Perturbative S-matrix of Marginally deformed Super Yang-Mills

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Introduction and Conclusion

We consider the world-sheet scattering of β-deformed SYM at leading orders of perturbation. In the dual string side, we start from single TsT-transformed $AdS_5 \times S^5$ about natural tori and consider the near BMN limit of the background. We compute tree-level amplitudes from effective quartic potential obtained by light-cone gauge fixing. The results are exactly matched with the exact results. In the gauge theory side, we apply "Coordinate Bethe Ansatz" on three spin state Hamiltonian. Here, the S-matrix and TBCs are not the same but we show that their BAEs are equivalent each other. We conclude that (a) perturbative S-matrix and appropriate TBCs at both sides are consistent with the exact S-matrix with the twisted boundary conditions and (b) in fact, the twisting of boundary conditions seems to be prerequisite essential to see integrability in β-deformed SYM.

TsT-transformation, $AdS_5 \times S^5$ and TBC

1) Frolov's derivation [Frolov '05]

- Lunin-Maldacena background which is dual to β-deformed SYM can be obtained by world-sheet transformation : T-dual, Shift and T-dual again.
- String dynamics on LM background is equivalent with that on $AdS_5 \times S^5$ if we compensate appropriate twisted boundary conditions for isometry angles of S^5.

$$\tilde{\tilde{\phi}}_i(2\pi) - \tilde{\tilde{\phi}}_i(0) = 2\pi(n_i + \epsilon_{ijk}\gamma_j J_k), \ (i,j,k = 1,2,3)$$

2) Another approach to dual string theory of β-deformed SYM

If we only consider single TsT transformation for any two angles among three angles, the resulting background is quite simple.

$$\begin{aligned} ds^2_{string}/R^2 &= ds^2_{AdS_5} + d\rho_1^2 + \rho_1^2 d\hat{\phi}_1^2 + \sum_{i=2}^{3}\left(d\rho_i^2 + \hat{G}\rho_i^2 d\hat{\phi}_i^2\right) \\ B_2 &= -\hat{\gamma}R^2\hat{G}\left(\rho_2^2\rho_3^2 d\hat{\phi}_2 \wedge d\hat{\phi}_3\right), \ \hat{G}^{-1} = 1 + \hat{\gamma}^2\rho_2^2\rho_3^2 \end{aligned}$$

We show that the string dynamics on the above background with little differnet TBC is equivalent with LM strings with PBC and also $AdS_5 \times S^5$ strings with the Frolov's TBC.

$$\begin{aligned} \hat{\phi}_1(2\pi) - \hat{\phi}_1(0) &= \tilde{\tilde{\phi}}_1(2\pi) - \tilde{\tilde{\phi}}_1(0) = P_{ws} \\ \hat{\phi}_2(2\pi) - \hat{\phi}_2(0) &= \tilde{\tilde{\phi}}_2(2\pi) - \tilde{\tilde{\phi}}_2(0) + 2\pi\gamma_1 J_3 = 2\pi(n_2 + \gamma_3 J_1) \\ \hat{\phi}_3(2\pi) - \hat{\phi}_3(0) &= \tilde{\tilde{\phi}}_3(2\pi) - \tilde{\tilde{\phi}}_3(0) - 2\pi\gamma_1 J_2 = 2\pi(n_3 - \gamma_2 J_1) \end{aligned}$$

Near BMN limit, Gauge fixing and WS Scattering

As in [KMRZ '06], one can compute string WS S-matrix from gauge fixed Lagrangian in near BMN limit. The most efficient way for gauge fixing is the 1st order formalism.

Single TsT background → **The 1st order formalism** → **Quartic interactions with deformed effects** → **Worldsheet S-matrix at Strong coupling**

Near BMN limit

$$t \to X^+ - \frac{X^-}{2R^2}, \quad \phi \to X^+ + \frac{X^-}{2R^2}, \quad Z_k \to \frac{Z_k}{R}, \quad Y_{k'} \to \frac{Y_{k'}}{R}$$

Curvature expansion

$$\begin{aligned} ds^2 &= 2dX^+dX^- + dY^2 + dZ^2 - dX^{+2}(Z^2+Y^2) \\ &+ \frac{1}{2R^2}\left(2dX^-dX^+(Z^2-Y^2) + dZ^2Z^2 - dY^2Y^2 + dX^+dX^+(Y^4 - Z^4)\right) \\ B &= \frac{1}{2R^2}\hat{\gamma}(Y_1Y_3dY_2\wedge dY_4 + Y_2Y_4dY_1\wedge dY_3 - Y_2Y_3dY_1\wedge dY_4 - Y_1Y_4dY_2\wedge dY_3) \end{aligned}$$

Feynman Diagram & Full Kinematic consideration

$$\begin{aligned} L_{g.f.} &= \frac{1}{2}(\partial_\mu Z\partial^\mu Z + \partial_\mu Y\partial^\mu Y) - \frac{1}{2}(Z^2+Y^2) + \frac{1}{4\sqrt{\lambda}}[(Y_{1\dot{1}}Y_{2\dot{2}} + Y_{1\dot{2}}Y_{2\dot{1}})(2\partial_\sigma Y_{1\dot{1}}\partial_\sigma Y_{2\dot{2}} + 2\partial_\sigma Y_{1\dot{2}}\partial_\sigma Y_{2\dot{1}} \\ &+ (\partial_\tau Z)^2 + (\partial_\sigma Z)^2) - Z^2(\partial_\tau Y_{1\dot{2}}\partial_\tau Y_{2\dot{1}} + \partial_\tau Y_{1\dot{1}}\partial_\tau Y_{2\dot{2}} + \partial_\sigma Y_{1\dot{2}}\partial_\sigma Y_{2\dot{1}} + \partial_\sigma Y_{1\dot{1}}\partial_\sigma Y_{2\dot{2}} + 2(\partial_\sigma Z)^2)] \\ &- \frac{1}{4\sqrt{\lambda}}\hat{\gamma}[Y_{1\dot{1}}Y_{1\dot{2}}\partial_\tau Y_{2\dot{2}}\partial_\sigma Y_{2\dot{1}} - Y_{1\dot{1}}Y_{1\dot{2}}\partial_\tau Y_{2\dot{1}}\partial_\sigma Y_{2\dot{2}} + Y_{1\dot{1}}Y_{2\dot{1}}\partial_\tau Y_{1\dot{2}}\partial_\sigma Y_{2\dot{2}} - Y_{1\dot{1}}Y_{2\dot{1}}\partial_\tau Y_{2\dot{2}}\partial_\sigma Y_{1\dot{2}} \\ &+ Y_{1\dot{2}}Y_{2\dot{2}}\partial_\tau Y_{2\dot{1}}\partial_\sigma Y_{1\dot{1}} - Y_{1\dot{2}}Y_{2\dot{2}}\partial_\tau Y_{1\dot{1}}\partial_\sigma Y_{2\dot{1}} + Y_{2\dot{1}}Y_{2\dot{2}}\partial_\tau Y_{1\dot{1}}\partial_\sigma Y_{1\dot{2}} - Y_{2\dot{1}}Y_{2\dot{2}}\partial_\tau Y_{1\dot{2}}\partial_\sigma Y_{1\dot{1}}] \end{aligned}$$

Tree level amplitudes can be easily computed from the above interactions.

As the full kinematic constant from relativistic and Jacobian factors is cancelled with the Feynman diagramatic contribution, the full T-matrices are constant shifts of results in $AdS_5 \times S^5$. This is exactly matched with the exact results. Also, twisted BCs which is related with our simple TsT background are those of [Ahn '11]. (See the Appendix.)

$$\begin{aligned} Y^{1\dot{1}}(\sigma,\tau) &= \int\frac{dp}{2\sqrt{\omega_p}}\left[a^{1\dot{1}}(p)e^{-i(\omega\tau-p\sigma)} + \epsilon^{12}\epsilon^{\dot{1}\dot{2}}a(p)^\dagger_{2\dot{2}}e^{i(\omega\tau-p\sigma)}\right] \\ Y^{1\dot{2}}(\sigma,\tau) &= \int\frac{dp}{2\sqrt{\omega_p}}\left[a^{1\dot{2}}(p)e^{-i(\omega\tau-p\sigma)} + \epsilon^{12}\epsilon^{\dot{2}\dot{1}}a(p)^\dagger_{2\dot{1}}e^{i(\omega\tau-p\sigma)}\right] \\ Y_{1\dot{1}}(\sigma,\tau) &= \int\frac{dp}{2\sqrt{\omega_p}}\left[\epsilon_{12}\epsilon_{\dot{1}\dot{2}}a^{2\dot{2}}(p)e^{-i(\omega\tau-p\sigma)} + a(p)^\dagger_{1\dot{1}}e^{i(\omega\tau-p\sigma)}\right] \\ Y_{1\dot{2}}(\sigma,\tau) &= \int\frac{dp}{2\sqrt{\omega_p}}\left[\epsilon_{12}\epsilon_{\dot{2}\dot{1}}a^{2\dot{1}}(p)e^{-i(\omega\tau-p\sigma)} + a(p)^\dagger_{1\dot{2}}e^{i(\omega\tau-p\sigma)}\right] \end{aligned}$$

$$\mathcal{S} = \mathbb{I} + \frac{2i\pi}{\sqrt{\lambda}}\mathbb{T}$$

$$\Lambda(p,p') = \frac{1}{\varepsilon' p - \varepsilon p'}$$

$$\begin{aligned} \mathbb{T} &= (\varepsilon_1 p_2 - \varepsilon_2 p_1)a_1(p)^\dagger_{1\dot{1}}a_2(p)^\dagger_{2\dot{1}}a_1(p)_{1\dot{1}}a_2(p)_{2\dot{1}} - (\varepsilon_1 p_2 - \varepsilon_2 p_1)a_1(p)^\dagger_{1\dot{1}}a_2(p)^\dagger_{1\dot{2}}a_1(p)_{1\dot{1}}a_2(p)_{1\dot{2}} \\ &+ (\varepsilon_1 p_2 - \varepsilon_2 p_1)a_1(p)^\dagger_{2\dot{2}}a_2(p)^\dagger_{1\dot{2}}a_1(p)_{2\dot{2}}a_2(p)_{1\dot{2}} - (\varepsilon_1 p_2 - \varepsilon_2 p_1)a_1(p)^\dagger_{2\dot{2}}a_2(p)^\dagger_{2\dot{1}}a_1(p)_{2\dot{2}}a_2(p)_{2\dot{1}} \end{aligned}$$

$$\begin{aligned} Y_{1\dot{1}}(2\pi)/Y_{1\dot{1}}(0) &= e^{2\pi i\gamma_3 J_1} = e^{2\pi i(\gamma_3-\gamma_2)J/2}e^{2\pi i(\gamma_3+\gamma_2)J/2} \\ Y_{1\dot{2}}(2\pi)/Y_{1\dot{2}}(0) &= e^{-2\pi i\gamma_2 J_1} = e^{2\pi i(\gamma_3-\gamma_2)J/2}e^{-2\pi i(\gamma_3+\gamma_2)J/2} \end{aligned}$$

$$\begin{aligned} \mathbb{T}_{\hat{\gamma}}|Y_{1\dot{1}}(p)Y_{1\dot{2}}(p')\rangle &= -\hat{\gamma}|Y_{1\dot{1}}(p)Y_{1\dot{2}}(p')\rangle \\ \mathbb{T}_{\hat{\gamma}}|Y_{1\dot{1}}(p)Y_{2\dot{1}}(p')\rangle &= +\hat{\gamma}|Y_{1\dot{1}}(p)Y_{2\dot{1}}(p')\rangle \\ \mathbb{T}_{\hat{\gamma}}|Y_{2\dot{2}}(p)Y_{1\dot{2}}(p')\rangle &= +\hat{\gamma}|Y_{2\dot{2}}(p)Y_{1\dot{2}}(p')\rangle \\ \mathbb{T}_{\hat{\gamma}}|Y_{2\dot{2}}(p)Y_{2\dot{1}}(p')\rangle &= -\hat{\gamma}|Y_{2\dot{2}}(p)Y_{2\dot{1}}(p')\rangle \end{aligned}$$

Deformed Spin-chains, S-matrix and BAE

Consider the $SU(3)_\beta$ chain which is valid at 1-loop order. The Hamiltonian can be obtained by direct computation as in [Berenstein, Cherkis '04] or by using Drinfeld-Reshetikhen twisted R-matrix as in [Beisert, Roiban '05]. When we apply the coordinate BAE to the mixed part of Hamiltonian, the correct Bethe-type Ansatz is not usual but phase shifted one.

$$\begin{aligned} \psi_{12}(p_1,p_2) &= A_{12}e^{i(p_1-2\pi\beta)n_1 + i(p_2+2\pi\beta)n_2} + \acute{A}_{12}e^{i(p_1+2\pi\beta)n_2 + i(p_2-2\pi\beta)n_1} \\ \psi_{21}(p_1,p_2) &= A_{21}e^{i(p_1+2\pi\beta)n_1 + i(p_2-2\pi\beta)n_2} + \acute{A}_{21}e^{i(p_1-2\pi\beta)n_2 + i(p_2+2\pi\beta)n_1} \end{aligned}$$

To compute S-matrix using these ansatze is straightforward as in usual. Interstingly, these ansatze naturally gives twisted boundary conditions for each excitations. Alternatively, we can take simple change of basis as in [B,C '04]. Then, we can easily obtain the $SU(3)_\beta$ S-matrix and in here, the change of basis gives $|\tilde{1}\rangle_{L+1} = e^{-2iL\pi\beta}|\tilde{1}\rangle_1, \ |\tilde{2}\rangle_{L+1} = e^{2iL\pi\beta}|\tilde{2}\rangle_1$ us twisted boundary conditions. Although this S-matrix and TBC look different with exact results, we can show that their BAEs are equivalent.

$$S = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & e^{3i\gamma}t & r & 0 \\ 0 & r & e^{-3i\gamma}t & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

To derive BAEs, we have to use "nested coordinate Bethe Ansatz" because of non-diagonal S-matrix.

Considered carefully twisted S-matrix and different TBCs, set of BAEs can be derived. Alternatively, we can do it by using "Algebraic Bethe Ansatz". For example, one can follow steps in [Kulish, Reshetikhen '83]. Concretely, the deformed effects in eigenvalues come from S-matrix deformation and ratios between TBCs of fields are appeared in front of each BAEs. Actually, these BAEs are just one loop & SU(3) reduction of Beisert-Roiban BAE [Frolov, Roiban, Tseytlin '05]. In [ABBN '10], BR BAEs are derived from exact S-matrix with TBCs.

$$\begin{aligned} e^{-2\pi i\beta L}\left(\frac{u_{1,k}+\frac{i}{2}}{u_{1,k}-\frac{i}{2}}\right)^L &= \prod_{i=1,i\neq k}^{M_1}\frac{u_{1,k}-u_{1,i}+i}{u_{1,k}-u_{1,i}-i}\prod_{j=1}^{M_2}e^{-6\pi i\beta}\frac{u_{1,k}-u_{2,j}-\frac{i}{2}}{u_{1,k}-u_{2,j}+\frac{i}{2}} \\ e^{4\pi i\beta L} &= \prod_{j=1,j\neq l}^{M_2}\frac{u_{2,l}-u_{2,j}+i}{u_{2,l}-u_{2,j}-i}\prod_{i=1}^{M_1}e^{6\pi i\beta}\frac{u_{1,i}-u_{2,l}+\frac{i}{2}}{u_{1,i}-u_{2,l}-\frac{i}{2}} \end{aligned}$$

Therefore, we conclude that even though perturbative and exact S-matrix look different, they are spectrally the same each other.

Appendix : the exact S-matrix with TBCs

$$M_{a\dot{a}} = \mathbb{I}\otimes e^{2iJ\beta h}$$

$$\tilde{\mathcal{S}}(p_1,p_2) = F\,\mathcal{S}(p_1,p_2)\,F \qquad F = e^{i\beta(h\otimes\mathbb{I}\otimes\mathbb{I}\otimes h - \mathbb{I}\otimes h\otimes h\otimes\mathbb{I})} \qquad h = \mathrm{diag}(\frac{1}{2}, -\frac{1}{2}, 0, 0)$$

$$\tilde{t}(\lambda) = \mathrm{str}_{a\dot{a}}\, M_{a\dot{a}}\tilde{S}_{a\dot{a}1\dot{1}}(\lambda,p_1)\ldots\tilde{S}_{a\dot{a}N\dot{N}}(\lambda,p_N)$$

$$\begin{aligned} F &= e^{\frac{2\pi i\hat{\gamma}_1}{\sqrt{\lambda}}} = (\mathbb{I} + 2\pi i\hat{\gamma}_1\frac{M}{\sqrt{\lambda}}) \\ \tilde{S}_{strong} &= F\,S_{strong}\,F = \mathbb{I} + 2\pi i\frac{(2M\gamma_1 + \mathbb{T})}{\sqrt{\lambda}} \end{aligned}$$

Core References

[1] Frolov, JHEP 0505:069 ('05)

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[4] Kulish, Reshetikhen, J. Phys. A: Math. Gen. 16 (1983) L591

[5] Berenstein, Cherkis, Nucl.Phys.B 731:1 ('05)

[6] Beisert, Roiban, JHEP 0508:039 ('05)

[7] Ahn et al, JHEP 1102:027 ('11)