# $(2,0)$ theory on $\mathbb{R} \times T^5$ at low energies


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*What does the spectrum of states look like at low energies (compared to the scale set by the inverse size of the $T^5$)?*

Despite our limited understanding of $(2,0)$ theory [Witten 1995], this question is tractable:

An $ADE$-type $(2,0)$ theory $\Phi$ on $\mathbb{R} \times T^5$ is an ultra-violet completion of Yang-Mills theory on $\mathbb{R} \times T^4$ with gauge group $G_{\text{adj}} = G/C$, where $G$ is a simply laced and simply connected group with center subgroup $C$:

| $\Phi$ | $G$ | $C$ |
|---|---|---|
| $A_{n-1}$ | $SU(n)$ | $\mathbb{Z}_n$ |
| $D_{2k+1}$ | $Spin(4k+2)$ | $\mathbb{Z}_4$ |
| $D_{4k+2}$ | $Spin(8k+4)$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |
| $D_{4k}$ | $Spin(8k)$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |
| $E_6$ | $E_6$ | $\mathbb{Z}_3$ |
| $E_7$ | $E_7$ | $\mathbb{Z}_2$ |
| $E_8$ | $E_8$ | 1. |

And we have at least some understanding of low-energy Yang-Mills theory...

The first few homotopy groups of the gauge group $G_{\text{adj}} = G/C$ are

$$\pi_k(G_{\text{adj}}) \simeq \begin{cases} 1, & k = 0 \\ C, & k = 1 \\ 1, & k = 2 \\ \mathbb{Z} & k = 3. \end{cases}$$

So the gauge bundle (a principal $G_{\text{adj}}$ bundle over $T^4$) is topologically classified by two characteristic classes :

The *discrete abelian magnetic 't Hooft flux* (second Stieffel-Whitney class)

$$m \in M = H^2(T^4, C)$$

and the *instanton number* (second Chern class)

$$k \in H^4(T^4, \mathbb{Q}) \simeq \mathbb{Q}.$$

These are correlated by

$$k - \frac{1}{2} m \cdot m \in \mathbb{Z} \subset \mathbb{Q}.$$

($m \cdot m$ is the tensor product of the inner product on $H^2(T^4, \mathbb{Z})$ and the pairing on $C \simeq \Gamma_{\text{weight}}/\Gamma_{\text{root}}$.)

Because of the magnetic contribution Tr$(F \wedge *F)$ to the Yang-Mills energy density, low-energy states are localized on flat connections, $F = 0$.

A necessary condition for a flat connection is that the (fractional part of the) instanton number $k = \frac{1}{2} m \cdot m$ vanishes in $H^0(T^4, \mathbb{Q})$ modulo $H^0(T^4, \mathbb{Z})$.

A flat connection is characterized by its holonomies

$$U \in \text{Hom}(\pi_1(T^4), G_{\text{adj}})$$

modulo simultaneous conjugation by elements of $G_{\text{adj}}$ (connected gauge transformations).

The holonomies $U_i$, $i = 1, \ldots 4$ commute in $G_{\text{adj}}$, but when lifted to

$$\widehat{U} \in \text{Hom}(\pi_1(T^4), G)$$

they are only almost commuting in the sense that

$$\widehat{U}_i \widehat{U}_j \widehat{U}_i^{-1} \widehat{U}_j^{-1} = m_{ij}.$$

(Here $m = m_{ij} dx^i \wedge dx^j$.)

Large gauge transformations are parametrized by $\Gamma = \mathrm{Hom}(\pi_1(T^4), \pi_1(G_{\mathrm{adj}})) \simeq H^1(T^4, C)$ and act on $\widehat{U}_i$ by multiplication. The transformation properties of a quantum state is described by the discrete abelian electric 't Hooft flux

$$e \in E = \mathrm{Hom}(\Gamma, U(1)) \simeq H^3(T^4, C),$$

where we have used the (canonical) isomorphism

$$C \simeq \mathrm{Hom}(C, U(1)).$$

Because of the almost commutation relations, certain large gauge transformations are equivalent to conjugation by the holonomies $\widehat{U}_i$, i.e to connected gauge transformations. Quantum states should be invariant under such transformations, which gives the conditions that

$$j = m \cdot e \in H^1(T^4, \mathbb{Q})$$

vanishes modulo $H^1(T^4, \mathbb{Z})$.

Our conclusion is that (2,0) theory on $T^5 = T^4 \times S^1$ has a characteristic class

$$f = m + e \in H^3(T^5, C) = H^2(T^4, C) \oplus H^3(T^4, C),$$

and a necessary condition for low-energy states is that

$$\begin{aligned} 0 &= \frac{1}{2} f \cdot f = \frac{1}{2} m \cdot m + m \cdot e = k + j \\ &\in H^1(T^5, \mathbb{Q}) = H^0(T^4, \mathbb{Q}) \oplus H^1(T^4, \mathbb{Q}). \end{aligned}$$

More generally, the spectrum of low-energy states should only depend on the orbit of $f$ under the $SL_5(\mathbb{Z})$ mapping class group of $T^5$. (Invariance under the $SL_4(\mathbb{Z})$ mapping class group of $T^4$, which does not mix $m$ and $e$, is manifest in the Yang-Mills theory interpretation.)

We will now verify these predictions by explicit computation in some cases, in complete analogy with previous computations on $\mathbb{R} \times T^3$. [Henningson-Wyllard 2007]

For a given $m \in H^2(T^4, C)$ such that $k = \frac{1}{2} m \cdot m = 0$, the structure of the moduli space of flat connections $\mathcal{M}$ can in principle be worked out, roughly as for bundles over $T^3$ [. . . , Borel-Friedman-Morgan 1999, . . . ].

It is of the form

$$\mathcal{M} = \bigcup_\alpha \mathcal{M}_\alpha.$$

Each connected component is of the form

$$\mathcal{M}_\alpha = (T^{r_\alpha} \times T^{r_\alpha} \times T^{r_\alpha} \times T^{r_\alpha})/W_\alpha,$$

for some rank $r_\alpha$ and some discrete group $W_\alpha$ acting on $T^{r_\alpha}$.

(Basic example: For $m = 0$, there is a component $\mathcal{M}_0$, such that $T^{r_0}$ is a maximal torus of $G$ and $W_0$ the corresponding Weyl group.)

The wave function of a low-energy state is supported on the moduli space $\mathcal{M}$ of flat connections.

At a point on the component $\mathcal{M}_\alpha$ of $\mathcal{M}$, the unbroken subalgebra has the form

$$h \simeq s \oplus u(1)^r$$

for some $r$, $0 \leq r \leq r_\alpha$, and some semi-simple Lie algebra $s$ of rank $r_\alpha - r$.

Given $h$, we let $\mathcal{M}^h$ denote the corresponding subspace of $\mathcal{M}$. In general, it consists of several connected components:

$$\mathcal{M}^h = \bigcup_a \mathcal{M}^h_a.$$

The components are permuted by large gauge transformations. Diagonalizing the action of these on the space of states supported on $\mathcal{M}^h$ gives a spectrum of electric 't Hooft flux $e \in H^3(T^4, C)$.

A connected component $\mathcal{M}_a^h$ is parametrized by the components of the holonomies $\widehat{U}_i$ that belong to the abelian term $u(1)^r$ of $h$.

The canonical conjugate to the $u(1)^r$ holonomy is the electric field strength $E_i$. Because of the electric contribution $E_i E_i$ to the Yang-Mills energy density, the wave function of a low-energy state must be constant on each component $\mathcal{M}_a^h$.

The Yang-Mills energy density also contains a term $\Pi\Pi$, where $\Pi$ are the canonical conjugates to the covariantly constant modes of the $5r$ abelian scalar fields. There is thus a $5r$-dimensional continuum of non-normalizable (unless $r = 0$) "eigenstates" of the $\Pi$ operators.

Quantizing the covariantly constant modes of the spinor fields gives a further finite degeneracy to the spectrum of low-energy states.

We must also quantize the degrees of freedom associated with the semi-simple term $s$ of the unbroken Lie algebra $h = s \oplus u(1)^r$. These parametrize the directions transverse to $\mathcal{M}^h$ in $\mathcal{M}$.

At low energies, this is modeled by $s$ quantum mechanics with 16 supercharges (i.e. the dimensional reduction to $0+1$ dimensions of maximally supersymmetric Yang-Mills theory).

This theory has no mass-gap, but is believed to have a finite-dimensional linear space $V_s$ of normalizable zero-energy states.

$V_s$ has an orthonormal basis with elements in one-to-one correspondence with the set of distinguished markings of the $s$ Dynkin diagram. (A marking of a Dynkin diagram defines a grading $s = \oplus_{n \in \mathbb{Z}} s_n$. This is distinguished if $\dim s_0 = \dim s_1 = \dim s_{-1}$.) [Kac-Smilga 1999]

We have

$$\dim V_s = \begin{cases} 1, & s \simeq su(n) \\ \#\text{ partitions of } n & \\ \text{into distinct odd parts,} & s \simeq so(n) \\ \#\text{ partitions of } 2n & \\ \text{into distinct even parts,} & s \simeq sp(2n) \\ 3, & s \simeq E_6 \\ 6, & s \simeq E_7 \\ 11, & s \simeq E_8 \\ 4, & s \simeq F_4 \\ 2, & s \simeq G_2. \end{cases}$$

We introduce the generating functions

$$\begin{aligned} P(q) &= P_{\text{even}}(q) + P_{\text{odd}}(q) \\ &= \sum_{n=0}^{\infty} \dim V_{so(n)} q^n = \prod_{k=1}^{\infty} (1 + q^{2k-1}) \\ &= 1 + q + q^3 + q^4 + q^5 + q^6 + q^7 + 2q^8 + \ldots \end{aligned}$$

and

$$\begin{aligned} Q(q) &= \sum_{n=0}^{\infty} \dim V_{so(2n)} q^{2n} = \prod_{k=1}^{\infty} (1 + q^{2k}) \\ &= 1 + q^2 + q^4 + 2q^6 + 2q^8 + 3q^{10} + \ldots \end{aligned}$$

The complete spectrum is obtained by summing the contributions for all possible $h \simeq s \oplus u(1)^r$ and all components $\mathcal{M}_a^h$, each of which contributes $\dim V_s$ rank $r$ continua of states.

The prediction from $(2,0)$ theory is that the number $N_f^r(\Phi)$ of rank $r$ continua with a certain value of $f = m + e \in H^3(T^5, C)$ only depends on the $SL_5(\mathbb{Z})$ orbit of $f$.

The $A_r$ cases are easiest to check, but do not give full justice to the subject.

Work is in progress on the $D_{4k+2}$ and $D_{4k}$ cases, which have the richest structure.

The exceptional $E_6$, $E_7$, and $E_8$ cases are left for the future.

So here we will only consider the...

... $D_{2k+1}$ cases, which have $C \simeq \mathbb{Z}_4$.

The 7 different $SL_4(\mathbb{Z})$ orbits of $m \in H^2(T^4, C)$ and how they together with the $4^4 = 256$ different values of $e \in H^3(T^4, C)$ build up the 6 different $SL_5(\mathbb{Z})$ orbits of $f$ are given by

| $[m]$ | $[f] = 0_0$ | $2_0$ | $2_2$ | $1_0$ | $1_2$ | $1_1 = 1_3$ | sum |
|---|---|---|---|---|---|---|---|
| $0_0$ | 1 | 15 | | 240 | | | $1 \cdot 256$ |
| $2_0$ | | 4 | 12 | 48 | 192 | | $35 \cdot 256$ |
| $2_2$ | | | 16 | | 240 | | $28 \cdot 256$ |
| $1_0$ | | | | 16 | 48 | 192 | $1120 \cdot 256$ |
| $1_2$ | | | | | 64 | 192 | $1120 \cdot 256$ |
| $1_1$ | | | | | | 256 | $896 \cdot 256$ |
| $1_3$ | | | | | | 256 | $896 \cdot 256$ |
| sum | 1 | 155 | 868 | 19840 | 138880 | 888832 | $4096 \cdot 256$ |

where

$$
\begin{aligned}
0_0 &= [0] \\
2_0 &= [2dx^1dx^2] \\
2_2 &= [2dx^1dx^2 + 2dx^3dx^4] \\
1_0 &= [dx^1dx^2] \\
1_2 &= [dx^1dx^2 + 2dx^3dx^4] \\
1_1 &= [dx^1dx^2 + dx^3dx^4] \\
1_3 &= [dx^1dx^2 + 3dx^3dx^4].
\end{aligned}
$$

We define the generating functions

$$Z_f(q,y) = \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} N_f^r(D_{2k+1}) q^{4k+2} y^r.$$

They are non-vanishing only for $f \cdot f = 0$:

| | $[f] = 0_0$ | $2_0$ | $2_2$ | $1_0$ | $1_2$ | $1_1 = 1_3$ |
|---|---|---|---|---|---|---|
| $[m] = 0_0$ | $Z_{0_0}$ | $Z_{2_0}$ | | $Z_{1_0}$ | | |
| $2_0$ | | $Z'_{2_0}$ | $Z_{2_2}$ | $Z'_{1_0}$ | 0 | |
| $2_2$ | | | $Z'_{2_2}$ | | 0 | |
| $1_0$ | | | | $Z''_{1_0}$ | 0 | 0 |
| $1_2$ | | | | | 0 | 0 |
| $1_1$ | | | | | | 0 |
| $1_3$ | | | | | | 0 |

The remaining entries can be computed as described above, and expressed in terms of the functions $P_{\text{even}}$, $P_{\text{odd}}$, $Q$, and $R$, where

$$\begin{aligned} R(q,y) &= \prod_{k=1}^{\infty} (1 - yq^{2k})^{-1} \\ &= 1 + yq^2 + (y + y^2)q^4 + (y + y^2 + y^3)q^6 \\ &\quad + (y + 2y^2 + y^3 + y^4)q^8 + \ldots \end{aligned}$$

They are given by (the $q^{4k+2}$ terms of):

$$
\begin{aligned}
Z_{0_0} &= \frac{1}{16}R(q,y)(P^{16}_{\text{even}}(q) + 30P^{8}_{\text{even}}(q)P^{8}_{\text{odd}}(q) + P^{16}_{\text{odd}}(q)) \\
&+\frac{15}{16}R(q,y)(P^{8}_{\text{even}}(q^2) + 14P^{4}_{\text{even}}(q^2)P^{4}_{\text{odd}}(q^2) + P^{8}_{\text{odd}}(q^2)) \\
Z_{2_0} &= \frac{1}{16}R(q,y)(P^{16}_{\text{even}}(q) + 30P^{8}_{\text{even}}(q)P^{8}_{\text{odd}}(q) + P^{16}_{\text{odd}}(q)) \\
&-\frac{1}{16}R(q,y)(P^{8}_{\text{even}}(q^2) + 14P^{4}_{\text{even}}(q^2)P^{4}_{\text{odd}}(q^2) + P^{8}_{\text{odd}}(q^2)) \\
Z'_{2_0} &= \frac{1}{16}R(q,y)(4P^{12}_{\text{even}}(q)P^{4}_{\text{odd}}(q) + 24P^{8}_{\text{even}}(q)P^{8}_{\text{odd}}(q) + 4P^{4}_{\text{even}}( \\
&+\frac{3}{16}R(q,y)(4P^{6}_{\text{even}}(q^2)P^{2}_{\text{odd}}(q^2) + 8P^{4}_{\text{even}}(q^2)P^{4}_{\text{odd}}(q^2) + 4P \\
Z_{2_2} &= \frac{1}{16}R(q,y)(4P^{12}_{\text{even}}(q)P^{4}_{\text{odd}}(q) + 24P^{8}_{\text{even}}(q)P^{8}_{\text{odd}}(q) + 4P^{4}_{\text{even}}( \\
&-\frac{1}{16}R(q,y)(4P^{6}_{\text{even}}(q^2)P^{2}_{\text{odd}}(q^2) + 8P^{4}_{\text{even}}(q^2)P^{4}_{\text{odd}}(q^2) + 4P \\
Z'_{2_2} &= \frac{1}{16}R(q,y)(16P^{10}_{\text{even}}(q)P^{6}_{\text{odd}}(q) + 16P^{6}_{\text{even}}(q)P^{10}_{\text{odd}}(q)) \\
Z_{1_0} &= \frac{1}{8}R(q^2,y)(P^{8}_{\text{even}}(q) + P^{8}_{\text{odd}}(q)) \\
&+\frac{7}{8}R(q^2,y)(P^{4}_{\text{even}}(q^2) + P^{4}_{\text{odd}}(q^2)) \\
Z'_{1_0} &= \frac{1}{4}R(q^2,y)P^{4}_{\text{even}}(q)P^{4}_{\text{odd}}(q) \\
&+\frac{3}{4}R(q^2,y)P^{2}_{\text{even}}(q^2)P^{2}_{\text{odd}}(q^2) \\
Z''_{1_0} &= R(q^2,y)Q(q^2)^4(P^{9}_{\text{even}}(q^2)P^{3}_{\text{odd}}(q^2) + 3P^{7}_{\text{even}}(q^2)P^{5}_{\text{odd}}(q^2) \\
&+3P^{5}_{\text{even}}(q^2)P^{7}_{\text{even}}(q^2) + P^{3}_{\text{even}}(q^2)P^{9}_{\text{odd}}(q^2))
\end{aligned}
$$

Happily, $Z_f$ only depends on the $SL_5(\mathbb{Z})$ orbit $[f]$ of $f$:

$$
\begin{aligned}
Z_{0_0} &= yq^2 \\
&\quad +(1+2y+y^2+y^3)q^6 \\
&\quad +(32+35y+4y^2+3y^3+y^4+y^5)q^{10} \\
&\quad +(528+285y+71y^2+39y^3+5y^4+3y^5+y^6+y^7)q^{14}+\dots
\end{aligned}
$$

$$
\begin{aligned}
Z_{2_0} &= Z'_{2_0} \\
&= (1+y)q^6 \\
&\quad +(32+12y+2y^2+y^3)q^{10} \\
&\quad +(528+198y+46y^2+13y^3+2y^4+y^5)q^{14}+\dots
\end{aligned}
$$

$$
\begin{aligned}
Z_{2_2} &= Z'_{2_2} \\
&= q^6 \\
&\quad +(32+7y+y^2)q^{10} \\
&\quad +(528+175y+40y^2+7y^3+y^4)q^{14}+\dots
\end{aligned}
$$

$$
\begin{aligned}
Z_{1_0} &= Z'_{1_0} = Z''_{1_0} \\
&\quad q^6 \\
&\quad +(10+y)q^{10} \\
&\quad +(67+11y+y^2)q^{14}+\dots
\end{aligned}
$$

A possible refinement is to decompose the spectrum into unitary representations of the stability subgroup of $f$ in $SL_5(\mathbb{Z})$.

To summarize, we have studied $Spin(4k+2)/\mathbb{Z}_4$ maximally supersymmetric Yang-Mills theory on $\mathbb{R} \times T^4$.

The low-energy spectrum consists of a set of continua of states, characterized by their dimensions $5r$ and their magnetic and electric 't Hooft fluxes $m$ and $e$.

In particular, we have verified the $SL_5(\mathbb{Z})$ covariance that follows from the interpretation of the theory as type $D_{2k+1}$ $(2,0)$ theory on $\mathbb{R} \times T^5$. This unifies $m$ and $e$ to a single class $f$.

For the other simply laced cases, this approach gives highly non-trivial predictions for the structure of the moduli space of flat connections over $T^4$.

But more interesting would be to understand the conceptual foundations of $(2,0)$ theory that underly these results.

Thank you!