CONFORMAL SIGMA MODELS ON SUPER TARGETS AND LATTICE MODELS

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Based on works with C. Candu, J. Jacobsen, T. Quella, V. Mitev, N. Read, V. Schomerus

Started with N. Read, HS, "Exact spectra of conformal supersymmetric nonlinear sigma models in two dimensions", hep-th/0106124

...

C. Candu, T. Quella, V. Mitev, HS and V. Schomerus, "The principal chiral field on projective superspace", to appear.

BACKGROUND

• Target space supersymmetry (dating back to Parisi Sourlas) appears in several important problems of condensed matter. Some these are (roughly, in order of difficulty):

⋆ Polymers, percolation, trees

⋆ Random bond Ising model

⋆ Spin quantum Hall effect

⋆ Ordinary quantum Hall effect

• Some of the things we we would like to know (roughly, in order of difficulty)

⋆ The possible types of CFTs

⋆ The associated critical exponents

⋆ Some information about the OPEs (logarithms?), the four point functions (probabilistic interpretations)

⋆ Stability of fixed points and RG flows

• So the archetype of the problem we are facing is:

which CFT does the $U(1,1|2)/U(1|1) \times U(1|1)$ sigma model at $\theta = \pi$ flow to?

• Why is this so difficult?

⋆ Strong non unitarity issues: logarithmic CFTs with indecomposable representations of chiral algebras

⋆ and probably left/right indecomposability

⋆ Continuous group symmetry does not imply Kac Moody symmetry any longer

⋆ Strong non unitarity issues: probabilities $p < 0$ or $p > 1$ appear in S matrix approaches

• More fundamentally maybe: the space of CFTs is very difficult to map out, and integrable cases seem less generic than for non super targets.

• Lattice models have been particularly useful in understanding:

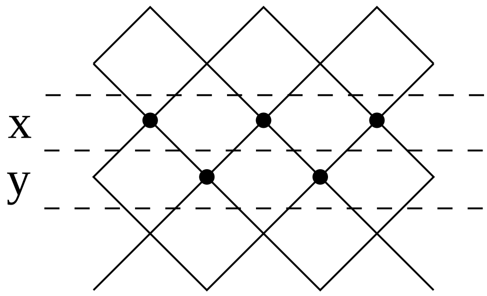
⋆ Algebraic features

⋆ the space of CFTs

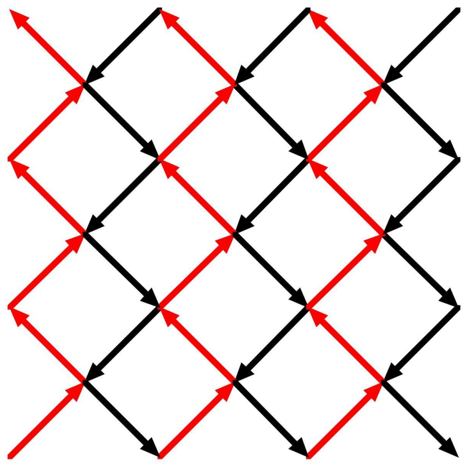
THE LATTICE MODELS: GENERALITIES

• In Euclidian space, the models carry representations of $SU(m+2n|2n)$ or $OSp(m+2n|2n)$ on every edge. Simplest cases: alternate $\Box$ and $\bar{\Box}$ for SU, take fundamental everywhere for OSp.

Think of transfer matrices propagating vertically



• The choice of alternating representations in the SU case means that the edges carry a fixed orientation (see Chalker Coddington model for IQHE transition):



In the hamiltonian limit we get H acting on $\left(\Box \otimes \bar{\Box}\right)^{\otimes 2L}$

• In the ordinary (non super) case, the integrable chain based on pure $\Box$ and the alternating one all flow to the SU WZW theory. In the super case, only the alternating chain has a conformal invariant limit, which is again a WZW theory. In the integrable case, $\Box$ and $\bar{\Box}$ can be passed through each other, and the geometry is not very meaningful.

• While interesting, the WZW models are not what we are after. So our models will not be given by the usual solutions of the Yang Baxter equation

ALGEBRA

• We start with $SU(m)$. We take $2L$ sites labelled $i = 0$, . . . , $2L - 1$. With odd sites associate the $\square$ of $SU(m)$ and with even sites $\bar{\square}$. Use a bosonic representation with vector space $V_i \cong \mathbf{C}^m$ at each site.

Represent states using b_i^a, $b_{ia}^\dagger$ for i even, $\bar{b}_{ia}$, $\bar{b}_i^{a\dagger}$ for i odd, with $[b_i^a, b_{jb}^\dagger] = \delta_{ij}\delta_b^a$ (a, $b = 1$, . . . , m), and similarly for i odd. The spaces V_i are defined by the constraints

$$
\begin{aligned}
b_{ia}^\dagger b_i^a &= 1 \quad (i \text{ even}), \\
\bar{b}_i^{a\dagger}\bar{b}_{ia} &= 1 \quad (i \text{ odd})
\end{aligned}
$$

of one boson per site. Generators of U(m) (or in fact of gl$_m$) acting in the spaces V_i are $J_{ia}^b = b_{ia}^\dagger b_i^b$ for i even, $J_{ia}^b = -\bar{b}_i^{b\dagger}\bar{b}_{ia}$ for i odd. The global gl$_m$ algebra $J_a^b = \sum_i J_{ia}^b$, acts in the tensor product $V = \left(\square \otimes \bar{\square}\right)^L$.

• $SU(m)$-invariant nearest-neighbor coupling in the chain is unique, up to additive and multiplicative constants: usual "Heisenberg coupling" of magnetism, can be written in terms of operators E_i

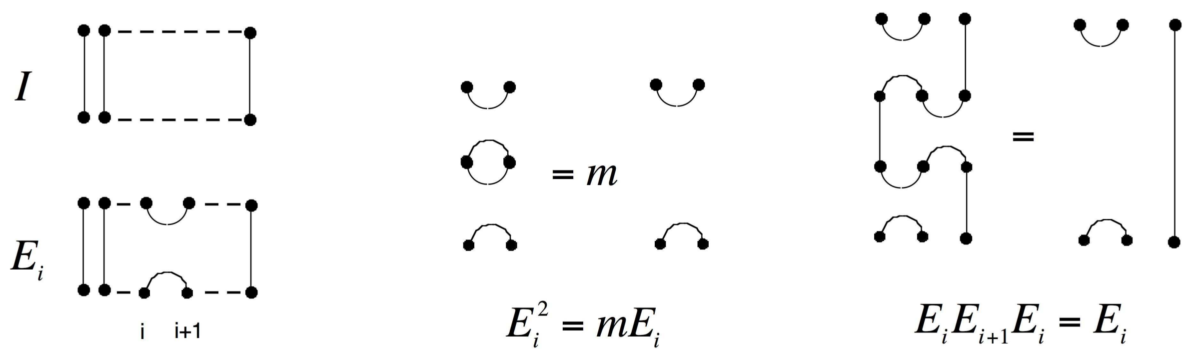
$$E_i = \begin{cases} \bar{b}^{a\dagger}_{i+1} b^\dagger_{ia} b^b_i \bar{b}_{i+1,b}, & i \text{ even}, \\ \bar{b}^{a\dagger}_i b^\dagger_{i+1,a} b^b_{i+1} \bar{b}_{ib}, & i \text{ odd}. \end{cases}$$

The E_i's are Hermitian, $E_i^\dagger = E_i$. Acting in the constrained space V, they satisfy

$$\begin{aligned} E_i^2 &= mE_i, \\ E_i\, E_{i\pm1}\, E_i &= E_i, \\ E_i\, E_j &= E_j\, E_i \qquad (j \neq i,\ i\pm1). \end{aligned}$$

which define the Temperley Lieb algebra $TL_N(m)$.

- Relations have well known graphical interpretation:



Transfer matrices propagating along the $(1,1)$ direction of the square lattice can be written in terms of elementary vertex interactions

$$T \equiv t_1 t_3 \cdots t_{2L-3} t_0 t_2 \cdots t_{2L-2},$$

witht $t = 1 + xe$.

By taking either of the two terms in t_i for each vertex in the graph, an expansion in terms space-filling loops is obtained with corresponding coefficients for each vertex, and a factor m for each loop.

• Generalize to the $SU(m+n|n)$ case. Each site now carries a $\mathbf{Z}_2$-graded vector space of dimensions $m+n$ for the even (bosonic), n for the odd (fermionic), subspace ($n \geq 0$ is an integer) $\equiv$ the fundamental of the Lie superalgebra $gl(m+n|n)$ for i even, and its dual for i odd. The chain is the graded tensor product of these V_i. It may be constructed using fermion operators in addition to the boson operators as in the $n = 0$ special case.

For i even: boson operators b_i^a, $b_{ia}^\dagger$, $[b_i^a, b_{jb}^\dagger] = \delta_{ij}\delta_b^a$ (a, $b = 1, \ldots, n+m$), and fermion operators f_i^α, $f_{i\alpha}^\dagger$, $\{f_i^\alpha, f_{j\beta}^\dagger\} = \delta_{ij}\delta_\beta^\alpha$ (α, $\beta = 1, \ldots, n$). For i odd, we have similarly boson operators $\bar{b}_{ia}$, $\bar{b}_i^{a\dagger}$, $[\bar{b}_{ia}, \bar{b}_j^{b\dagger}] = \delta_{ij}\delta_a^b$ (a, $b = 1, \ldots, n+m$), and fermion operators $\bar{f}_{i\alpha}$, $\bar{f}_i^{\alpha\dagger}$, $\{\bar{f}_{i\alpha}, \bar{f}_j^{\beta\dagger}\} = -\delta_{ij}\delta_\alpha^\beta$ (α, $\beta = 1, \ldots, n$).

The minus sign implies that the norm-square of any two states that are mapped onto each other by the action of a single $\overline{f}_{i\alpha}$ or $\overline{f}_i^{\alpha\dagger}$ have opposite signs, and the "Hilbert" space has an indefinite inner product.

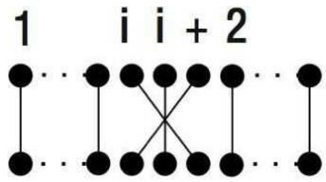
The space V is now defined as the subspace of states that obey the constraints

$$\begin{aligned}\sum_a b_{ia}^\dagger b_i^a + \sum_\alpha f_{i\alpha}^\dagger f_i^\alpha &= 1 \quad (i \text{ even}),\\ \sum_a \overline{b}_i^{a\dagger}\overline{b}_{ia} - \sum_\alpha \overline{f}_i^{\alpha\dagger}\overline{f}_{i\alpha} &= 1 \quad (i \text{ odd}).\end{aligned}$$

The generators of the Lie superalgebra gl$(m+n|n)$ acting on each site of the chain are the bilinear forms $J_{ia}^b = b_{ia}^\dagger b_i^b$, $f_{i\alpha}^\dagger f_i^\beta$, $b_{ia}^\dagger f_i^\beta$, $f_{i\alpha}^\dagger b_i^b$ for i even, and similarly for i odd.

• The TL generators are constructed similarly. When signs are properly handled, one gets a representation of the same algebra where each loop gets a factor $(n+m)-m = \text{str}\,1 = m$ (the evaluation of contributions for each loop can be viewed in terms of states in V flowing around the loop).

• Of course additional interactions can be added. For three neighbors, apart from $E_i E_{i+1} + E_{i+1} E_i$, we have $P_{i,i+2}$:

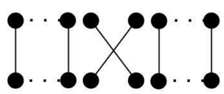


The resulting algebra is called Walled Brauer.

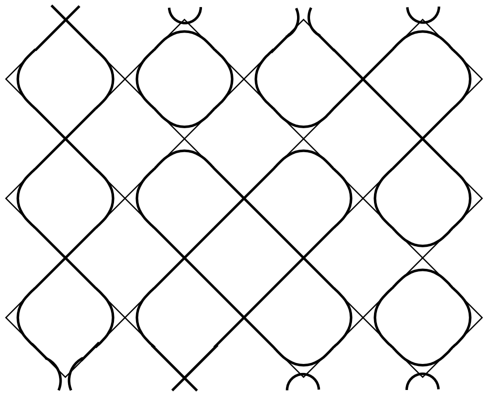
• A similar construction is possible for $OSp(m+2n|2n)$. Since

$$V \otimes V \ = \ 1 + \text{Sym} + \text{Antisym} \quad O(m)$$

Going over steps similar to $SU(m)$ we now have three possible interactions in the chain, hence we must add to the algebra $P_{i,i+1} \equiv P_i$



giving the Brauer algebra. When expanding the partition function, configurations will look generically as



• All this is for open boundary conditions. The periodic case involves affine versions of these algebras.

WHY INSIST ON LATTICE ALGEBRAS

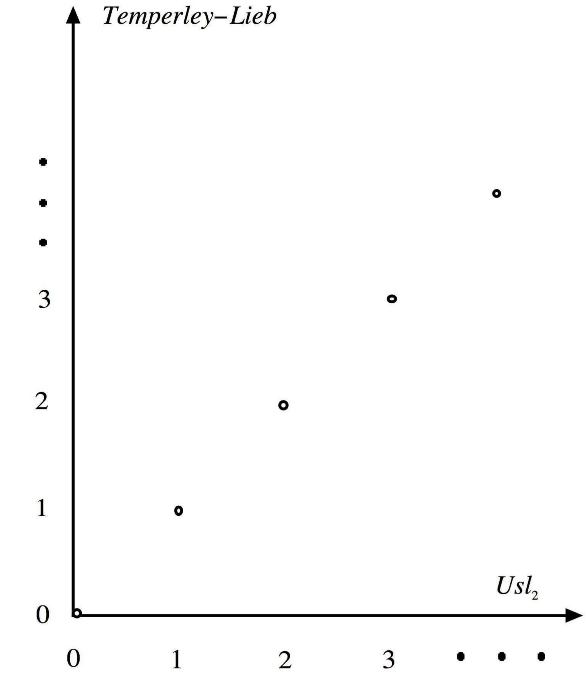
• Indecomposability is difficult to tackle within the CFTs, especially when it mixes $\text{Vir} \otimes \overline{\text{Vir}}$. Meanwhile, the study of non semi-simple associative algebras is somewhat more advanced in the math literature (and involves issues of wilderness).

• The general conjecture that the structure of the chain (as a bimodule) under

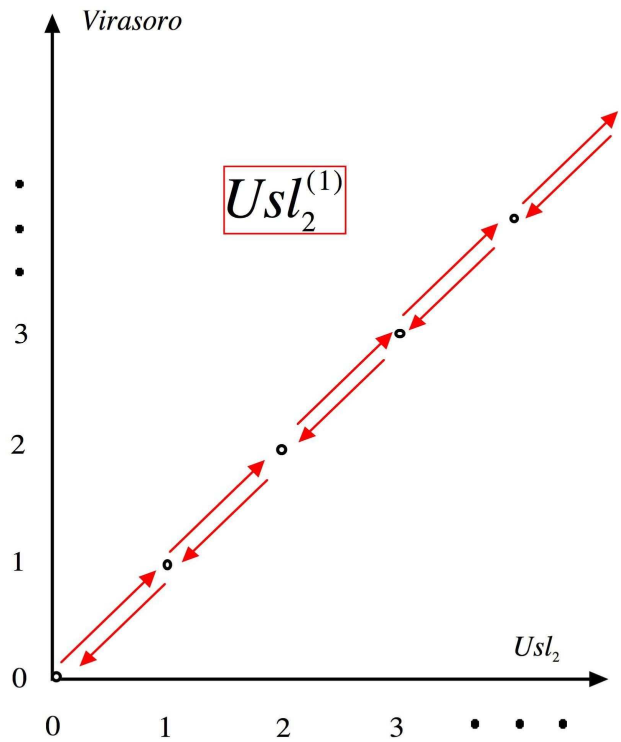
(extended) superalgebra $\otimes$ commutant

"carries over" to the continuum limit seems to hold. It involves the fact that the lattice algebras are cellular, and finer statements about Morita equivalence.

• Let us illustrate this by some examples. For the $SU(2)$ XXX chain we have the following kind of structure:

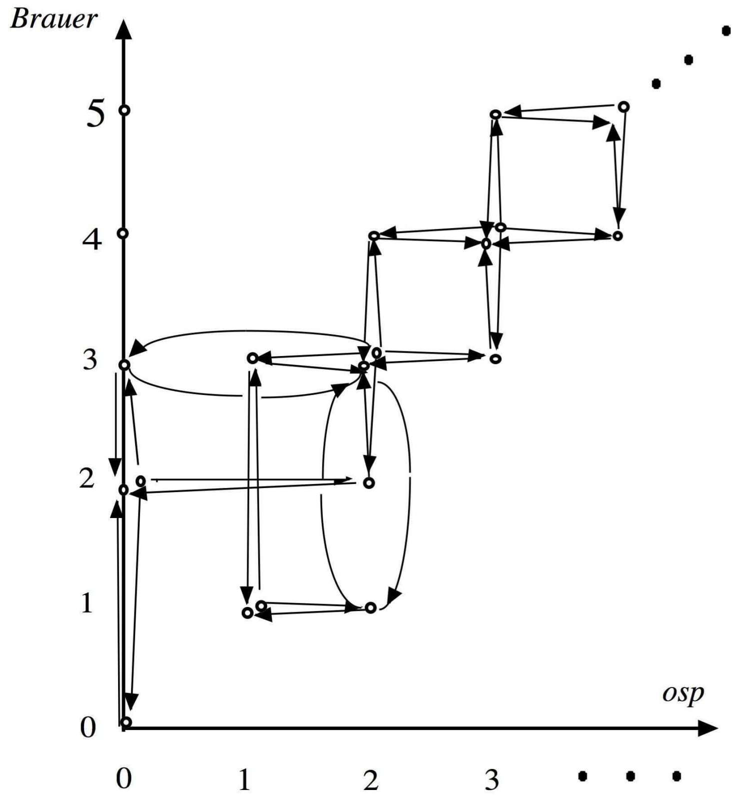


and in the continuum limit it becomes



so the current algebra "joins the dots".

Let us now compare with the $OSp(4|2)$ chain, where we have for instance (the exact shape depends on the "type" of representation)



The conjecture is in this case that all states in this diagram belong to a unique indecomposable representation of some chiral algebra (with the same Jordan cell structures), and have conformal weights that differ by integers. This applies directly to the $OSp(4|2)/OSp(3|2)$ sigma model, irrespective of g_σ^2.

Whenever indecomposability appears, it seems to be in some sense maximal, involving projective representations.

• Even if the models are not derived from Yang Baxter, the spectrum of H can often be determined block by block using other representations of the same underlying algebra that belong to Yang Baxter models (e.g. Temperley Lieb $\rightarrow$ XXZ).

• A lot of information can be gained about fusion as well through induction on the lattice $L \times L' \hookrightarrow L + L'$ (LCFT community: Pearce Rasmussen, Gaberdiel Runkel, Ridout Kitola).

• New symmetries show up, which are variants of Yangians (Read HS).

THE SIGMA MODEL ON $CP^{m-1|m}$

• Coherent state quantization: describe spins via a path integral. For $SU(2)$, (antiferromagnetic) interactions and large spin limit $\rightarrow$ $O(3) = SU(2)/U(1)$ sigma model at $\theta = 2\pi s$ + bare coupling constant $g_\sigma^2 \propto \frac{1}{s}$ + flow to large coupling. Physics is described by the $s = \frac{1}{2}$ XXX antiferromagnetic spin chain, and thus the $SU(2)_1$ WZW model.

$SU(m+n|n)$ and alternating reps (antiferromagnetic) $\rightarrow$ $U(n+m|n)/U(1) \times U(n+m-1|n)$ or $CP^{n+m-1|n}$ model at $\theta = \pi$.

• Fields : complex components z^a ($a = 1, \ldots, n+m$), ζ^α ($\alpha = 1, \ldots, n$), with $z^a(\zeta^\alpha)$ commuting (anticommuting). Constraint: $z_a^\dagger z^a + \zeta_\alpha^\dagger \zeta^\alpha = 1$ modulo U(1) phase transformations $z^a \mapsto e^{iB} z^a$, $\zeta^\alpha \mapsto e^{iB}\zeta^\alpha$. Lagrangian density:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2g_\sigma^2}\left[(D_\mu z_a^\dagger D_\mu^\dagger z^a + D_\mu \zeta_\alpha^\dagger D_\mu^\dagger \zeta^\alpha\right] \\ &+ \frac{i\theta}{2\pi}(\partial_\mu a_\nu - \partial_\nu a_\mu)\end{aligned}$$

where a_μ ($\mu = 1, 2$):

$$a_\mu = \frac{i}{2}[z_a^\dagger \partial_\mu z^a + \zeta_\alpha^\dagger \partial_\mu \zeta^\alpha - (\partial z_a^\dagger) z^a - (\partial \zeta_\alpha^\dagger)\zeta^\alpha],$$

• Coupling constants: g_σ^2 (target = supersymmetric space), and θ mod. 2π.

• β function:

$$\frac{dg_\sigma^2}{dl} = \beta(g_\sigma^2) = mg_\sigma^4 + O(g_\sigma^6)$$

For $m = 0$ is vanishes to leading order. But β function is independent of n, and for $n = 1$ $CP^{0|1}$ = symplectic fermions

$$\mathcal{L} = \frac{1}{2g_\sigma^2}\partial_\mu\zeta^\dagger\partial_\mu\zeta$$

where g_σ^2 is redundant. Thus $\beta = 0$ to all orders! $\rightarrow$ conformal sigma model (keep $\theta = \pi$ for now, more about topological angle later) $\equiv$ non abelian extension of symplectic fermions.

Now what?: spectrum of the boundary theory

for Neumann boundary conditions (volume filling branes)

- $g_\sigma^2 \to 0$ limit:

monopole numbers   Euler fct   fermionic contr.

$$Z^{\mathrm{CP};\nu\mu}_{R=\infty} = q^{\frac{1}{12}}\phi(q)\underbrace{\oint \frac{du}{u^{\mu-\nu}}}_{\text{U(1) gauging}}\underbrace{\phi(q)\lim_{t\to 1}(1-t^2)}_{\text{constraint}}\prod_{n=0}^{\infty}\prod_{\alpha\beta=\pm\frac{1}{2}}\frac{1+y^{\alpha}(zu^{-1})^{\beta}q^{n}}{1-x^{\alpha}(zu)^{\beta}q^{n}}$$

bosonic contr.

$$= \sum_{\Lambda}\psi^{\mathrm{CP}}_{\Lambda;\nu\mu}(q)\,\chi_{\Lambda}(x,y) \longleftarrow \text{u(2|2) characters}$$

- Spectrum at finite g_σ^2:

Branching fcts from R = ∞   universal indep of Λ   Characters of u(2|2) reps

$$\boxed{Z^{\mathrm{CP};\nu\mu}_{R} = q^{\frac{1}{2}\lambda(\lambda-1)}\sum_{\Lambda}\psi^{\mathrm{CP}}_{\Lambda;\nu\mu}(q)\,q^{f_{\nu\mu}(R,\theta)C^{(2)}_{\Lambda}}\,\chi_{\Lambda}(x,y)}$$

λ = λ(Θ₁,Θ₂) from symplectic Fermions   Casimir of U(2|2)

$\Theta_{1,2} = \theta + 2\pi\nu(\mu)$ Dirac monopole term (ν,μ number of 'edge states')

• Quasi abelian evolution of weights. For free boson:

$$\Delta_{\Phi}^{g_\sigma^2} = \Delta_{\Phi}^{0} + f(g_\sigma^2) q^2$$

where q is $U(1)$ charge. For $CP^{1|2}$ we have strong arguments that

$$\Delta_{\Phi}^{g_\sigma^2} = \Delta_{\Phi}^{0} + f(g_\sigma^2) C_{\Phi}^{(2)}$$

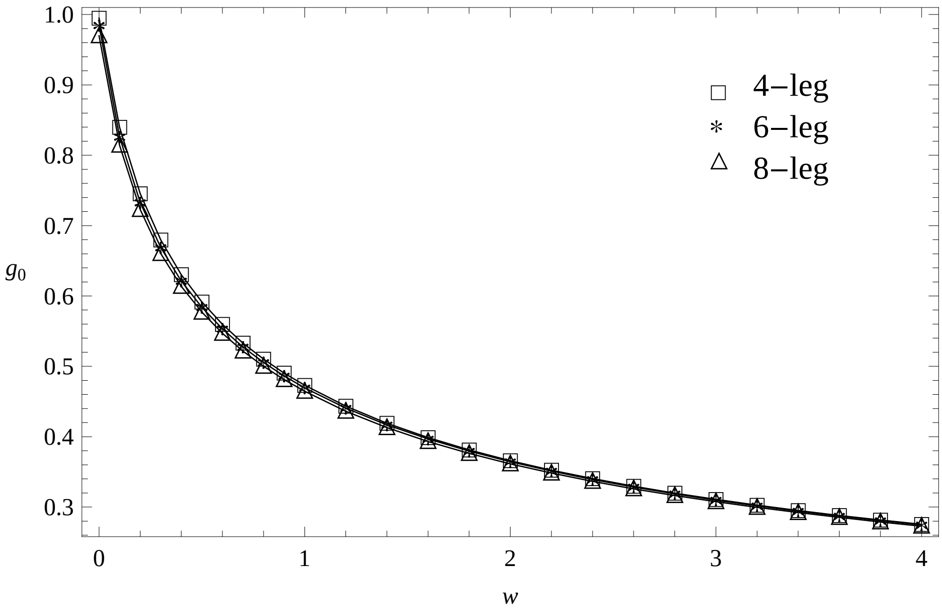
where $C_{\Phi}^{(2)}$ is quadratic Casimir (Bershadsky et al., Candu HS, Mitev et al.).

• Study of lattice model. First, blocks under Brauer $\otimes$ $gl(2|2)$ are made of reps. with the same Casimir

• Numerics:

$$H = -\sum_{j=1}^{2L} E_j + w \sum_{j=1}^{2L-2} P_{i,i+2} + H_{bdry}$$

acting on $\Box^{\otimes \mu} \otimes (\Box \otimes \bar{\Box})^{\otimes L} \otimes \bar{\Box}^{\otimes \nu}$. For instance



where $2l$ legs = invariant symmetric tensor of rank $2l$.

- All orders perturbative calculations.

Note: there are no observable instanton effects. Role of θ still a bit unclear.

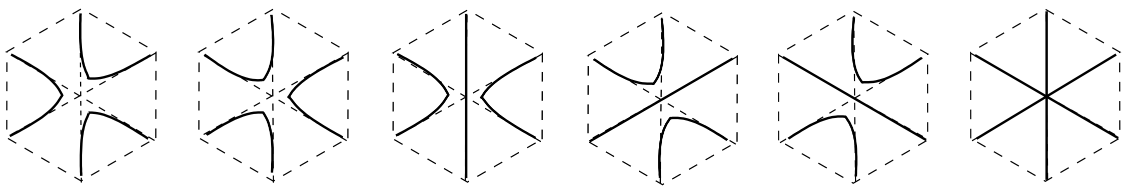
CONCLUSIONS AND OPEN PROBLEMS

• Similar results can be obtained for $\frac{OSp(2n+1|2n)}{OSp(2n|2n)}$ [Candu, HS; Mitev, Quella, Schomerus]. In this case there is a WZW point at $g_\sigma^2 = 1$. Is there such a point here? (eg $psu(n|n)$ at level 1?)

• Casimir algebras ?

• The periodic case ?

• Geometrical applications:



1 2 3 4 5 6

• Integrability?