Integrability in Gauge & String Theory

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The AdS_4/CFT_3 worldsheet S-matrix

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Introduction

Exact S-matrices play a fundamental role in integrable 1+1 dim relativistic QFTs

- Can often be “guessed” from symmetries, Yang-Baxter equation, etc. [Zamolodchikov^2 ’79, ...]
- Can be used to compute other quantities of interest, such as:
  - finite-size effects [Zamolodchikov ’90, ...]
  - form factors & correlation functions [Karowski ’79, Smirnov ’90,...]

⋮

Also true in AdS_5/CFT_4 :

[Maldacena '97, ...]

"harmonic oscillator"

Although the all-loops Hamiltonian (dilatation operator) is not known, an all-loops S-matrix has been proposed

[Staudacher '04, Beisert '05, Janik '06, Beisert, Eden & Staudacher '06,...]

- Leads to all-loops asymptotic Bethe ansatz eqs (BAEs)

[Beisert & Staudacher '05, Beisert '05, Martins & Melo '07, ...]

- Leads to exact finite-size results

[Janik & Lukowski '07, Bajnok & Janik '08, ...]

- Correlation functions in free string theory on $AdS_5 \times S^5$?

Last summer's big breakthrough: AdS_4/CFT_3 [Aharony, Bergman, Jafferis & Maldacena '08]

"hydrogen atom" ?

An exact S-matrix is again expected to play a fundamental role

Goals of this talk:

- "guessed" S-matrix
- checks

Outline

1. Integrability in AdS_4/CFT_3
2. Symmetries & elementary excitations
3. S-matrix
4. Checks
   - All-loops asymptotic BAEs
   - Direct 2-loop test
5. Discussion

1. Integrability in AdS_4/CFT_3

AdS_4	CFT_3
type IIA string in $AdS_4 \times CP^3$	planar $\mathcal{N}=6$ Chern-Simons in 2+1 dims
classical sigma model is integrable	growing evidence: dilatation operator is an integrable quantum spin chain Hamiltonian
[Arutyunov & Frolov '08, Stefanski '08, Gromov & Vieira '08, Zarembo '09, Kalousios, Vergu & Volovich '09, ...]	[Minahan & Zarembo '08, Zwiebel '09, Minahan, Schulgin & Zarembo '09, Bak, Min & Rey '09, ...]
quantum integrable ?	

all-loop BAEs proposed [Gromov & Vieira '08]

$\mathcal{N}=6$ Chern-Simons theory

[ABJM '08, BKKS '08, ...]

$$S = \frac{k}{4\pi}\int d^3x\,\mathrm{tr}\Big[\epsilon^{\mu\nu\lambda}\big(A_\mu\partial_\nu A_\lambda + \frac{2}{3}A_\mu A_\nu A_\lambda$$

$$-\hat{A}_\mu\partial_\nu\hat{A}_\lambda - \frac{2}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\big)$$ gauge

$+D_\mu Y_A^\dagger D^\mu Y^A + Y^6$ terms — scalars

$+$ fermions $\Big]$

k integer

gauge symmetry: $SU(N)\times SU(N)$

A_μ $\quad\hat{A}_\mu$

R - symmetry: $SU(4)\supset SU(2)\times SU(2)$

$A_a\ (N,\bar{N};2) \qquad B_{\dot{a}}\ (\bar{N},N;2)$

$Y^A = (A_1, A_2, B_{\dot{1}}^\dagger, B_{\dot{2}}^\dagger) \quad (N,\bar{N};4) \qquad D_\mu Y = \partial_\mu Y + A_\mu Y - Y\hat{A}_\mu$

$Y_A^\dagger = (A_1^\dagger, A_2^\dagger, B_{\dot{1}}, B_{\dot{2}}) \quad (\bar{N},N;\bar{4}) \qquad D_\mu Y^\dagger = \partial_\mu Y^\dagger + \hat{A}_\mu Y^\dagger - Y^\dagger A_\mu$

Scale invariant: $\Delta_0(A_\mu) = \Delta_0(\hat{A}_\mu) = 1, \quad \Delta_0(Y) = 1/2$ 3 dims!

$\mathcal{N} = 6$ superconformal symmetry for $k > 2$

$$Osp(2,2|6) \supset SO(2,3) \times SO(6)$$

$$\uparrow$$

also isometry group of $AdS_4 \times CP^3$

CP symmetry, with $A_\mu \leftrightarrow \hat{A}_\mu\,, \quad A_a \leftrightarrow B_{\dot{a}}$

Planar limit: $N, k \to \infty, \quad \lambda \equiv N/k = \text{fixed}$

2-loop BAEs

[Minahan & Zarembo '08, ...]

- Scalar sector

Local, gauge-invariant, single-trace operators:

$$\mathrm{tr}\, Y^{A_1}(x) Y^\dagger_{B_1}(x) Y^{A_2}(x) Y^\dagger_{B_2}(x) \cdots Y^{A_L}(x) Y^\dagger_{B_L}(x)$$

$\longleftrightarrow$

states of closed $SU(4)$ quantum spin chain with $2L$ sites

$|A_1 B_1 A_2 B_2 \cdots A_L B_L\rangle$ alternating $4 \; \bar{4} \; \cdots$

2-loop dilatation operator (mixing matrix)

$$\Gamma = \lambda^2 H\,, \qquad H = \sum_{l=1}^{2L} \left(1 - \mathcal{P}_{l,l+2} + \frac{1}{2} \{ \mathcal{K}_{l,l+1}\,, \mathcal{P}_{l,l+2} \} \right) .$$

$\mathcal{P}_{ij}$ permutation matrix $\qquad \mathcal{K}_{ij} = \mathcal{P}_{ij}^{t_2}$ $\qquad$ Integrable!

Eigenvalues (anomalous dimensions) given by Bethe ansatz:

$$\gamma = \lambda^2 \left(\sum_{j=1}^{M_u} \frac{1}{u_j^2 + 1/4} + \sum_{j=1}^{M_v} \frac{1}{v_j^2 + 1/4} \right)$$

$\{u_j, v_j, r_j\}$ are solutions of BAEs:

$$e_1(u_j)^L = \prod_{\substack{k=1 \\ k\neq j}}^{M_u} e_2(u_j - u_k) \prod_{k=1}^{M_r} e_{-1}(u_j - r_k)$$

$$1 = \prod_{k=1}^{M_r} e_2(r_j - r_k) \prod_{k=1}^{M_u} e_{-1}(r_j - u_k) \prod_{k=1}^{M_v} e_{-1}(r_j - v_k)$$

$$e_1(v_j)^L = \prod_{\substack{k=1 \\ k\neq j}}^{M_u} e_2(v_j - v_k) \prod_{k=1}^{M_r} e_{-1}(v_j - r_k)$$

where

$$e_n(u) \equiv \frac{u + in/2}{u - in/2}$$

For integrable spin chain with simple Lie algebra symmetry,

$$e_{V_l}(u_j^{(l)})^{L_l} = \prod_{l'=1}^{rank(g)} \prod_{\substack{j'=1 \\ (l',j')\neq(l,j)}}^{M^{(l')}} e_{A_{l,l'}}(u_j^{(l)} - u_{j'}^{(l')})$$

[Ogievetsky & Wiegmann '86]

$A_{l,l'}$ Cartan matrix of Lie algebra g

V_l Dynkin labels of representation

$su(4)$:

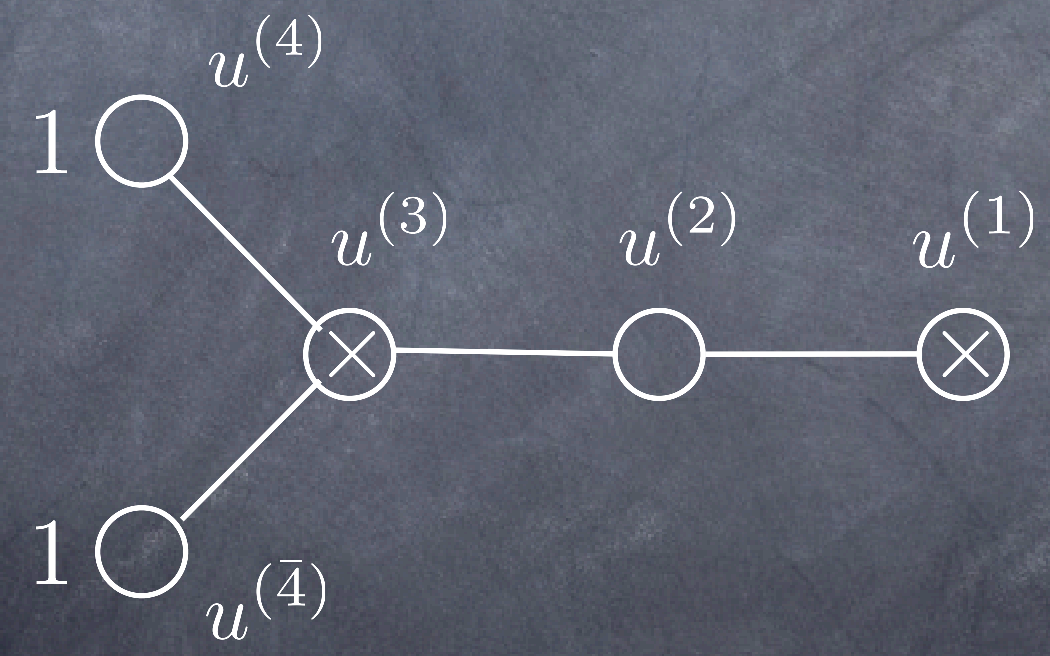
$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Reproduces above set of BAEs

- Full theory: “guess” from group theory

$$osp(2,2|6)$$

Dynkin diagram not unique. Two fermionic roots:



$\Rightarrow$ BAEs

$$
\begin{aligned}
e_1(u_j^{(4)})^L &= \prod_{\substack{k=1\\k\neq j}}^{M^{(4)}} e_2(u_j^{(4)} - u_k^{(4)}) \prod_{k=1}^{M^{(3)}} e_{-1}(u_j^{(4)} - u_k^{(3)}) \\
e_1(u_j^{(\bar 4)})^L &= \prod_{\substack{k=1\\k\neq j}}^{M^{(\bar 4)}} e_2(u_j^{(\bar 4)} - u_k^{(\bar 4)}) \prod_{k=1}^{M^{(3)}} e_{-1}(u_j^{(\bar 4)} - u_k^{(3)}) \\
1 &= \prod_{k=1}^{M^{(4)}} e_{-1}(u_j^{(3)} - u_k^{(4)}) \prod_{k=1}^{M^{(\bar 4)}} e_{-1}(u_j^{(3)} - u_k^{(\bar 4)}) \prod_{k=1}^{M^{(2)}} e_1(u_j^{(3)} - u_k^{(2)}) \\
1 &= \prod_{\substack{k=1\\k\neq j}}^{M^{(2)}} e_{-2}(u_j^{(2)} - u_k^{(2)}) \prod_{k=1}^{M^{(3)}} e_1(u_j^{(2)} - u_k^{(3)}) \prod_{k=1}^{M^{(1)}} e_1(u_j^{(2)} - u_k^{(1)}) \\
1 &= \prod_{k=1}^{M^{(2)}} e_1(u_j^{(1)} - u_k^{(2)})
\end{aligned}
$$

[Minahan, Schulgin & Zarembo '09]

- 4 loops [Bak, Min & Rey '09, ...]
- Assume all-loop integrability

2. Symmetries
&
elementary excitations

Global symmetry is partially broken by vacuum!

Analogy: Heisenberg ferromagnet ($XXX_{1/2}$ chain)

Vacuum: $|\uparrow \cdots \uparrow\rangle$

Breaks $SU(2) \to U(1)$

Elementary excitations:

$$\sum_n e^{ipn} |\uparrow \cdots \uparrow \overset{n}{\downarrow} \uparrow \cdots \uparrow\rangle$$ "magnons"

$$E = 4 \sin^2 \frac{p}{2}$$

Classified by unbroken $U(1)$ symmetry

$\mathcal{N}=6$ Chern-Simons:

[Nishioka & Takayanagi '08,
Gaiotto, Giombi & Yin '08,
Grignani, Harmark & Orselli '08, ...]

Vacuum: $\mathrm{tr}\,(A_1 B_{\dot{1}} A_1 B_{\dot{1}} \cdots A_1 B_{\dot{1}})$

$$\gamma = 0$$

(True to all orders in λ , since is chiral primary operator.)

$$\Delta = L = J \quad \Rightarrow \quad \Delta - J = 0$$

$$\begin{aligned} J(A_1) = J(B_{\dot{1}}) &= 1/2 \\ J(A_2) = J(B_{\dot{2}}) &= 0 \end{aligned}$$

CP invariant

Breaks $SU(4) \rightarrow SU(2)$ (rotates $A_2, B_{\dot{2}}^{\dagger}$)

$Osp(2,2|6) \rightarrow SU(2|2)$

Elementary excitations:

$$\sum_n e^{ipn} | \overset{\overset{1}{\downarrow}}{(A_1 B_{\dot{1}})} \cdots \overset{\overset{n}{\downarrow}}{(\chi B_{\dot{1}})} \cdots \overset{\overset{L}{\downarrow}}{(A_1 B_{\dot{1}})} \rangle$$

"A - particles"

$$\chi \in \{A_2, B^\dagger_{\dot{2}}, \text{fermions}\}$$

CP $\updownarrow$

$$\sum_n e^{ipn} | \overset{\overset{1}{\downarrow}}{(A_1 B_{\dot{1}})} \cdots \overset{\overset{n}{\downarrow}}{(A_1 \chi)} \cdots \overset{\overset{L}{\downarrow}}{(A_1 B_{\dot{1}})} \rangle$$

"B - particles"

$$\chi \in \{B_{\dot{2}}, A^\dagger_2, \text{fermions}\}$$

$$\Delta_0 - J = \frac{1}{2}$$

Fundamental reps $(2|2)$ of $SU(2|2)$

Elementary excitations:

ZF operators:

$$\sum_n e^{ipn} |\, (\overset{1}{\downarrow}{A_1 B_{\dot 1}}) \cdots (\overset{n}{\downarrow}{\chi B_{\dot 1}}) \cdots (\overset{L}{\downarrow}{A_1 B_{\dot 1}}) \rangle$$

"A - particles" $A_i^\dagger(p)$

$$\chi \in \{A_2, B_{\dot 2}^\dagger, \text{fermions}\}$$

CP ↕

$$\sum_n e^{ipn} |\, (\overset{1}{\downarrow}{A_1 B_{\dot 1}}) \cdots (\overset{n}{\downarrow}{A_1 \chi}) \cdots (\overset{L}{\downarrow}{A_1 B_{\dot 1}}) \rangle$$

"B - particles" $B_i^\dagger(p)$

$$\chi \in \{B_{\dot 2}, A_2^\dagger, \text{fermions}\}$$

$i = 1, \ldots, 4$

Acting on $|0\rangle$ create asymptotic particle states of momentum p $(L \to \infty)$

$$\Delta_0 - J = \frac{1}{2}$$

Fundamental reps $(2|2)$ of $SU(2|2)$

One-particle states form representation of centrally-extended $SU(2|2)$ algebra

[Beisert '05, Arutyunov, Frolov & Zamaklar '06]

$\Rightarrow$

$$\Delta - J = \sqrt{\frac{1}{4} + 4g^2 \sin^2 \frac{p}{2}}$$

$$\begin{aligned} g = h(\lambda) \quad &\sim \quad \lambda && \text{for } \lambda \text{ small} \\ &\sim \quad \sqrt{\lambda/2} && \text{for } \lambda \text{ large} \end{aligned}$$

Determining $h(\lambda)$ remains open problem

3. S-matrix

- A-A scattering:

$$A_i^\dagger(p_1)\,A_j^\dagger(p_2) = S^{AA}{}_{i\,j}^{\,i'j'}(p_1,p_2)\,A_{j'}^\dagger(p_2)\,A_{i'}^\dagger(p_1)$$

Associativity $A_i^\dagger(p_1)A_j^\dagger(p_2)A_k^\dagger(p_3)$ $\Rightarrow$ Yang-Baxter equation

$$S_{12}^{AA}(p_1,p_2)\,S_{13}^{AA}(p_1,p_3)\,S_{23}^{AA}(p_2,p_3) = S_{23}^{AA}(p_2,p_3)\,S_{13}^{AA}(p_1,p_3)\,S_{12}^{AA}(p_1,p_2)$$

$SU(2|2)$ symmetry determines S^{AA} up to scalar factor

[Beisert '05, AFZ '06]

$$S^{AA}(p_1,p_2) = S_0(p_1,p_2)\,\widehat{S}(p_1,p_2)$$

↑ scalar ↑ matrix

Besides YBE, $\widehat{S}(p_1, p_2)$ satisfies unitarity:

$$\widehat{S}_{12}(p_1, p_2)\, \widehat{S}_{21}(p_2, p_1) = \mathbb{I}$$

and crossing: [Janik '06]

$$\widehat{S}_{12}^{t_2}(p_1, p_2)\, C_2\, \widehat{S}_{12}(p_1, \bar{p}_2)\, C_2^{-1} = \widehat{S}_{12}^{t_1}(p_1, p_2)\, C_1\, \widehat{S}_{12}(\bar{p}_1, p_2)\, C_1^{-1} = f(p_1, p_2)\, \mathbb{I}$$

$$f(p_1, p_2) = \frac{\left(\frac{1}{x_1^+} - x_2^-\right)(x_1^+ - x_2^+)}{\left(\frac{1}{x_1^-} - x_2^-\right)(x_1^- - x_2^+)}$$

$$x^\pm(\bar{p}) = \frac{1}{x^\pm(p)}$$

$$\frac{x^+}{x^-} = e^{ip} \qquad\qquad x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g}$$

- B-B scattering:

$$B_i^\dagger(p_1)\, B_j^\dagger(p_2) = S^{BB\;\;i'j'}_{\;\;\;\;\;\;\;ij}(p_1,p_2)\, B_{j'}^\dagger(p_2)\, B_{i'}^\dagger(p_1)$$

- A-B and B-A scattering:

$$A_i^\dagger(p_1)\, B_j^\dagger(p_2) = S^{AB\;\;i'j'}_{\;\;\;\;\;\;\;ij}(p_1,p_2)\, B_{j'}^\dagger(p_2)\, A_{i'}^\dagger(p_1)$$

N.B. reflectionless !

- B-B scattering:

$$B_i^\dagger(p_1)\, B_j^\dagger(p_2) = S^{BB}{}_{i\,j}^{\,i'j'}(p_1, p_2)\, B_{j'}^\dagger(p_2)\, B_{i'}^\dagger(p_1)$$

- A-B and B-A scattering:

$$A_i^\dagger(p_1)\, B_j^\dagger(p_2) = S^{AB}{}_{i\,j}^{\,i'j'}(p_1, p_2)\, B_{j'}^\dagger(p_2)\, A_{i'}^\dagger(p_1)$$

N.B. reflectionless !

Symmetry suggests

$$\begin{aligned} S^{BB}(p_1,p_2) &= S^{AA}(p_1,p_2) = S_0(p_1,p_2)\,\widehat{S}(p_1,p_2) \\ S^{AB}(p_1,p_2) &= S^{BA}(p_1,p_2) = \tilde{S}_0(p_1,p_2)\,\widehat{S}(p_1,p_2) \end{aligned}$$

"matrix part" fixed - remains only to determine scalar factors $S_0,\ \tilde{S}_0$

Assume unitarity:

$$S_{12}^{AA}(p_1,p_2)\,S_{21}^{AA}(p_2,p_1) = S_{12}^{AB}(p_1,p_2)\,S_{21}^{AB}(p_2,p_1) = \mathbb{I}$$

$\Rightarrow$

$$S_0(p_1,p_2)\,S_0(p_2,p_1) = 1\,, \qquad \tilde{S}_0(p_1,p_2)\,\tilde{S}_0(p_2,p_1) = 1$$

Assume crossing:

$$S_{12}^{AA\ t_2}(p_1,p_2)\,C_2\,S_{12}^{AB}(p_1,\bar{p}_2)\,C_2^{-1} = S_{12}^{AA\ t_1}(p_1,p_2)\,C_1\,S_{12}^{AB}(\bar{p}_1,p_2)\,C_1^{-1} = \mathbb{I}$$

$\Rightarrow$

$$S_0(p_1,p_2)\,\tilde{S}_0(p_1,\bar{p}_2) = S_0(p_1,p_2)\,\tilde{S}_0(\bar{p}_1,p_2) = \frac{1}{f(p_1,p_2)}$$

Satisfied by

$$S_0(p_1\,,p_2) = \frac{1-\frac{1}{x_1^+x_2^-}}{1-\frac{1}{x_1^-x_2^+}}\sigma(p_1\,,p_2)$$

$$\tilde{S}_0(p_1\,,p_2) = \frac{x_1^- - x_2^+}{x_1^+ - x_2^-}\sigma(p_1\,,p_2)$$

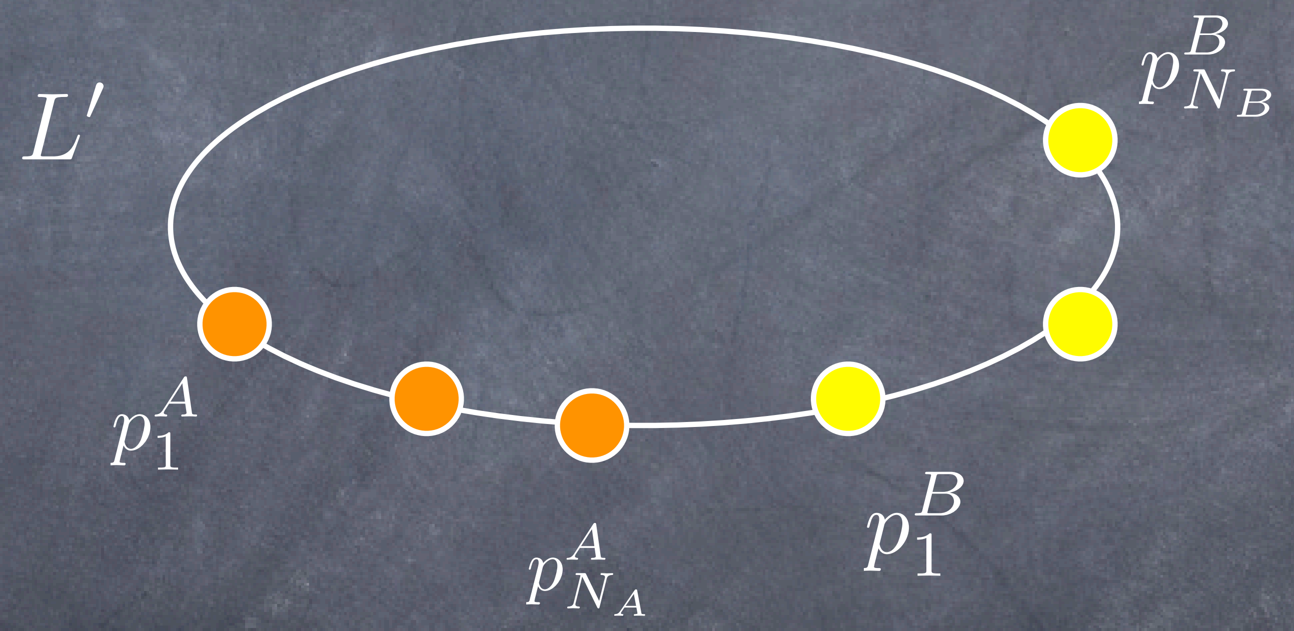
$\sigma(p_1\,,p_2)$ dressing factor

[Beisert, Eden & Staudacher '06]

4. Checks

All-loop asymptotic BAEs

Consider a set of A - particles $\{p_1^A, \ldots, p_{N_A}^A\}$ and a set of B - particles $\{p_1^B, \ldots, p_{N_B}^B\}$ that are widely separated on a ring of length L'



Periodic boundary conditions on wavefunction

$\Rightarrow$ quantization conditions for momenta

"Bethe-Yang"

A - particles:

$$e^{-ip_k^A L'} = \Lambda(\lambda = p_k^A, \{p_i^A, p_i^B\})$$

$\Lambda(\lambda, \{p_i^A, p_i^B\})$ eigenvalues of inhomogeneous transfer matrix

$$t(\lambda, \{p_i^A, p_i^B\}) = \mathrm{str}_a\, S^{AA}_{a\,1}(\lambda, p_1^A) \cdots S^{AA}_{a\,N_A}(\lambda, p_{N_A}^A)\, S^{AB}_{a\,N_A+1}(\lambda, p_1^B) \cdots S^{AB}_{a\,N_A+N_B}(\lambda, p_{N_B}^B)$$

$$= (\text{ scalar factors })(\text{ “matrix part” })$$

Eigenvalues of ``matrix part”:

[Beisert '06, Martins & Melo '07]

$$\widehat{\Lambda}(\lambda, \{p_i^A, p_i^B\}; \{\lambda_j, \mu_j\}) = \prod_{i=1}^{N_A}\left[\frac{x^+(\lambda) - x^-(p_i^A)}{x^-(\lambda) - x^+(p_i^A)}\frac{\eta(p_i^A)}{\eta(\lambda)}\right]\prod_{i=1}^{N_B}\left[\frac{x^+(\lambda) - x^-(p_i^B)}{x^-(\lambda) - x^+(p_i^B)}\frac{\eta(p_i^B)}{\eta(\lambda)}\right]$$

$$\times \prod_{j=1}^{m_1}\left[\eta(\lambda)\frac{x^-(\lambda) - x^+(\lambda_j)}{x^+(\lambda) - x^+(\lambda_j)}\right] + \text{ terms which vanish if } \lambda = p_k^A$$

$$\eta(\lambda) \equiv e^{i\lambda/2}$$

where $\{\lambda_j, \mu_j\}$ are solutions of BAEs

$$e^{i(P^A+P^B)/2} \prod_{i=1}^{N_A} \frac{x^+(\lambda_j) - x^-(p_i^A)}{x^+(\lambda_j) - x^+(p_i^A)} \prod_{i=1}^{N_B} \frac{x^+(\lambda_j) - x^-(p_i^B)}{x^+(\lambda_j) - x^+(p_i^B)} = \prod_{l=1}^{m_2} \frac{x^+(\lambda_j) + \frac{1}{x^+(\lambda_j)} - \tilde{\mu}_l + \frac{i}{2g}}{x^+(\lambda_j) + \frac{1}{x^+(\lambda_j)} - \tilde{\mu}_l - \frac{i}{2g}}$$

$$\prod_{j=1}^{m_1} \frac{\tilde{\mu}_l - x^+(\lambda_j) - \frac{1}{x^+(\lambda_j)} + \frac{i}{2g}}{\tilde{\mu}_l - x^+(\lambda_j) - \frac{1}{x^+(\lambda_j)} - \frac{i}{2g}} = \prod_{\substack{k=1\\k\neq l}}^{m_2} \frac{\tilde{\mu}_l - \tilde{\mu}_k + \frac{i}{g}}{\tilde{\mu}_l - \tilde{\mu}_k - \frac{i}{g}} \qquad (1)$$

Taking into account also scalar factors, Bethe-Yang eqs. for A-particles become:

$$e^{ip_k^A L} = \prod_{\substack{i=1\\i\neq k}}^{N_A} \left[\frac{x^+(p_k^A) - x^-(p_i^A)}{x^-(p_k^A) - x^+(p_i^A)}\right] \left[\frac{1 - \frac{1}{x^+(p_k^A)x^-(p_i^A)}}{1 - \frac{1}{x^-(p_k^A)x^+(p_i^A)}} \sigma(p_k^A, p_i^A)\right]$$

$$\times \prod_{i=1}^{N_B} \sigma(p_k^A, p_i^B) \prod_{j=1}^{m_1} \left[\frac{x^-(p_k^A) - x^+(\lambda_j)}{x^+(p_k^A) - x^+(\lambda_j)}\right] \qquad (2)$$

Bethe-Yang eqs. for B-particles:

$$e^{ip_k^B L} = \prod_{\substack{i=1 \\ i\neq k}}^{N_B} \left[\frac{x^+(p_k^B) - x^-(p_i^B)}{x^-(p_k^B) - x^+(p_i^B)}\right] \left[\frac{1 - \frac{1}{x^+(p_k^B)x^-(p_i^B)}}{1 - \frac{1}{x^-(p_k^B)x^+(p_i^B)}} \sigma(p_k^B, p_i^B)\right]$$

$$\times \prod_{i=1}^{N_A} \sigma(p_k^B, p_i^A) \prod_{j=1}^{m_1} \left[\frac{x^-(p_k^B) - x^+(\lambda_j)}{x^+(p_k^B) - x^+(\lambda_j)}\right] \tag{3}$$

Can map (1)-(3) to all-loop BAEs:

$$\begin{aligned}
x^\pm(p_k^A) &= x^\pm_{4,k}\,, \quad k = 1, \ldots, K_4 \equiv N_A\,, \\
x^\pm(p_k^B) &= x^\pm_{\bar{4},k}\,, \quad k = 1, \ldots, K_{\bar{4}} \equiv N_B\,, \\
x^+(\lambda_j) &= \frac{1}{x_{1,j}}\,, \quad j = 1, \ldots, K_1\,, \\
x^+(\lambda_{K_1+j}) &= x_{3,j}\,, \quad j = 1, \ldots, K_3\,, \quad K_1 + K_3 \equiv m_1\,, \\
\tilde{\mu}_j &= \frac{u_{2,j}}{g}\,, \quad j = 1, \ldots, K_2 \equiv m_2
\end{aligned}$$

$$e^{ip_{4,k}L} = \prod_{\substack{j=1\\ j\neq k}}^{K_4} \frac{u_{4,k}-u_{4,j}+i}{u_{4,k}-u_{4,j}-i}\,\sigma(u_{4,k},u_{4,j}) \prod_{j=1}^{K_{\bar{4}}} \sigma(u_{4,k},u_{\bar{4},j})$$

$$\times \prod_{j=1}^{K_1} \frac{1-\frac{1}{x^-_{4,k}x_{1,j}}}{1-\frac{1}{x^+_{4,k}x_{1,j}}} \prod_{j=1}^{K_3} \frac{x^-_{4,k}-x_{3,j}}{x^+_{4,k}-x_{3,j}}$$

$$e^{ip_{\bar{4},k}L} = \prod_{\substack{j=1\\ j\neq k}}^{K_{\bar{4}}} \frac{u_{\bar{4},k}-u_{\bar{4},j}+i}{u_{\bar{4},k}-u_{\bar{4},j}-i}\,\sigma(u_{\bar{4},k},u_{\bar{4},j}) \prod_{j=1}^{K_4} \sigma(u_{\bar{4},k},u_{4,j})$$

$$\times \prod_{j=1}^{K_1} \frac{1-\frac{1}{x^-_{\bar{4},k}x_{1,j}}}{1-\frac{1}{x^+_{\bar{4},k}x_{1,j}}} \prod_{j=1}^{K_3} \frac{x^-_{\bar{4},k}-x_{3,j}}{x^+_{\bar{4},k}-x_{3,j}}$$

$$\prod_{i=1}^{K_4} \frac{1-\frac{1}{x_{1,j}x^-_{4,i}}}{1-\frac{1}{x_{1,j}x^+_{4,i}}} \prod_{i=1}^{K_{\bar{4}}} \frac{1-\frac{1}{x_{1,j}x^-_{\bar{4},i}}}{1-\frac{1}{x_{1,j}x^+_{\bar{4},i}}} = \prod_{l=1}^{K_2} \frac{u_{1,j}-u_{2,l}+\frac{i}{2}}{u_{1,j}-u_{2,l}-\frac{i}{2}}$$

$$\prod_{i=1}^{K_4} \frac{x_{3,j}-x^-_{4,i}}{x_{3,j}-x^+_{4,i}} \prod_{i=1}^{K_{\bar{4}}} \frac{1-\frac{1}{x_{3,j}x^-_{\bar{4},i}}}{1-\frac{1}{x_{3,j}x^+_{\bar{4},i}}} = \prod_{l=1}^{K_2} \frac{u_{3,j}-u_{2,l}+\frac{i}{2}}{u_{3,j}-u_{2,l}-\frac{i}{2}}$$

$$\prod_{\substack{j=1\\ j\neq l}}^{K_2} \frac{u_{2,l}-u_{2,j}+i}{u_{2,l}-u_{2,j}-i} = \prod_{j=1}^{K_1} \frac{u_{2,l}-u_{1,j}+\frac{i}{2}}{u_{2,l}-u_{1,j}-\frac{i}{2}} \prod_{j=1}^{K_3} \frac{u_{2,l}-u_{3,j}+\frac{i}{2}}{u_{2,l}-u_{3,j}-\frac{i}{2}}$$

Same as ✓

[Gromov & Vieira '08]

$$e^{ip_{4,k}L} = \prod_{\substack{j=1\\ j\neq k}}^{K_4} \frac{u_{4,k}-u_{4,j}+i}{u_{4,k}-u_{4,j}-i}\,\sigma(u_{4,k},u_{4,j}) \prod_{j=1}^{K_{\bar{4}}} \sigma(u_{4,k},u_{\bar{4},j})$$

$$\times \prod_{j=1}^{K_1} \frac{1-\frac{1}{x_{4,k}^- x_{1,j}}}{1-\frac{1}{x_{4,k}^+ x_{1,j}}} \prod_{j=1}^{K_3} \frac{x_{4,k}^- - x_{3,j}}{x_{4,k}^+ - x_{3,j}}$$

$$e^{ip_{\bar{4},k}L} = \prod_{\substack{j=1\\ j\neq k}}^{K_{\bar{4}}} \frac{u_{\bar{4},k}-u_{\bar{4},j}+i}{u_{\bar{4},k}-u_{\bar{4},j}-i}\,\sigma(u_{\bar{4},k},u_{\bar{4},j}) \prod_{j=1}^{K_4} \sigma(u_{\bar{4},k},u_{4,j})$$

$$\times \prod_{j=1}^{K_1} \frac{1-\frac{1}{x_{\bar{4},k}^- x_{1,j}}}{1-\frac{1}{x_{\bar{4},k}^+ x_{1,j}}} \prod_{j=1}^{K_3} \frac{x_{\bar{4},k}^- - x_{3,j}}{x_{\bar{4},k}^+ - x_{3,j}}$$

$$\prod_{i=1}^{K_4} \frac{1-\frac{1}{x_{1,j} x_{4,i}^-}}{1-\frac{1}{x_{1,j} x_{4,i}^+}} \prod_{i=1}^{K_{\bar{4}}} \frac{1-\frac{1}{x_{1,j} x_{\bar{4},i}^-}}{1-\frac{1}{x_{1,j} x_{\bar{4},i}^+}} = \prod_{l=1}^{K_2} \frac{u_{1,j}-u_{2,l}+\frac{i}{2}}{u_{1,j}-u_{2,l}-\frac{i}{2}}$$

$$\prod_{i=1}^{K_4} \frac{x_{3,j}-x_{4,i}^-}{x_{3,j}-x_{4,i}^+} \prod_{i=1}^{K_{\bar{4}}} \frac{1-\frac{1}{x_{3,j} x_{\bar{4},i}^-}}{1-\frac{1}{x_{3,j} x_{\bar{4},i}^+}} = \prod_{l=1}^{K_2} \frac{u_{3,j}-u_{2,l}+\frac{i}{2}}{u_{3,j}-u_{2,l}-\frac{i}{2}}$$

$$\prod_{\substack{j=1\\ j\neq l}}^{K_2} \frac{u_{2,l}-u_{2,j}+i}{u_{2,l}-u_{2,j}-i} = \prod_{j=1}^{K_1} \frac{u_{2,l}-u_{1,j}+\frac{i}{2}}{u_{2,l}-u_{1,j}-\frac{i}{2}} \prod_{j=1}^{K_3} \frac{u_{2,l}-u_{3,j}+\frac{i}{2}}{u_{2,l}-u_{3,j}-\frac{i}{2}}$$

Same as ✓

[Gromov & Vieira '08]

Weak-coupling limit:

$$g \to 0\,, \quad x \to \frac{u}{g}$$

$$x^{\pm} \to \frac{1}{g}(u \pm i/2)$$

Recover 2-loop BAEs ✓

Direct 2-loop test

Compute two-particle S-matrix from definition; i.e., solve

$$H|\psi\rangle = E|\psi\rangle$$

H 2-loop scalar-sector Hamiltonian

$|\psi\rangle$ all possible two-particle eigenstates

Simplest example: two "A" particles of same type

$$|\psi\rangle = \sum_{x_1<x_2} \left[e^{i(p_1x_1+p_2x_2)} + S(p_2\,,p_1)\, e^{i(p_2x_1+p_1x_2)} \right] |\, (A_1 \overset{\overset{1}{\downarrow}}{B}_{\mathrm{i}}) \cdots (\chi \overset{\overset{x_1}{\downarrow}}{B}_{\mathrm{i}}) \cdots (\chi \overset{\overset{x_2}{\downarrow}}{B}_{\mathrm{i}}) \cdots (A_1 \overset{\overset{L}{\downarrow}}{B}_{\mathrm{i}})\rangle$$

$$\chi \in \{A_2\,, B_2^\dagger\}$$

$$S(p_2\,,p_1) = \frac{u_2 - u_1 + i}{u_2 - u_1 - i}\,, \quad u_j = u(p_j) = \frac{1}{2}\cot(p_j/2)$$

[Bethe '31]

Hardest: one "A" and one "B" of type

$$|(A_1 B_{\dot{1}}) \cdots (\chi B_{\dot{1}}) \cdots (A_1 \chi^\dagger) \cdots (A_1 B_{\dot{1}})\rangle \qquad \chi \in \{A_2\,, B^\dagger_{\dot{2}}\}$$

since mixes with

$$|(A_1 B_{\dot{1}}) \cdots (A_k A^\dagger_k) \cdots (A_1 B_{\dot{1}})\rangle$$

and $\qquad\qquad k = 1, 2$

$$|(A_1 B_{\dot{1}}) \cdots (B^\dagger_{\dot{k}} B_{\dot{k}}) \cdots (A_1 B_{\dot{1}})\rangle$$

Result agrees with weak-coupling limit of proposed S-matrix ! ✓

In particular, "A"-"B" scattering is reflectionless

5. Discussion

Alternative S-matrix with reflection?

$A^{\dagger}_{a\,i}(p)$ a=1,2 flavor a=1: "A", a=2: "B"

i=1,...,4 SU(2|2)

Consider S-matrix with tensor product structure:

$$A^{\dagger}_{a\,i}(p_1)\,A^{\dagger}_{b\,j}(p_2) = S_0(p_1,p_2)\,S_{a\,b}^{a'\,b'}(p_1,p_2)\,\widehat{S}_{i\,j}^{i'\,j'}(p_1,p_2)\,A^{\dagger}_{b'\,j'}(p_2)\,A^{\dagger}_{a'\,i'}(p_1)$$

↑ flavor (6- or 8-vertex)

↑ SU(2|2)

Factorizable and admits "A"-"B" reflection,
but does NOT lead to correct BAEs

Origin of reflectionless property

Occurs in other integrable QFTs, e.g. [P. Dorey]

thermal perturbation of 3-state Potts model (A_2 affine Toda field theory) [Köberle & Swieca '79, Zamolodchikov '88]

Spectrum: s, $\bar{s}$ (same mass)

s - $\bar{s}$ scattering is reflectionless due to existence of higher local integral of motion, which acts differently on s, $\bar{s}$

Perhaps similar mechanism is at work in AdS_4/CFT_3 ?

Related further developments

- String-theory (strong-coupling) computation of S-matrix

  [Zarembo '09,
  Kalousios, Vergu & Volovich '09]

- Finite-size corrections

  [Bombardelli & Fioravanti '08, Lukowski & Sax '08,
  Ahn & Bozhilov '08, Gromov, Kazakov & Vieira '09, ...]

⋮