusp}}(a) \ln^2\left(\frac{x^2_{13}}{x^2_{24}}\right) + \text{ const },$$

$$F_5 = -\frac{1}{8}\Gamma_{\text{cusp}}(a) \sum_{i=1}^{5} \ln\left(\frac{x^2_{i,i+2}}{x^2_{i,i+3}}\right) \ln\left(\frac{x^2_{i+1,i+3}}{x^2_{i+2,i+4}}\right) + \text{ const}$$

Exactly the BDS ansatz for the 4- and 5-point MHV amplitudes!

✔ Starting from $n = 6$ there are conformal invariants in the form of cross-ratios $u_{ijkl} = \frac{x^2_{il} x^2_{jk}}{x^2_{ik} x^2_{jl}}$

General solution to the Ward identity contains an arbitrary function of the conformal cross-ratios.

✔ Crucial test - go to six points at two loops where the answer is not determined by conformal symmetry [Drummond,Henn,GK,Sokatchev] [Bern,Dixon,Kosower,Roiban,Spradlin,Vergu,Volovich]

$$F_6^{(\text{WL})} = F_6^{(\text{MHV})} \neq F_6^{(\text{BDS})}$$

The Wilson loop/MHV amplitude duality holds at $n = 6$ to two loops!

Dual conformal symmetry beyond MHV

One-loop NMHV superamplitudes

- ✔ $n-$gluon one-loop NMHV amplitudes are known [Bern,Dixon,Kosower]
- ✔ New result for one-loop NMHV superamplitude: [Henn,Drummond,GK,Sokatchev]

$$\mathcal{A}_n^{\mathrm{NMHV};1} = \mathcal{A}_n^{\mathrm{MHV};1} \times \left[R_n^{\mathrm{NMHV}}(x_i, \lambda_i, \theta_i^A) + O(\epsilon) \right]$$

IR divergences break symmetries of NMHV and MHV but they cancel inside the 'ratio function'

One-loop NMHV 'ratio function' $=$ ***sum of tree-level dual superconformal invariants:***

$$R_n^{\mathrm{NMHV}} = \sum_{p,q,r} R_{pqr}(\lambda, \tilde{\lambda}, \theta)\, V_{pqr}(x_{ij}^2)$$

- ✗ Helicity structure is invariant under both (conventional and dual) superconformal symmetries
- ✗ Loop corrections are described by scalar functions

$$V_{pqr} = 1 + a\, V^{(1)}(\{u_{pqr}\}) + O(a^2)\,, \qquad u_{pqr} = \text{dual conformal cross-ratios}$$

  they are dual conformal invariants made of IR finite combinations of 1-loop scalar box integrals [Henn,Drummond,GK,Sokatchev],[Brandhuber,Heslop,Travaglini],[Elvang,Freedman,Kiermaier]

- ✗ R_n^{NMHV} is free from spurious singularities and has correct collinear limit

The ratio function is dual conformal invariant but it is not superconformal invariant, why ?

Dual supersymmetry $\overline{Q}-$anomaly

- Main idea: Instead of $\bar{Q}\mathcal{A}_n$ let us compute its discontinuity $\mathrm{Disc}_s\left(\bar{Q}\mathcal{A}_n\right) = \bar{Q}(\mathrm{Disc}_s\mathcal{A}_n)$

$$\mathrm{Disc}_{s_{123}}\mathcal{A}_6^{\mathrm{MHV};1} = \mathcal{A}^{\mathrm{MHV};0}(-\ell_1, 1, 2, 3, -\ell_2) \star \mathcal{A}^{\mathrm{MHV};0}(\ell_2, 4, 5, 6, \ell_1)\,,$$

Two tree amplitudes are integrated over the phase space of on-shell states ℓ_1 and ℓ_2

$$\begin{aligned}\delta_{\bar{Q}}\mathrm{Disc}_{s_{123}}\mathcal{A}_6^{\mathrm{MHV};1} &= \delta_{\bar{Q}}\mathcal{A}^{\mathrm{MHV};0}(-\ell_1, 1, 2, 3, -\ell_2) \star \mathcal{A}^{\mathrm{MHV};0}(\ell_2, 4, 5, 6, \ell_1) \\ &+ \mathcal{A}^{\mathrm{MHV};0}(-\ell_1, 1, 2, 3, -\ell_2) \star \delta_{\bar{Q}}\mathcal{A}^{\mathrm{MHV};0}(\ell_2, 4, 5, 6, \ell_1)\end{aligned}$$

- If tree supeamplitudes were exactly invariant, $\delta_{\bar{Q}}\mathcal{A}^{\mathrm{MHV};0} = 0$, then $\delta_{\bar{Q}}\mathrm{Disc}_{s_{123}}\mathcal{A}_6^{\mathrm{MHV};1} = 0$.
  But they are not due to holomorphic anomaly! [Cachazo,Svrcek,Witten],[Bena,Bern,Kosower,Roiban]

- $\bar{Q}-$anomaly of one-loop NMHV ratio function

$$\begin{aligned}\bar{Q}^A_{\dot\alpha}\left(\mathrm{Disc}_{x_{14}^2} R_6^{\mathrm{NMHV};1}\right) &\sim (\eta_1[23] + \eta_2[31] + \eta_3[12])^A R_{146} \\ &\times \left(\frac{\tilde\lambda_{1\dot\alpha}[6|x_{63}|3\rangle}{x_{14}^2[61][12]} + \frac{\tilde\lambda_{3\dot\alpha}[4|x_{41}|1\rangle}{x_{14}^2[23][34]}\right) + (i \to i+3) \neq 0\end{aligned}$$

Holomorphic anomaly is responsible for the breakdown of $\bar{Q} = \bar{s}$ symmetry of the ratio function (but not of the dual conformal symmetry!) [GK,Sokatchev]

Symmetry of all-loop superamplitudes

DHKS proposal for all-loop superamplitude in $\mathcal{N}=4$ SYM:

$$\mathcal{A}_n(x_i,\lambda_i,\theta_i^A) = \mathcal{A}_n^{\rm MHV} + \mathcal{A}_n^{\rm NMHV} + \mathcal{A}_n^{\rm N^2MHV} + \ldots + \mathcal{A}_n^{\overline{\rm MHV}}$$

- ✔ At tree level, $\mathcal{A}_n$ is fixed by conventional and dual symmetries + analyticity conditions
- ✔ At loop level, both symmetries become anomalous due to IR divergences + holomorphic anomaly
- ✔ The dual superconformal symmetry is restored in the ratio of superamplitudes $\mathcal{A}_n$ and $\mathcal{A}_n^{\rm MHV}$

$$\mathcal{A}_n(x_i,\lambda_i,\theta_i^A) = \mathcal{A}_n^{\rm MHV} \times \Big[R_n(x_i,\lambda_i,\theta_i^A) + O(\epsilon)\Big]$$

The ratio function

$$R_n = 1 + R_n^{\rm NMHV} + R_n^{\rm N^2MHV} + \ldots$$

is *IR finite* and, most importantly, it is *dual conformal invariant* [Drummond,Henn,GK,Sokatchev]

$$K^{\alpha\dot\alpha} R_n^{(\text{all loops})} = 0$$

The conjecture was recently proven to one loop [Brandhuber,Heslop,Travaglini]

Conclusions and open questions

- ✔ Tree amplitudes in $\mathcal{N}=4$ SYM respect conventional and dual superconformal symmetries but their combined action is not sufficient to fix the amplitudes. The additional information needed comes from analyticity properties of the amplitudes.

- ✔ At loop level, both symmetries are broken by IR divergences + holomorphic anomaly. The dual conformal anomaly is well understood but how to control the remaining anomalies?

  We need the dual model for $\mathcal{N}=4$ superamplitude:

  Dual model for the MHV amplitude = light-like Wilson loop

  Dual model for the MHV+NMHV+ $\ldots +$ $\overline{\text{MHV}}$ amplitude = ???

  $\bar{Q}-$anomaly indicates that the dual model does not respect Poincaré supersymmetry.

  How could it be?

- ✔ Weak/strong coupling paradox:

  At weak coupling, the $\bar{Q}-$anomaly is present to all loops $\bar{Q}R_n = af_1 + a^2 f_2 + \ldots$ but it is not seen at strong coupling !?

$$\mathcal{A}_n^{\mathrm{N^pMHV}} \sim \exp\left(-\sqrt{a}S_{\mathrm{min}}\right)\left[1+O(1/\sqrt{a})\right] \qquad \Rightarrow \qquad R_n^{\mathrm{N^pMHV}} \sim 1 + O(1/\sqrt{a})$$

  What is the meaning of holomorphic anomaly in string theory?