(Open) $\mathcal{N} = 4$ Topological Amplitudes and Harmonicity Relations

Stefan Hohenegger

ETH Zürich
Institute for Theoretical Physics

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work in collaboration with I. Antoniadis (CERN), K.S. Narain (ICTP Trieste) and E. Sokatchev (LAPTH Annecy)

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What is a Topological Amplitude?

In a very wide context, a topological amplitude is a string amplitude which can be rewritten as a correlator in a topologically twisted theory

$$F = \langle V_1 \dots V_n \rangle_{\text{string}} = \langle \tilde{V}_1 \dots \tilde{V}_m e^{-S} \rangle_{\text{top}}$$

Special focus in this talk

- string theories with extended supersymmetry (e.g. $\mathcal{N} = (4,4)$)
- open and closed string theories
- twisted correlators “as simple as possible“

Best Studied Example: $\mathcal{N} = 2$ Topological Amplitudes

In order to motivate the further discussion, let me illustrate this abstract definition by probably the best known example:

$$F_g = \langle R_{(+)}^2 T_{(+)}^{2g-2} \rangle_{g-\text{loop}} = \int_{\mathcal{M}_g} \langle \prod_{a=1}^{3g-3} |G^-(\mu_a)|^2 \rangle_{\text{top}}.$$

Antoniadis, Gava, Narain, Taylor, 1993

- g-loop correlator in type II string theory on CY_3 (insertions from $\mathcal{N} = 2$ SUGRA multiplet)
- genus g partition function of the $\mathcal{N} = 2$ (closed) topological string

Example: $\mathcal{N} = 2$ Topological Amplitudes

In more details

$$F_g = \langle R_{(+)}^2 T_{(+)}^{2g-2} \rangle_{g-\text{loop}} = \int_{\mathcal{M}_g} \langle \prod_{a=1}^{3g-3} |G^-(\mu_a)|^2 \rangle_{\text{top}}.$$

The string correlator contains two insertions of the self-dual part of the Riemann tensor

$$R_{(+)}^{\mu\nu\rho\tau} = R^{\mu\nu\rho\tau} - \frac{i}{2}\epsilon^{\mu\nu\sigma\lambda} R_{\sigma\lambda}{}^{\rho\tau},$$

and are contracted in the following manner

$$R_{(+)}^2 = R_{(+)}^{\mu\nu\rho\tau} R_{(+),\mu\nu\rho\tau}.$$

Example: $\mathcal{N} = 2$ Topological Amplitudes

In more details

$$F_g = \langle R_{(+)}^2 T_{(+)}^{2g-2} \rangle_{g-\text{loop}} = \int_{\mathcal{M}_g} \langle \prod_{a=1}^{3g-3} |G^-(\mu_a)|^2 \rangle_{\text{top}}.$$

Moreover, it holds $(2g-2)$ insertions of the self-dual part of the field strength tensor of the graviphoton

$$T_{(+)}^{\mu\nu} = T^{\mu\nu} - \frac{i}{2} \epsilon^{\mu\nu\rho\tau} T_{\rho\tau},$$

which are contracted in pairs of two

$$T_{(+)}^{2g-2} = (T_{(+)}^{\mu\nu} T_{(+),\mu\nu})^{g-1}.$$

Example: $\mathcal{N} = 2$ Topological Amplitudes

In more details

$$F_g = \langle R_{(+)}^2 T_{(+)}^{2g-2} \rangle_{g-\text{loop}} = \int_{\mathcal{M}_g} \langle \prod_{a=1}^{3g-3} |G^-(\mu_a)|^2 \rangle_{\text{top}}.$$

On the topological side we have the genus g partition function

- The G^- are sewed with the Beltrami differentials μ_a
- They provide a measure in the moduli space $\mathcal{M}_g$ of a genus g Riemann surface

Example: $\mathcal{N} = 2$ Topological Amplitudes

In more details

$$F_g = \langle R^2_{(+)} T^{2g-2}_{(+)} \rangle_{g-\text{loop}} = \int_{\mathcal{M}_g} \langle \prod_{a=1}^{3g-3} |G^-(\mu_a)|^2 \rangle_{\text{top}}.$$

To understand the G^- we start with an $\mathcal{N} = (2,2)$ SCFT spanned by the operators

$$\{T, G^\pm, J | \overline{T}, \overline{G}^\pm, \overline{J}\}.$$

The twist is performed in the following manner

Witten, 1992
Bershadsky, Cecotti, Ooguri, Vafa, 1993
Cecotti, Vafa, 1993

$$T \to T - \frac{1}{2}\partial J, \qquad \overline{T} \to \overline{T} \mp \frac{1}{2}\overline{\partial J}.$$

The sign in the right moving sector distinguishes between the A- and B-model.

Example: $\mathcal{N}=2$ Topological Amplitudes

In more details

$$F_g = \langle R^2_{(+)} T^{2g-2}_{(+)} \rangle_{g-\text{loop}} = \int_{\mathcal{M}_g} \langle \prod_{a=1}^{3g-3} |G^-(\mu_a)|^2 \rangle_{\text{top}}.$$

The absolute square is a shorthand notation to treat left- and right movers at the same time.

- A-model:

$$|G^-(\mu_a)|^2 = G^-(\mu_a)\overline{G}^+(\overline{\mu}_a),$$

- B-model:

$$|G^-(\mu_a)|^2 = G^-(\mu_a)\overline{G}^-(\overline{\mu}_a).$$

Uses of ($\mathcal{N} = 2$) Topological Amplitudes

Besides being interesting in their own right, these topological amplitudes have been of use in a number of instances

- They provide us a window to study certain aspects of (special) string amplitudes to all orders (e.g. moduli dependence)
  Bershadsky, Cecotti, Ooguri, Vafa, 1993
- The topological correlators can be computed iteratively to very high order ($g \sim 51$), Huang, Klemm, Quackenbush 2006
  Klemm, Huang, 2006
  Grimm, Klemm, Mariño, Weiss 2007
  Bershadsky, Cecotti, Ooguri, Vafa 1993
- The corresponding effective couplings on the string side usually have some interesting properties on their own (e.g. useful for black hole entropy) Ooguri, Strominger, Vafa 2004

Motivation for Extended Supersymmetry

In this talk, I will be mainly interested in amplitudes with more supersymmetry (i.e. $\mathcal{N} = 4$). There are several reasons why one might want to study such objects

- Extended amount of supersymmetry gives more possibilities to formulate amplitudes
- Interesting theories on the string side, e.g. type II on $K3 \times T^2$ or heterotic on T^6
- new higher derivative effective couplings
  - closed string couplings useful for SUGRA
  - open string couplings interesting for phenomenology

    Antoniadis, Narain, Taylor 2005
- new viewpoints on properties of the $\mathcal{N} = 2$ theories (e.g. holomorphic anomaly)

Earlier works in 6 dimensions (type II on $K3$) by Berkovits, Vafa 1994, 1998; Ooguri, Vafa 1995

Outline

1. $\mathcal{N} = 4$ Topological Amplitudes
   - $\mathcal{N} = 4$ Closed Topological Amplitudes
   - $\mathcal{N} = 4$ Open Topological Amplitudes

2. Harmonicity Relation
   - Closed String Relations
   - Open String Relations

1. $\mathcal{N} = 4$ Topological Amplitudes
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Closed $\mathcal{N} = 4$ Amplitudes: String Theory Setup

The setup on the string theory side will be characterized by

- I consider type II string theory compactified on $K3 \times T^2$ (or its dual version heterotic string on T^6).
- $\mathcal{N} = (4, 4)$ space-time supersymmetry
- From a space-time/SUGRA point of view this can be described by coupling the $\mathcal{N} = 4$ SUGRA multiplet to 22+6 vector multiplets
- I will consider couplings between the bosons of the SUGRA multiplet: Riemann tensor $R^{\mu\nu\rho\tau}_{(\pm)}$, graviphotons $T^{(ij)}_{(\pm),\mu\nu}$ and graviscalars Φ
- the moduli are vector multiplet scalars and will appear as arguments of the coupling functions.

The Twisted Theory

The world-sheet theory of type II string theory on $K3 \times T^2$ is made up from an $\mathcal{N} = (2, 2)$ SCFT on the torus

$$\{T_{T^2}, G^{\pm}_{T^2}, J_{T^2} | \overline{T}_{T^2}, \overline{G}^{\pm}_{T^2}, \overline{J}_{T^2}\}$$

along with an $\mathcal{N} = (4, 4)$ SCFT from $K3$

$$\{T_{K3}, G^{\pm}_{K3}, \tilde{G}^{\pm}_{K3}, J_{K3}, J^{\pm\pm}_{K3} | \overline{T}_{K3}, \overline{G}^{\pm}_{K3}, \overline{\tilde{G}}^{\pm}_{K3}, \overline{J}_{K3}, \overline{J}^{\pm\pm}_{K3}\}$$

Banks, Dixon 1988
Berkovits, Vafa 1994, 1998

Twisting of this theory is done by picking an $\mathcal{N} = (2, 2)$ subalgebra

$$T_{T^2} + T_{K3} \to T_{T^2} + T_{K3} - \frac{1}{2}\partial J_{T^2} - \frac{1}{2}\partial J_{K3},$$
$$\overline{T}_{T^2} + \overline{T}_{K3} \to \overline{T}_{T^2} + \overline{T}_{K3} \mp \frac{1}{2}\overline{\partial J}_{T^2} \mp \frac{1}{2}\overline{\partial J}_{K3}$$

The Topological Amplitudes

In this setup, there are two types of topological amplitudes

Antoniadis, Narain, SH, 2006

- Series $\mathcal{F}_g^{(1)}$

$$\mathcal{F}_g^{(1)} = \langle R_{(+)}^2 R_{(-)}^2 T_{(+)}^{2g-2} \rangle_{g-\text{loop}} =$$

$$= \int_{\mathcal{M}_g} \langle \prod_{a=1}^{g} |G_{T^2}^{-}(\mu_a)|^2 \prod_{b=g+1}^{3g-3} |G_{K3}^{-}(\mu_b)|^2 \int |J_{K3}|^2 \int |J_{T^2}|^2 \rangle_{\text{top}}$$

- Series $\mathcal{F}_g^{(2)}$

$$\mathcal{F}_g^{(2)} = \langle R_{(+)}^2 (\partial\partial\Phi)^2 T_{(+)}^{2g-2} \rangle_{g+1-\text{loop}} =$$

$$= \int_{\mathcal{M}_{g+1}} \langle \prod_{a=1}^{g+1} |G_{T^2}^{-}(\mu_a)|^2 \prod_{b=g}^{3g-1} |G_{K3}^{-}(\mu_b)|^2 |J_{K3}^{--}(\mu_{3g})|^2 |\psi_3(\alpha)|^2 \rangle_{\text{top}}$$

Manifestly Supersymmetric Effective Action Couplings

We want to find a manifestly supersymmetric formulation of the effective action couplings corresponding to the topological amplitudes. We therefore introduce (standard) $\mathcal{N} = (4,4)$ superspace

$$\mathbb{R}^{4|4} = \{x^\mu, \theta^i_\alpha, \bar{\theta}^{\dot{\alpha}}_i\} \qquad \text{with} \qquad i = 1\dots 4 \in SU(4) \qquad \text{R-symmetry}$$

However, this is not suitable for a consistent formulation of the couplings and therefore I introduce harmonic superspace

Galperin, Ivanov, Kalitsyn, Ogievetsky, Sokatchev 1984, 1985
Hartwell, Howe 1994, 1995

$$\mathbb{HR}^{4|4} = \{x^\mu, \theta^i_\alpha, \bar{\theta}^{\dot{\alpha}}_i, u^{+a}_i, u^{-\dot{a}}_i\}$$

where the additional harmonic variables live on the coset

$$\frac{SU(4)}{S(U(2)\times U(2))} = \{u^{+a}_i,\ u^{-\dot{a}}_i\} \quad \text{with} \quad \begin{cases} i = 1\dots 4 & \in & SU(4) \\ a, \dot{a} = 1,2 & \in & SU(2) \\ \pm & \dots & U(1) \end{cases}$$

Properties of the Harmonic Variables

The harmonic variables have some important properties

- They are complex $SU(4)$ matrices with the complex conjugates

$$\bar{u}^i{}_{+a} = \overline{(u_i^{+a})}, \qquad \text{and} \qquad \bar{u}^i{}_{-\dot{a}} = \overline{(u_i^{-\dot{a}})}$$

- They satisfy the unitarity conditions

$$u_i^{+a}\,\bar{u}^i{}_{+b} = \delta^a_b \qquad u_i^{-\dot{a}}\,\bar{u}^i{}_{-\dot{b}} = \delta^{\dot{a}}_{\dot{b}},$$

$$u_i^{+a}\,\bar{u}^i{}_{-\dot{b}} = u_i^{-\dot{a}}\,\bar{u}^i{}_{+b} = 0,$$
$$u_i^{+a}\,\bar{u}^j{}_{+a} + u_i^{-\dot{a}}\,\bar{u}^j{}_{-\dot{a}} = \delta^j_i,$$

- They satisfy the unit determinant condition

$$\epsilon^{ijkl} u_i^{+a} u_j^{+b} u_k^{-\dot{a}} u_l^{-\dot{b}} = \epsilon^{ab}\epsilon^{\dot{a}\dot{b}}$$

Harmonic Projection

For convenience we also introduce the following objects

$$u_{ij}^{++} = u_i^{+a}\epsilon_{ab}u_j^{+b},$$
$$u_{ij}^{--} = u_i^{-\dot{a}}\epsilon_{\dot{a}\dot{b}}u_j^{-\dot{b}},$$
$$u_{ij}^{a\dot{a}} = u_{[i}^{+a}u_{j]}^{-\dot{a}}.$$

Moreover, we can use the harmonic coordinates to project all $SU(4)$ indices. In particular, we can define the new Grassmann variables

$$\theta_\alpha^{+a} = \theta_\alpha^i\, u_i^{+a}, \qquad \text{and} \qquad \bar{\theta}^{\dot{\alpha}}_{-\dot{a}} = \bar{u}^i_{-\dot{a}}\,\bar{\theta}^{\dot{\alpha}}_i.$$

The measure on the harmonic superspace is given by

$$\int d\zeta^{(-4,-4)} = \int\; d^4x\; du\; d^4\theta^+ d^4\bar{\theta}_-$$

SUGRA Multiplet in Harmonic Superspace

With the help of these harmonic coordinates, we can now introduce the relevant multiplets for the couplings. We start with the SUGRA multiplet (only bosons)

$$W = \Phi + \theta^i\theta^j\sigma_{\mu\nu}T^{\mu\nu}_{(+)[ij]} + \theta^i\theta^j\theta^k\theta^l\sigma_{\mu\nu}\sigma_{\rho\tau}\epsilon_{ijkl}R^{\mu\nu\rho\tau}_{(+)} \ .$$

with the super descendant

$$K^{++}_{\mu\nu} = (\sigma_{\mu\nu})_{\alpha\beta}\bar{u}^i_{-\dot{a}}\bar{u}^j_{-\dot{b}}D^\alpha_i D^\beta_j \epsilon^{\dot{a}\dot{b}} \ W$$

It satisfies the following relations

- H-analyticity

$$D^{+b}_{-\dot{a}}K^{++}_{\mu\nu} = \left(u^{+b}_i\frac{\partial}{\partial u^{-\dot{a}}_i} - \bar{u}^i_{-\dot{a}}\frac{\partial}{\partial \bar{u}^i_{+b}}\right)K^{++}_{\mu\nu} = 0$$

- G-analyticity

$$D^\alpha_{-\dot{a}}K^{++}_{\mu\nu} = \bar{u}^i_{-\dot{a}}D^\alpha_i K^{++}_{\mu\nu} = 0 = \bar{D}^{+a}_{\dot{\alpha}}K^{++}_{\mu\nu} = u^{+a}_i\bar{D}^i_{\dot{\alpha}}K^{++}_{\mu\nu}.$$

Vector Multiplet in Harmonic Superspace

Similarly, we define the vector multiplet (only bosons)

$$Y_A^{++} = \phi_A^{ij} u_{ij}^{++} + \theta^{+a} \sigma^{\mu\nu} \theta^{+b} \epsilon_{ab} F_{(+),A\mu\nu} + \bar{\theta}_{-\dot{a}} \tilde{\sigma}^{\mu\nu} \bar{\theta}_{-\dot{b}} \epsilon^{\dot{a}\dot{b}} F_{(-),A\mu\nu}.$$

which also satisfies

- H-analyticity

$$D_{-\dot{a}}^{+b} K_{\mu\nu}^{++} = \left(u_i^{+b} \frac{\partial}{\partial u_i^{-\dot{a}}} - \bar{u}^i_{-\dot{a}} \frac{\partial}{\partial \bar{u}^i_{+b}} \right) Y_A^{++} = 0$$

- G-analyticity

$$D_{-\dot{a}}^{\alpha} K_{\mu\nu}^{++} = \bar{u}^i_{-\dot{a}} D_i^{\alpha} Y_A^{++} = 0 = \bar{D}_{\dot{\alpha}}^{+a} Y_A^{++} = u_i^{+a} \bar{D}_{\dot{\alpha}}^i Y_A^{++}.$$

Higher-Derivative Couplings

The couplings corresponding to the two topological amplitudes are then given by Antoniadis, Narain, SH, Sokatchev 2007

- Series $\mathcal{F}_g^{(1)}$

$$S_1 = \int d\zeta^{(-4,-4)} \, (\bar{K}^{++}_{\mu\nu} \bar{K}^{++\,\mu\nu}) \, (K^{++}_{\rho\sigma} K^{++\,\rho\sigma})^g \, \mathcal{F}_g^{(1)}(Y_A^{++}, u)$$

- Series $\mathcal{F}_g^{(2)}$

$$S_2 = \int d\zeta^{(-4,-4)} \, (K^{++}_{\mu\nu} K^{++\,\mu\nu})^{g+1} \, \mathcal{F}_g^{(2)}(Y_A^{++}, u)$$

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Open $\mathcal{N} = 4$ Amplitudes: String Theory Setup

For the open topological amplitudes, there are two dual setups on the string theory side

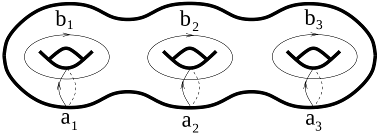
- type I on $K3 \times T^2$
  These are truly open strings and we will compute the correlators in this theory as world sheet involutions of the closed counterparts
- heterotic on $K3 \times T^2$
  This theory is dual to the previous one.

The advantage of the heterotic setup is the fact that we can still compute in the closed string channel, which saves us from dealing with the open string moduli explicitly.

Genus g Riemann Surfaces

The idea for computing open topological amplitudes will be to construct them from closed string amplitudes via world-sheet involutions. To this end, we recall the structure of a genus g Riemann surface

- The surface is endowed with a canonical homology basis of 1-cycles $(\mathbf{a}_i, \mathbf{b}_i)$, with $i = 1, \ldots, g$.



- The surface can furthermore be equipped with a set of g holomorphic 1-differentials ω_i, whose integrals over the homology cycles is given by

$$\int_{\mathbf{a}_i} \omega_j = \delta_{ij}, \qquad \text{and} \qquad \int_{\mathbf{b}_i} \omega_j = \tau_{ij}.$$

The symmetric $g \times g$ matrix τ_{ij} is called the period matrix.

$\mathbb{Z}_2$ World-sheet Involutions

From a closed Riemann surface, we can construct an open surface by viewing Σ_g as a double cover and take the quotient with respect to some involution I

$$I = \begin{pmatrix} \Gamma & 0 \\ 0 & -\Gamma \end{pmatrix},$$

Bianchi, Sagnotti, 1988
Blau, Clements, Della Pietra, Carlip, Della Pietra, 1988

with Γ a matrix enjoying the following properties

$$\Gamma^2 = \mathbb{1}, \qquad \text{and} \qquad \det\Gamma = \pm 1.$$

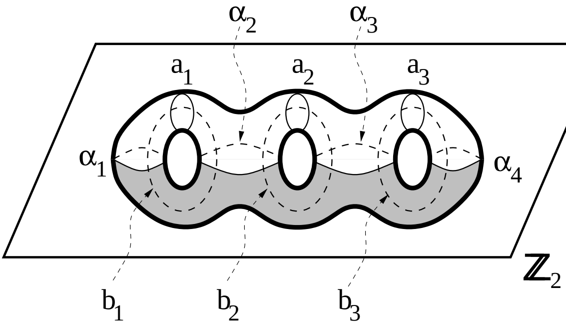
The action of I on the homology cycles is given by

$$I\mathbf{a}_i = \Gamma_{ij}\mathbf{a}_j, \qquad \text{and} \qquad I\mathbf{b}_i = -\Gamma_{ij}\mathbf{b}_j,$$

that is, I takes **a**-cycles to **a**-cycles and **b**-cycles to **b**-cycles

$\mathbb{Z}_2$ World-sheet Involutions

Pictorially, this means that the closed Riemann surface is cut into two along a plane



The fixed points in this identification will become the boundaries of the quotient Σ_g/I.

Open String Correlators

For the calculation of the correlators I additionally make the following assumptions

- We will choose $\Gamma = \mathbb{1}$ such that a maximum number of boundaries is generated. A genus g surface leads to $g+1$ boundary components.
- All boundary components have the same boundary condition

Under these assumption, the open string amplitude is given by the holomorphic part of the closed string amplitude, however, decorated by a factor

$$R = \begin{cases} \det(\mathrm{Im}\tau) & \text{Dirichlet conditions} \\ \frac{1}{\det(\mathrm{Im}\tau)} & \text{Neumann conditions} \end{cases}$$

Blau, Clements, Della Pietra, Carlip, Della Pietra, 1988

Detailed Correlators

Following the closed string results we also find two different topological amplitudes Antoniadis, Narain, SH, Sokatchev, to appear

- Series $\mathcal{F}^{(1)}_{g+1,0}$

$$\mathcal{F}^{(1)}_{(0,g+1)} = \langle F^2_{(+)} F^2_{(-)} (\lambda_\alpha \lambda^\alpha)^{g-1} \rangle =$$

$$= \int_{\mathcal{M}_{(0,g+1)}} \langle \prod_{a=1}^{g-1} G^-_{T^2}(\mu_a) \prod_{b=g}^{3g-3} G^-_{K3}(\mu_b) \int_{\partial\Sigma} J_{K3} \int_{\partial\Sigma} J_{T^2} \rangle$$

- Series $\mathcal{F}^{(2)}_{(0,g+1)}$

$$\mathcal{F}^{(2)}_{(0,g+1)} = \langle F^2_{(+)} (\partial\Phi)^2 (\lambda_\alpha \lambda^\alpha)^{g-1} \rangle =$$

$$= \int_{\mathcal{M}_{(0,g+1)}} \langle \prod_{a=1}^{g} G^-_{T^2}(\mu_a) \prod_{b=g+1}^{3g-4} G^-_{K3}(\mu_b) J^{--}_{K3}(\mu_{3g-3}) \psi_3(\alpha) \rangle,$$

Superspace Couplings in Open String Theory

Similar to the closed string amplitudes, also the open string couplings can be written in a manifest supersymmetric manner

- The couplings are part of the gauge sector of the open theory
- We use again harmonic superspace for the description

We introduce specially adapted harmonic variables

$$\frac{SU(2)}{U(1)} = \{v_i^+,\, v_i^-\} \qquad \text{with} \qquad \begin{cases} i = 1\ldots 2 & \in \quad SU(2) \\ \pm & \ldots \quad U(1) \end{cases}$$

They satisfy the relations

$$\epsilon^{ij} v_i^+ v_i^- = 1, \qquad\qquad v_i^+ v_j^- - v_i^- v_j^+ = \epsilon_{ij}.$$

Harmonic Superspace

Similar to the closed string case we can use the harmonic coordinates to project the $SU(2)$ indices of the Grassmann variables of the standard superspace

$$\theta^{\pm}_{\alpha} = \theta^{i}_{\alpha}\, v^{\pm}_{i}, \qquad \text{and} \qquad \bar{\theta}^{\dot{\alpha}}_{\pm} = \bar{\theta}^{\dot{\alpha}}_{i} \bar{v}^{i}_{\pm}.$$

The measure on the harmonic superspace is given by

$$\int d\zeta^{(-2,-2)} = \int \; d^4x \; dv \; d^2\theta^{+} d^2\bar{\theta}_{-}$$

Multiplets in Harmonic Superspace

With the help of these harmonic coordinates, we can now introduce the relevant multiplets for the open string couplings

- hyper multiplet

$$q_A^+ = f_A^i v_i^+ + \theta_\alpha^+ \chi_A^\alpha + \bar{\psi}_{A\dot{\alpha}} \bar{\theta}_-^{\dot{\alpha}} + \dots$$
$$\tilde{q}_{A-} = \bar{f}_{Ai} \bar{v}_-^i + \bar{\theta}_-^{\dot{\alpha}} \bar{\chi}_{A\dot{\alpha}} + \psi_A^\alpha \theta_\alpha^+ + \dots$$

  We can combine both of these multiplets into a single one by introducing an additional $SU(2)$ label

$$(q_A^+, \tilde{q}_{A-}) \leftrightarrow q_{Aa}^+, \qquad a = 1, 2.$$

- vector multiplet (only bosons)

$$W = \varphi + \theta_\alpha^i \lambda_i^\alpha + \theta_\alpha^i \theta_\beta^j \epsilon_{ij} F_{(+)}^{(\alpha\beta)}$$

  with the superdescendant

$$K_-^\alpha = \bar{v}_-^i D_i^\alpha\, W = \lambda_i^\alpha \bar{v}_-^i + (\sigma^\mu)^{\alpha\dot{\alpha}} \bar{\theta}_{\dot{\alpha}}^+ i\partial_\mu \varphi + \theta_\beta^+ F_{(+)}^{\alpha\beta}.$$

Higher-Derivative Couplings

The couplings corresponding to the two topological amplitudes are then given by Antoniadis, Narain, SH, Sokatchev, to appear

- Series $\mathcal{F}^{(1)}_{(0,g+1)}$

$$S_1 = \int d\zeta^{(-2,-2)}\,(\tilde{K}^+_{\dot{\alpha}}\epsilon^{\dot{\alpha}\dot{\beta}}\tilde{K}^+_{\dot{\beta}})\,(K^{\alpha}_{-}\epsilon_{\alpha\beta}K^{\beta}_{-})^g\,\mathcal{F}^{(1)}_g(q^+_{A\,a},v)$$

- Series $\mathcal{F}^{(2)}_{(0,g+1)}$

$$S_2 = \int d\zeta^{(-2,-2)}\,(K^{\alpha}_{-}\epsilon_{\alpha\beta}K^{\beta}_{-})^{g+1}\,\mathcal{F}^{(2)}_g(q^+_{A\,a},v)$$

Differential Equations

We now wish to characterize properties of the topological amplitudes in a more formal way.

- We are interested in formulating differential equations with respect to the arguments of the $\mathcal{F}_g$: the moduli (scalar fields) and the harmonic coordinates
- In contrast to the $\mathcal{N} = (2, 2)$ theory (holomorphic anomaly equation) the differential equations will be of second order instead of first order.
- For simplicity I will focus on the amplitudes $\mathcal{F}_g^{(2)}$ and $\mathcal{F}_{(0,g+1)}^{(2)}$
- The basic idea is to make use of analyticity properties of the amplitudes.

Analyticity Properties of the Topological Amplitudes

Let us recall the superspace couplings

$$\int d\zeta^{(-4,-4)}\,(K^{++}_{\mu\nu}K^{++\,\mu\nu})^{g+1}\,\mathcal{F}^{(2)}_{g}(Y^{++}_{A},u)$$
$$\int d\zeta^{(-2,-2)}\,(K^{\alpha}_{-}\epsilon_{\alpha\beta}K^{\beta}_{-})^{g+1}\,\mathcal{F}^{(2)}_{(0,g+1)}(q^{+}_{A\,a},v)$$

It is crucial to notice that in both cases the $\mathcal{F}_g$ depends only on one specific projection of the moduli, namely

$$Y^{++}_{A}=Y^{ij}_{A}u^{++}_{ij} \quad \dots \quad \text{closed string case}$$
$$q^{+}_{A\,a}=q^{i}_{A\,a}v^{+}_{i} \quad \dots \quad \text{open string case}$$

Analyticity Properties of the Topological Amplitudes

With this moduli dependence, $\mathcal{F}_g^{(2)}$ can be expanded in the following manner ($m = 2g - 2$)

$$\mathcal{F}_g^{(2)}(Y_A^{++}, u) = \sum_{n=0}^{\infty} \xi_{(i_1 \dots i_{m+n})(j_1 \dots j_{m+n})} \bar{u}_{++}^{i_1 j_1} \dots \bar{u}_{++}^{i_{m+n} j_{m+n}} \cdot \\ \cdot \phi^{(k_1 (l_1} \dots \phi^{k_n) l_n)} u_{k_1 l_1}^{++} \dots u_{k_n l_n}^{++} .$$

Using the relations fulfilled by the harmonic variables we can simplify this expression

$$\mathcal{F}_g^{(2)}(Y_A^{++}, u) = \sum_{n=0}^{\infty} \xi_{(i_1 \dots i_{m+n})(j_1 \dots j_{m+n})} \bar{u}_{++}^{i_1 j_1} \dots \bar{u}_{++}^{i_m j_m} \phi^{i_{m+1} j_{m+1}} \dots \phi^{i_{m+n} j_{m+n}} .$$

Harmonicity Relation

The symmetries of this expansion suggests the following two relations

- harmonicity relation

$$\epsilon^{ijkl}\frac{\partial}{\partial \bar{u}_j^{+a}}\frac{\partial}{\partial \phi_A^{kl}}\mathcal{F}_g^{(2)} = 0.$$

- second order constraint

$$\epsilon^{ijkl}D_{ij}^A D_{km}^B \mathcal{F}_g^{(2)} = 0.$$

Anomalies

In fact, checking these relations explicitly on the string theory side one finds that they are not quite satisfied but are violated in a rather subtle way. To be precise Antoniadis, Narain, SH, Sokatchev 2007

- harmonicity relation

$$\epsilon^{ijkl}\frac{\partial}{\partial \bar{u}_j^{+a}}\frac{\partial}{\partial \phi_A^{kl}}\mathcal{F}_g^{(2)} = (2g-2)\bar{u}^i_{+a}\frac{\partial}{\partial \phi_A^{++}}\mathcal{F}_{g-1}^{(2)}.$$

  The right hand side depends on the same $\mathcal{F}_g^{(2)}$ as the left hand side, however, of lower genus. We can therefore interpret the contribution as an anomaly, stemming from divergences at the boundary of $\mathcal{M}_g$.

- second order constraint

$$\epsilon^{ijkl}D^A_{ij}D^B_{km}\mathcal{F}_g^{(2)} = 4D^A_{++}D^B_{++}\mathcal{F}_{g-1}^{(2)} - 4(g+1)\delta^{AB}\mathcal{F}_g^{(2)}.$$

  Besides the anomaly, there is also the term proportional to δ^{AB}. It plays the role of a connection term owing to the fact that the space of the ϕ^{ij} is not flat.

1. $\mathcal{N} = 4$ Topological Amplitudes
   - $\mathcal{N} = 4$ Closed Topological Amplitudes
   - $\mathcal{N} = 4$ Open Topological Amplitudes

2. Harmonicity Relation
   - Closed String Relations
   - Open String Relations

$\mathcal{N} = (4,0)$ Relations

The relations for the open string couplings can be obtained in exactly the same manner and I will therefore only state the results

- harmonicity relation

$$\epsilon^{ij} \frac{\partial^2 \mathcal{F}^{(2)}_{(0,g)}}{\partial u_i^+ \partial f^j_{Aa}} = -(2g-1)(2g-2) C_{\text{sphere}} \cdot \cdot \sum_{g_1+g_2=g} D_+^{Aa} D_+^{Bb} F^{(2)}_{(0,g_1+1)} \epsilon_{bc} D_+^{Bc} F^{(2)}_{(0,g_2+1)} .$$

- second order relation

$$\epsilon^{ij} D_i^{Aa} D_j^{Bb} \mathcal{F}^{(2)}_{(0,g+1)} = 2g \delta^{AB} \epsilon^{ab} F^{(2)}_{(0,g+1)} + \text{anomaly}.$$

Conclusions

In this seminar I have reviewed the status of (open and closed) topological amplitudes in theories with extended supersymmetry. In particular I have

- established the precise relation to the topological correlators
- determined the corresponding effective action couplings in harmonic superspace
- uncovered special properties of the moduli dependence of these couplings by establishing two types of second order differential equations