Regge Limit of Gauge Theories & Gravity

(with Links to Integrability)

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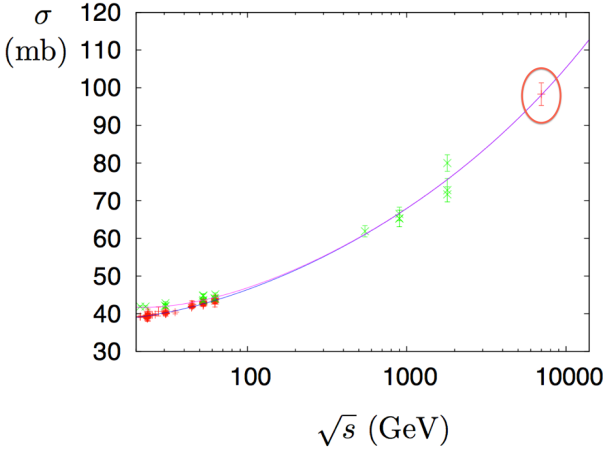
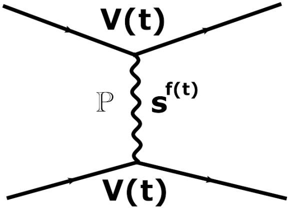
Let us complete the picture ...

... with a Pomeron exchange

1. Brief Introduction
2. BFKL as a Matrix Random Walk
3. Five-point Amplitude in Einstein-Hilbert Gravity
4. N-SUGRA - Double Logs in Energy

Hadron-hadron total cross-sections rise @ $\sqrt{s} = 7$ TeV @ LHC:
$\sigma^{\mathrm{TOT}} = 98.3 \pm 0.2\,(\mathrm{stat}) \pm 2.8\,(\mathrm{syst})$ mb

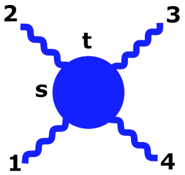




Consistent with Regge theory (Soft Pomeron Exchange): $\sigma^{\mathrm{TOT}} \simeq s^{0.1}$

Regge theory is pre-QCD

Microscopic picture of the Pomeron in terms of quarks & gluons?



Elastic amplitude with Mandelstam invariants s, t

Regge limit in perturbative QCD: $s \gg t, Q^2$

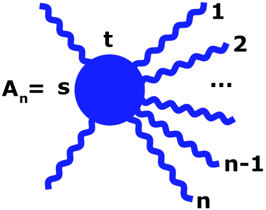
We need a hard scale $Q^2 \gg \Lambda^2_{\rm QCD}$ to allow for a perturbative expansion $\alpha_s(Q^2) \ll 1$

All-orders resummation of $\alpha_s(Q^2)\log\left(\frac{s}{Q^2}\right)$ terms: How?

Multi-Regge limit: $s \gg s_i \gg t_i \sim Q^2$ (Regge limit in all sub-channels)

Equivalent to strong ordering in rapidity of emitted particles:

$$Y \sim \log s \qquad y_i \gg y_{i-1} \qquad \mathbf{k}_i^2 \sim \mathbf{k}_{i-1}^2 \sim Q^2$$



Allows for resummation of longitudinal phase space:

$$\alpha_s^n(Q^2)\int_0^Y dy_1 \int_0^{y_1} dy_2 \dots \int_0^{y_{n-1}} dy_n \sim \frac{\left(\alpha_s(Q^2)Y\right)^n}{n!}$$

$$\sigma = \sum_n |A_n|^2$$

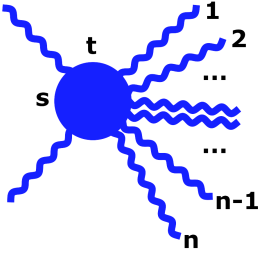
1. Brief Introduction

Regge limit in perturbative QCD: $s \gg t, Q^2$

Multi-Regge limit: Regge limit in all sub-channels [Only gluons]

Quasi-multi-Regge limit: one sub-channel with no limit [Quarks appear]
No ordering in rapidity in two nearby emitted particles:

$$Y \sim \log \frac{s}{s_0} \qquad y_i \gg y_{i-1} \qquad y_l \sim y_{l+1} \qquad \mathbf{k}_i^2 \sim \mathbf{k}_{i-1}^2 \sim Q^2$$



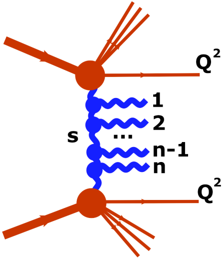
Fixes "factorization scale" s_0; $b \sim \log \frac{s_0'}{s_0}$:

$$\alpha_s^n(Q^2) \int_0^{Y'+b} dy_1 \int_0^{y_1} dy_2 \ldots \int_0^{y_{n-1}} dy_n$$

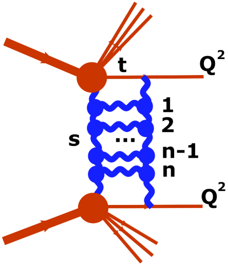
$$\sim \underbrace{\frac{\left(\alpha_s(Q^2)Y'\right)^n}{n!}}_{\text{LO}} + \underbrace{b\frac{\alpha_s^n(Q^2)}{(n-1)!}Y'^{n-1}}_{\text{NLO}} + \cdots$$

Rich Phenomenology:

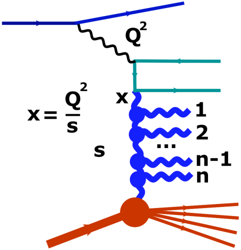
Multijets



Rapidity Gaps



Deep Inelastic Scattering

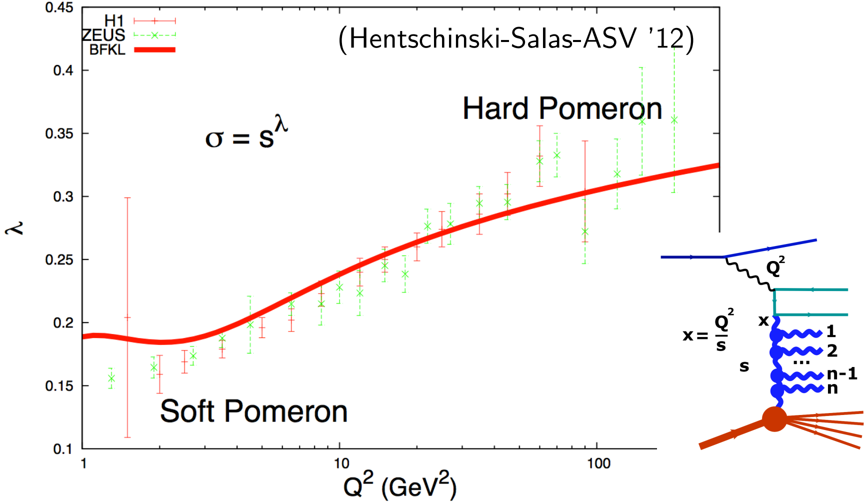


For these we obtain $\sigma \simeq s^{0.3}$

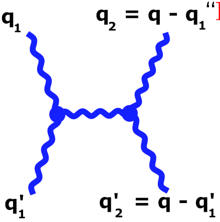
Hadron-Hadron (Soft Pomeron is Non-Perturbative): $\sigma^{\rm TOT} \simeq s^{0.1}$

We have calculated a Perturbative or Hard (BFKL) Pomeron.

To fit data: BFKL+Collinear Resummation+Optimal Renormalization



1. Brief Introduction

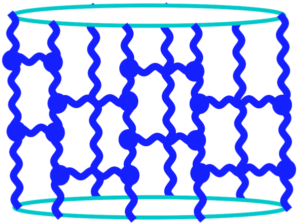


$$\text{“Reggeized Propagators”} \simeq g^2 N_c \delta^{(2)}\left(\vec{q}_1 - \vec{q}_1{}'\right)\delta^{(2)}\left(\vec{q}_2 - \vec{q}_2{}'\right)$$
$$\times \left(\int d^2\vec{r}\,\frac{\vec{q}_1{}^2}{\vec{r}^{\,2}(\vec{q}_1 - \vec{r})^2} + \int d^2\vec{r}\,\frac{\vec{q}_2{}^2}{\vec{r}^{\,2}(\vec{q}_2 - \vec{r})^2}\right)$$

$$\text{“Emission”} \simeq \delta^{(2)}\left(\vec{q}_1 + \vec{q}_2 - \vec{q}_1{}' - \vec{q}_2{}'\right)\frac{g^2 N_c}{\vec{q}_1{}^2\vec{q}_2{}^2}$$
$$\times \left(\frac{\vec{q}_1{}^2\vec{q}_2{}'^2 + \vec{q}_2{}^2\vec{q}_1{}'^2}{\vec{k}^2} - (\vec{q}_1 + \vec{q}_2)^2\right)$$

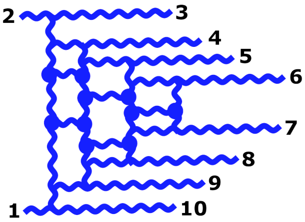
SL(2,$\mathbb{C}$) invariance (Lipatov)

Unitarization procedure: iterate the kernel (BKP):
Mapping to Periodic XXX Heisenberg
Spin Chain (Lipatov, Faddeev-Korchemsky)



1. Brief Introduction

Calculation of Infrared Finite Remainder Function in MHV, planar, $\mathcal{N} = 4$ SYM Scattering Amplitudes for arbitrary number of loops and legs in the Multi-Regge limit (Bartels-Lipatov-ASV '08, Fadin, Prygarin, Kormilitzin):



- A similar XXX Open Heisenberg Spin Chain appears (Lipatov)
- Progress MRK+Symbols (Dixon, Drummond, Duhr, Henn, Pennington)
- Strong coupling MRK amplitudes in (Bartels-Schomerus-Sprenger)

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2. BFKL as a matrix Random walk
   with Romagnoni (Madrid)
   arXiv:1111.4553
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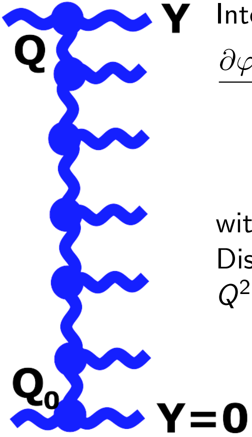
Closer look at the ladder: Forward case: $\vec{q} = 0$



Integrate over azimuthal angle, BFKL equation:

$$\frac{\partial \varphi(Q^2, Y)}{\alpha \partial Y} = \int_0^\infty \frac{dq^2}{|q^2 - Q^2|} \left\{ \varphi(q^2, Y) - \frac{2 \min(q^2, Q^2)}{q^2 + Q^2} \varphi(Q^2, Y) \right\}$$

with initial condition $\varphi(Q^2, Y = 0) = \delta(Q^2 - Q_0^2)$

Discretize virtuality space:

$Q^2 = N\,\Delta,\ q^2 = n\,\Delta,\ \phi_n \equiv \varphi(l^2, Y)$

$$\frac{\partial \phi_N}{\alpha \partial Y} = \underbrace{\sum_{n=1}^{N-1} \frac{\phi_n}{N-n} + \sum_{n=N+1}^{\overbrace{\infty}^{Unbounded\ Virtualities}} \frac{\phi_n}{n-N}}_{\text{Gluon Emission}} - \underbrace{2h(N-1)\phi_N}_{\text{No Emission}}$$

$h(N) = \sum_{l=1}^{N} \frac{1}{l} = \psi(N+1) - \psi(1)$

2. BFKL as a matrix Random walk

(Minahan/Zarembo)

Dilatation operator of planar $\mathcal{N} = 4$ SYM $\leftrightarrow$ XXX spin chain Hamiltonian

(Beisert)

Hamiltonian for 1-loop AD of spin $S-1$ operators of sl(2) closed sector:

$$\mathcal{H}_{1,2}^{\mathrm{sl}(2)} (a_1^\dagger)^{N-1} (a_2^\dagger)^{S-N} |00\rangle = -\lambda \sum_{l=1}^{S} \left(\frac{(1-\delta_l^N)}{|l-N|} - \left(h(N-1) + h(S-N)\right) \delta_l^N \right) (a_1^\dagger)^{l-1} (a_2^\dagger)^{S-l} |00\rangle$$

$$\lambda = \frac{g^2 N_c}{8\pi^2}$$

$(a^\dagger)^n |0\rangle = \frac{1}{n!} (\mathcal{D})^n \Phi \quad \rightarrow \quad$ 1 site in a 1-dim lattice $\quad \rightarrow \quad$ n-th excited state

Excitations classified in $s = -\frac{1}{2}$ representation of sl(2)

$XXX_{-\frac{1}{2}}$ Hamiltonian for $S = 2N - 1$ acts on two $(N-1)$-th excited states:

$$\mathcal{H}_{1,2}^{\mathbf{sl}(2)} (a_1^+)^{N-1} (a_2^+)^{N-1} |00\rangle \quad = \quad -\lambda \sum_{l=1}^{\overbrace{2N-1}^{\text{Bounded Excitations}}} \left(\frac{(1-\delta_l^N)}{|l-N|} - 2h(N-1)\delta_l^N \right) (a_1^+)^{l-1} (a_2^+)^{2N-1-l} |00\rangle$$

BFKL equation can be written as

$$\mathcal{H}^{\mathrm{BFKL}} \phi_N \quad = \quad \alpha \sum_{l=1}^{\overbrace{\infty}^{\text{Unbounded Virtualities}}} \left(\frac{(1-\delta_l^N)}{|l-N|} - 2h(N-1)\delta_l^N \right) \phi_l$$

What is the link between these two systems?

2. BFKL as a matrix Random walk

$$\alpha = -\lambda \qquad \vec{\phi} \equiv (\phi_1, \ldots, \phi_N)$$

$$\hat{\mathcal{H}}_N^{\mathbf{sl}(2)} = \begin{pmatrix} -2h(0) & 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{N-1} \\ 1 & -2h(1) & 1 & \frac{1}{2} & \cdots & \frac{1}{N-2} \\ \frac{1}{2} & 1 & -2h(2) & 1 & \cdots & \frac{1}{N-3} \\ \frac{1}{3} & \frac{1}{2} & 1 & -2h(3) & \cdots & \frac{1}{N-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N-1} & \frac{1}{N-2} & \frac{1}{N-3} & \frac{1}{N-4} & \cdots & -2h(N-1) \end{pmatrix}$$

Construct "BFKL-like Equation" for the sl(2) chain:

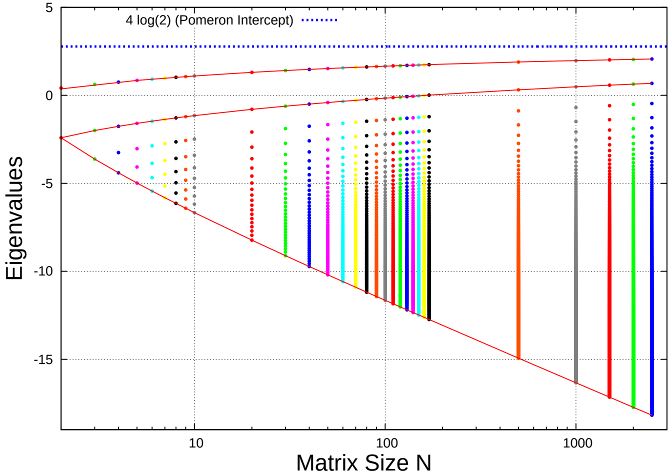
$$\frac{\partial \vec{\phi}}{\alpha \partial Y} = \hat{\mathcal{H}}_N^{\mathbf{sl}(2)} \cdot \vec{\phi} \qquad \rightarrow \qquad \vec{\phi} = e^{\alpha Y \hat{\mathcal{H}}_N^{\mathbf{sl}(2)}} \cdot \vec{\phi}_0$$

with Initial Condition $\vec{\phi}_0$

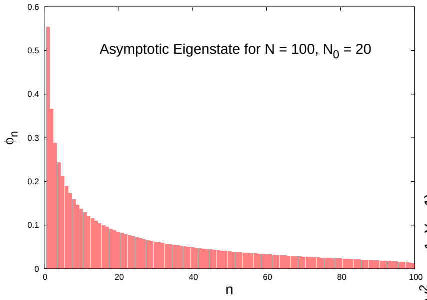
2. BFKL as a matrix Random walk

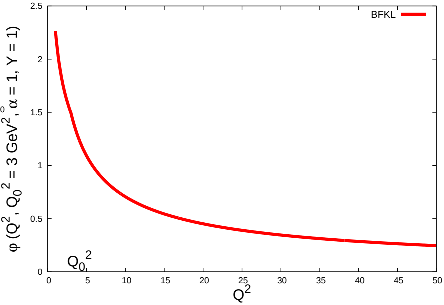
$$\alpha Y \to \infty$$

$$\vec{\phi} = e^{\alpha Y \hat{\mathcal{H}}_N^{\mathbf{sl}(2)}} \cdot \vec{\phi}_0 = \sum_{L=1}^{N} c_L^{(N)} e^{\alpha Y \lambda_L^{(N)}} \psi_L^{(N)} \to c_{\rm as}^{(N)} e^{\alpha Y \lambda_{\rm as}^{(N)}} \psi_{\rm as}^{(N)}$$



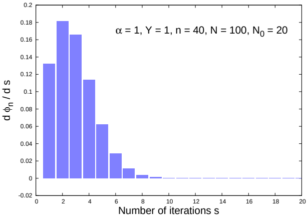
The Same "Virtuality" Distribution: $\vec{\phi} \equiv (\phi_1, \ldots, \phi_N)$

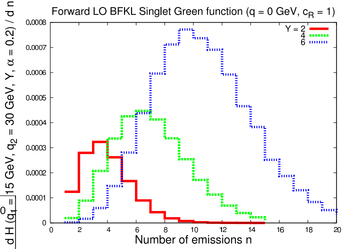




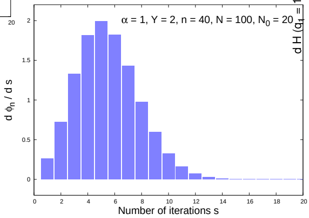
2. BFKL as a matrix Random walk

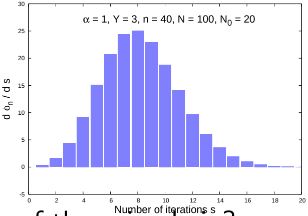
The Same "Multiplicity" Distribution: $\vec{\phi}_n \rightarrow c_{\rm as}^{(N)} \sum_{s=0}^{\infty} \frac{(\alpha Y \lambda_{\rm as}^{(N)})^s}{s!} \psi_{\rm as,n}^{(N)}$





(Chachamis, ASV)





- Is this connected to the coherent state pictures of the spin chain?

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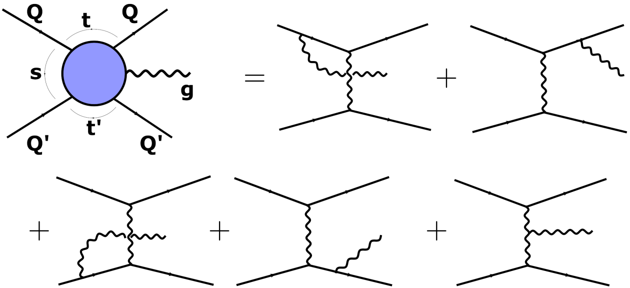
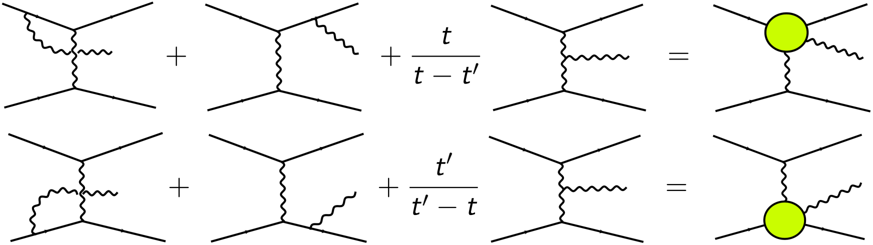
   with Serna Campillo & Vazquez-Mozo (Salamanca)
   JHEP 03 (2012) 005
4. $\mathcal{N}$-SUGRA - Double Logs in Energy

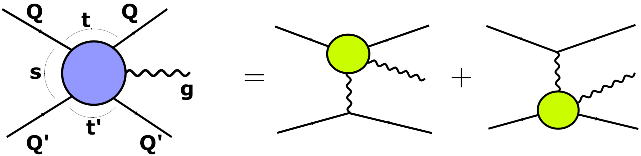
A Simple Review Calculation in QCD:



Nice Trick:



Exact Amplitude is the Sum of Two Gauge Invariant Effective Vertices:

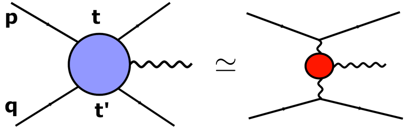


$t = k_1^2, t' = k_2^2$ Sudakov expansion

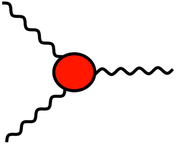
$$k_1 \; = \; \alpha_1 p + \beta_1 q + k_1^{\perp} \qquad k_2 \; = \; \alpha_2 p + \beta_2 q + k_2^{\perp}.$$

Multi-Regge kinematics (MRK):

$$1 \;\gg\; |\alpha_1| \;\gg\; |\alpha_2| \;=\; -\frac{t'}{s} \qquad 1 \;\gg\; |\beta_2| \;\gg\; |\beta_1| \;=\; -\frac{t}{s}$$



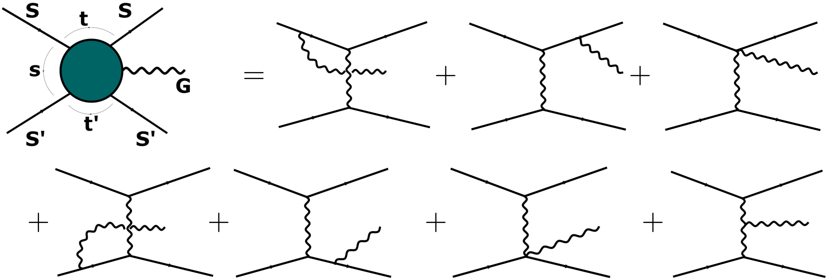
Universal Reggeized g - Reggeized g - g Effective Vertex (Lipatov) in MRK:



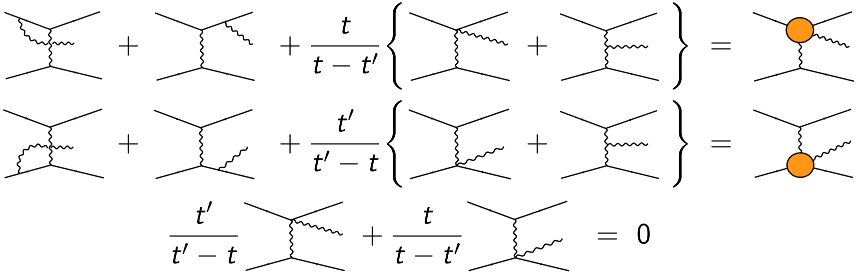
$$= ig\eta_{\mu\sigma}\left\{\left(\alpha_1 - \frac{2t}{s\beta_2}\right)p^{\nu} + \left(\beta_2 - \frac{2t'}{s\alpha_1}\right)q^{\nu} - \left(k_1^{\perp} + k_2^{\perp}\right)^2\right\}$$

3. Five-point Amplitude in Einstein-Hilbert gravity

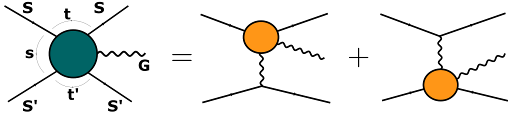
The Closest Calculation in Einstein-Hilbert Gravity:



Nice Trick is Still Nice but More Tricky:



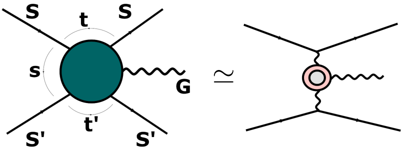
Exact Amplitude is the Sum of Two Gauge Invariant Sub-Amplitudes:



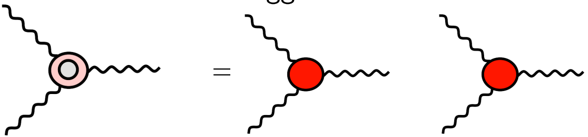
These Sub-Amplitudes are Very Useful in Loop Calculations

3. Five-point Amplitude in Einstein-Hilbert gravity

Using the same Sudakov expansion and Multi-Regge kinematics:



Universal Reggeized G - Reggeized G - G Effective Vertex (Lipatov):



$$+ \quad 4\beta_1\alpha_2 \left\{ \frac{p^\mu p^\nu}{\beta_2^2} + \frac{q^\mu q^\nu}{\alpha_1^2} + \frac{p^\mu q^\nu + q^\mu p^\nu}{\alpha_1\beta_2} \right\}$$

Subtraction Term to Fullfil Steinman Relations (no simultaneous singularities in overlapping channels).

- Is integrability present in gravity?

4. N-SUGRA - Double Logs in Energy

Four-graviton scattering in N-SUGRA $\boxed{\mathcal{A}_{4,(N)} = \mathcal{A}_4^{\rm Born}\mathcal{M}_{4,(N)}}$

At 1-loop there are three contributions ($\alpha = {\rm Newton's\, constant}/\pi$):

$$
\begin{aligned}
\mathcal{M}^{(1)}_{4,(N=8)} &= \underbrace{\alpha\, t \ln\left(\frac{-s}{-t}\right)\ln\left(\frac{-u}{-t}\right)}_{\text{Double Logs}} \\
&+ \underbrace{\alpha\,\frac{t}{2}\ln\left(\frac{-t}{\lambda^2}\right)\left(\ln\left(\frac{-s}{-t}\right)+\ln\left(\frac{-u}{-t}\right)\right)}_{\text{Trajectory}} \\
&- \underbrace{\alpha\,\frac{(s-u)}{2}\ln\left(\frac{-t}{\lambda^2}\right)\ln\left(\frac{-s}{-u}\right)}_{\text{Eikonal}}
\end{aligned}
$$

4. N-SUGRA - Double Logs in Energy

In the Regge limit $u \simeq -s$

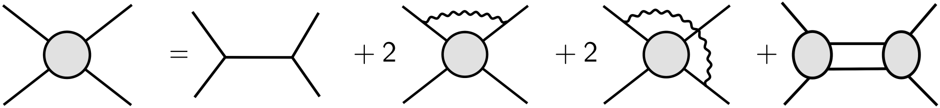
$$\mathcal{M}^{(1)}_{4,(N=8)} \simeq \underbrace{(\alpha\, t) \ln^2 \left(\frac{s}{-t}\right)}_{\text{Double Logs}} + \underbrace{(\alpha\, t) \ln \left(\frac{-t}{\lambda^2}\right) \ln \left(\frac{s}{-t}\right)}_{\text{Trajectory}} + i\,\pi\, \underbrace{(\alpha\, s) \ln \left(\frac{-t}{\lambda^2}\right)}_{\text{Eikonal}}$$

Can we resum the Double Log contributions to all-orders? Let us use

$$\mathcal{A}_{4,(N)} = \mathcal{A}_4^{\text{Born}} \left(\frac{s}{-t}\right)^{\alpha\, t\, \ln\left(\frac{-t}{\lambda^2}\right)} \int_{\delta-i\infty}^{\delta+i\infty} \frac{d\,\omega}{2\pi i} \left(\frac{s}{-t}\right)^{\omega} \frac{f_\omega^{(N)}}{\omega}$$

4. N-SUGRA - Double Logs in Energy

Double Logs are dominated by the contributions



with (1) softest virtual gravitons

(2) pair of softest gravitons/gravitinos in t-channel (Ladder)

The corresponding equation is

$$f_\omega^{(N)} = 1 - (\alpha\, t) \frac{d}{d\,\omega} \left(\frac{f_\omega^{(N)}}{\omega} \right) + (\alpha\, t) \left(\frac{N-6}{2} \right) \left(\frac{f_\omega^{(N)}}{\omega} \right)^2$$

With perturbative solution:

$$\begin{aligned} f_\omega^{(N)} &= 1 + (\alpha\, t) \frac{(N-4)}{2\omega^2} + (\alpha\, t)^2 \frac{(N-4)(N-3)}{2\omega^4} \\ &- (\alpha\, t)^3 \frac{(N-4)\left(5N^2 - 26N + 36\right)}{8\omega^6} + \dots \end{aligned}$$

4. $\mathcal{N}$-SUGRA - Double Logs in Energy

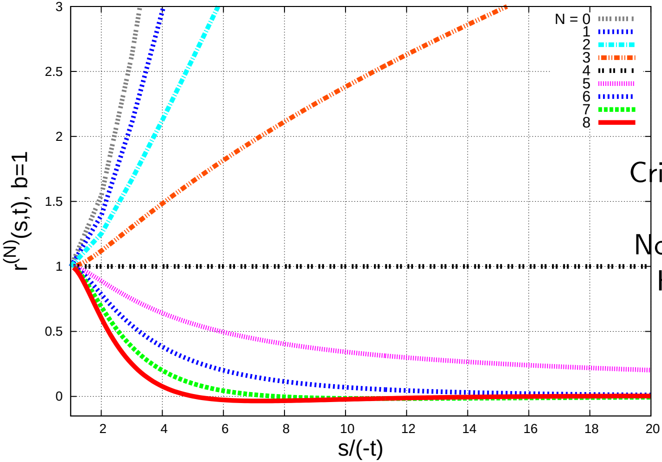
Agreement with exact 2-loop results for $N = 4, \ldots, 8$ SUGRA obtained with the conjecture that gravity is a double copy of two gauge theories (Boucher-Veronneau, Dixon, Bern, Carrasco, Johansson, Dunbar, Norridge).

We make All-Orders predictions, *e.g.* $N = 8$:

$$\begin{aligned}
\mathcal{A}_{4,(N=8)} &= \mathcal{A}_4^{\rm Born} \left(\frac{-t}{\lambda^2}\right)^{\alpha t \left(\ln\left(\frac{s}{-t}\right) + i\pi\left(\frac{s}{t}\right)\right)} \\
&\times \left\{ 1 + 2\left(\frac{\alpha\, t}{2}\right) \ln^2\left(\frac{s}{-t}\right) + \frac{5}{3}\left(\frac{\alpha\, t}{2}\right)^2 \ln^4\left(\frac{s}{-t}\right) \right. \\
&+ \frac{37}{45}\left(\frac{\alpha\, t}{2}\right)^3 \ln^6\left(\frac{s}{-t}\right) + \frac{353}{1260}\left(\frac{\alpha\, t}{2}\right)^4 \ln^8\left(\frac{s}{-t}\right) \\
&\left. + \frac{583}{8100}\left(\frac{\alpha\, t}{2}\right)^5 \ln^{10}\left(\frac{s}{-t}\right) + \ldots \right\}
\end{aligned}$$

4. $\mathcal{N}$-SUGRA - Double Logs in Energy

All-orders Resummation:
Solution of a linear Schrödinger eqn as a Parabolic Cylinder function.



$\mathcal{N} = 4$ SUGRA:
Critical theory separating
Convergent ($\mathcal{N} > 4$)
Non-convergent ($\mathcal{N} < 4$)
High Energy behaviour

- Is there a simple representation to all-orders of the full amplitude?

1. Brief Introduction
2. BFKL as a matrix Random walk
3. Five-point Amplitude in Einstein-Hilbert gravity
4. *N*-SUGRA - Double Logs in Energy

... still many things to be learnt from this limit ...