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Defects in Classical and Quantum WZW models

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joint work with

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Outline

- Defects in classical sigma models
- Jump defects in WZW models
- Defects in conformal quantum field theory & defect junctions
- Comparison of classical and quantum WZW model with jump defects
- Orbifolds

Applications of defects in 2d CFT

- (continuum limit of) order-disorder duality in lattice models

- perturbative symmetries of string theory, in particular T-duality

- functors between D-brane categories

  . . .

Classical σ-models

world sheet Σ (metric γ)

target space M (metric G , closed 3-form H)

$X : \Sigma \to M$ once differentiable

$S[X] = S_{kin}[X] + S_{top}[X]$

Topological term

If H = dB : $S_{\text{top}}[X] = i\int_\Sigma X^* B$

If $[H] \in H^3(M, 2\pi\mathbb{Z})$: $\exp(-S_{\text{top}})$

$$= \text{“} \exp\left(-i\int_\Sigma X^* \mathcal{G} \right) \text{”}$$

gerbe

Alvarez ‘85
Gawedzki ‘88
Brylinski ‘93

U_i : good open cover of M

$\mathcal{G}$ = (B_i : 2-form on U_i , $dB_i = H$

A_{ij} : 1-form on $U_i \cap U_j$

g_{ijk} : function $U_i \cap U_j \cap U_k \to U(1)$)

<u>Boundaries</u>

Target space : D-brane $\iota : D \hookrightarrow M$

2-form ω on D s.t. $\iota^* H = d\omega$

If H=dB : twisted line bundle with connection ∇ on D ,
ω = B + F

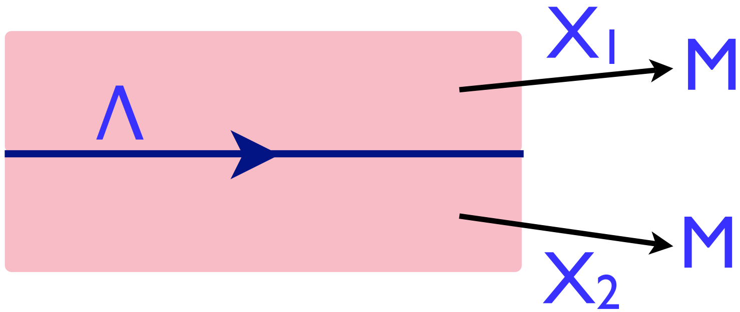
(bulk action) + log Hol_∇(X(∂Σ))

Gawedzki '99, Kapustin '99
Carey, Johnson, Murray '02

General: gerbe module on D

= stable isomorphism $\Phi : \iota^* \mathcal{G} \longrightarrow I_\omega$ on D

Defects



Fuchs, Schweigert, Waldorf '07

Target space : "bi-brane" $(\iota_1, \iota_2) : Q \hookrightarrow M \times M$

2-form ω on Q s.t. $\iota_1^* H - \iota_2^* H = d\omega$

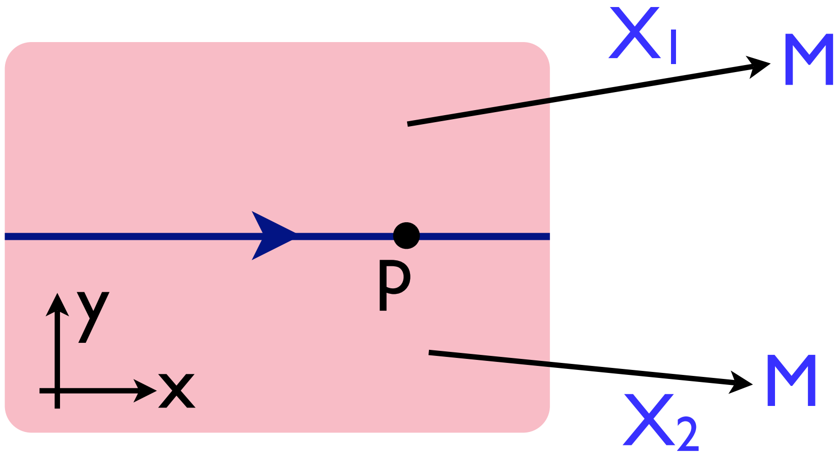
If H=dB : twisted line bundle with connection ∇ on Q,
$\omega = \iota_1{}^*B - \iota_2{}^*B + F$

(bulk action) + log $\mathrm{Hol}_\nabla((X_1,X_2)(\Lambda))$

General: gerbe bimodule on Q

= stable isomorphism $\Phi : \iota_1^* \mathcal{G} \to \iota_2^* \mathcal{G} \star I_\omega$ on Q

Defect condition for fields



1) $(X_1(p), X_2(p)) \in Q \subset M \times M$

2) for all $v_1 \oplus v_2 \in T_{(X_1,X_2)}Q$

Suszek, IR '08

$$G_{X_1(p)}(v_1, \partial_y X_1) - G_{X_2(p)}(v_2, \partial_y X_2)$$
$$= \frac{i}{2}\omega_{(X_1,X_2)(p)}(v_1 \oplus v_2, \partial_x (X_1, X_2))$$

. . . defect condition for fields

$$G_{X_1(p)}(v_1, \partial_y X_1) - G_{X_2(p)}(v_2, \partial_y X_2)$$
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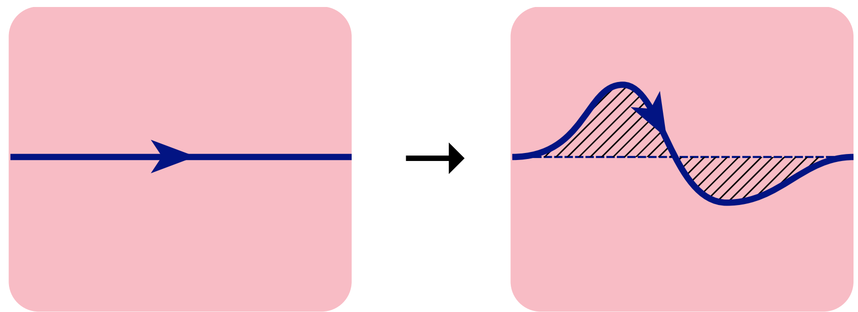
- boundary term in variation of action
- world sheet boundary : usual mixed D/N condition
- with $T = G(\partial X, \partial X)$, $\bar{T} = G(\bar{\partial} X, \bar{\partial} X)$:

  $T_1(p) - \bar{T}_1(p) = T_2(p) - \bar{T}_2(p)$ on defect

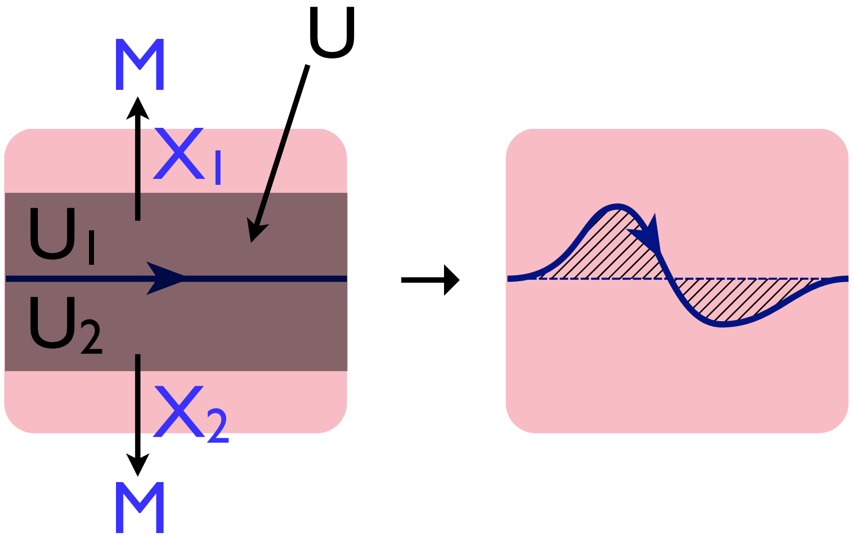
  $\rightarrow$ defect is conformal

Moving defects



Moving defects

Choose



$$\widehat{X} : U \to Q \subset M \times M$$

such that

1) $\iota_1 \circ \widehat{X}|_{\bar{U}_1} = X_1 \;,\; \iota_2 \circ \widehat{X}|_{\bar{U}_2} = X_2$

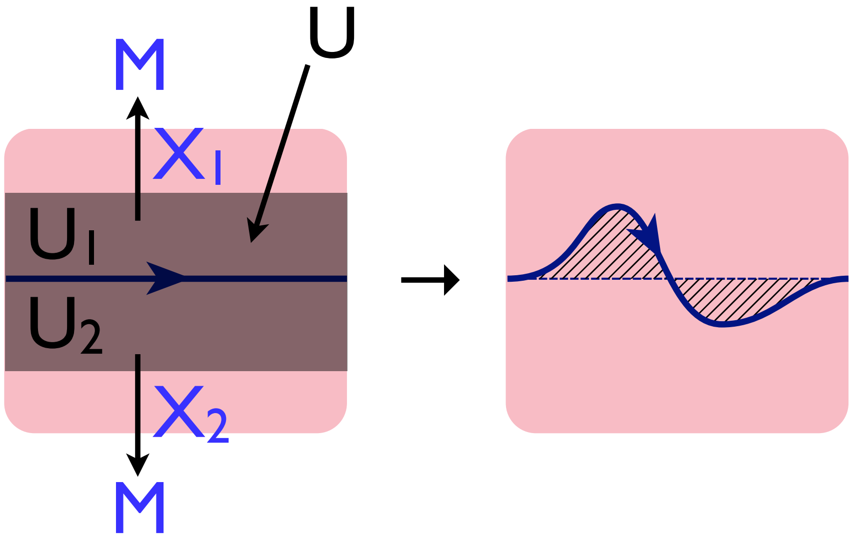
2) for all $v \in T_{\hat{X}(q)} Q$, q ∈ U , u_1,u_2 oriented ON-basis at q

$$\Delta G_{\hat{X}(q)}(v, \widehat{X}_* u_2) = \frac{i}{2} \omega_{\hat{X}(q)}(v, \widehat{X}_* u_1)$$

where $\Delta G = \iota_1^* G - \iota_2^* G$

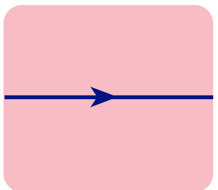
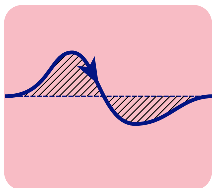
. . . moving defects

$$\Delta G_{\widehat{X}(q)}(v, \widehat{X}_* u_2)$$
$$= \frac{i}{2}\omega_{\hat{X}(q)}(v, \widehat{X}_* u_1)$$



- if $\widehat{X}$ exists it is unique
- $T_1 = T_2$ and $\bar{T}_1 = \bar{T}_2$ on defect line

  → defect is topological

- S[] = S[]

Jump defects in WZW model

M = G : compact, simple, connected and simply connected Lie group

H : Cartan 3-form

$\mathcal{G}$: basic gerbe on G

Gawedzki, Reis '02
Meinrenken '02

$k \in \mathbb{Z}_{>0}$: level

use gerbe $\mathcal{G}^{\star k}$, curvature k · H

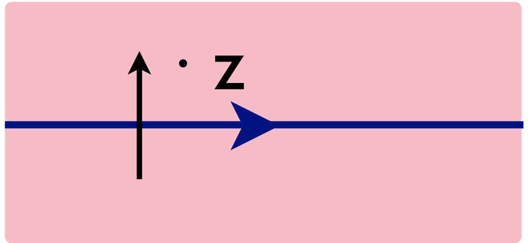
. . . jump defects in WZW model

Z(G) : centre of G

jump defects : for z ∈ Z(G)

$$Q_z = \{(zg, g) \,|\, g \in G\}$$

$$\omega = 0$$



stable isomorphism $\iota_1^* \mathcal{G}^{\star k} \to \iota_2^* \mathcal{G}^{\star k}$ from Gawedzki, Reis '03

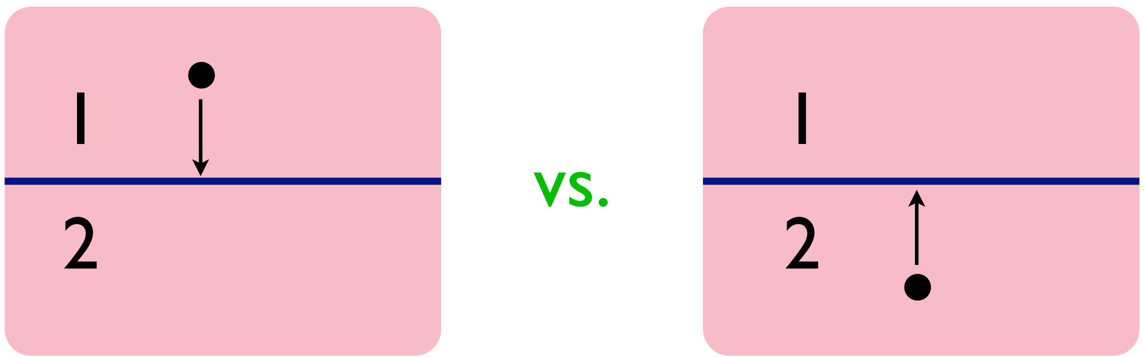
Defects in 2d conformal quantum field theory

Afflek, Oshikawa '96

Conformal defects $T_1(p) - \bar{T}_1(p) = T_2(p) - \bar{T}_2(p)$

Petkova, Zuber '00

Topological defects $T_1 = T_2 \qquad \bar{T}_1 = \bar{T}_2$



Defects in quantum WZW model

g : (complexified) Lie algebra of Lie group G

$\widehat{g}_k$: affine Lie algebra at level k

$\widehat{V}_\lambda$: irreducible highest weight representations, $\lambda \in P_k^+$

Space of states on the circle $\mathcal{H} = \bigoplus_{\lambda \in P_k^+} \widehat{V}_\lambda \otimes_{\mathbb{C}} \widehat{V}_{\bar{\lambda}}$

(Fusion product $\widehat{V}_\lambda \,\hat{\otimes}\, \widehat{V}_\mu \cong \bigoplus_{\nu \in P_k^+} (\widehat{V}_\nu)^{\oplus N_{\lambda\mu}^{\nu}}$)

. . . defects in quantum WZW model

Conformal defects :

Bachas, de Boer, Dijkgraaf, Ooguri '01
Quella, Schomerus '02
Bachas, Gaberdiel '04
Quella, Watts, IR '06
Bachas, Brunner '07

difficult, mostly examples

full set only known for $\widehat{su}(2)_1$

Fuchs, Gaberdiel, Schweigert, IR '07

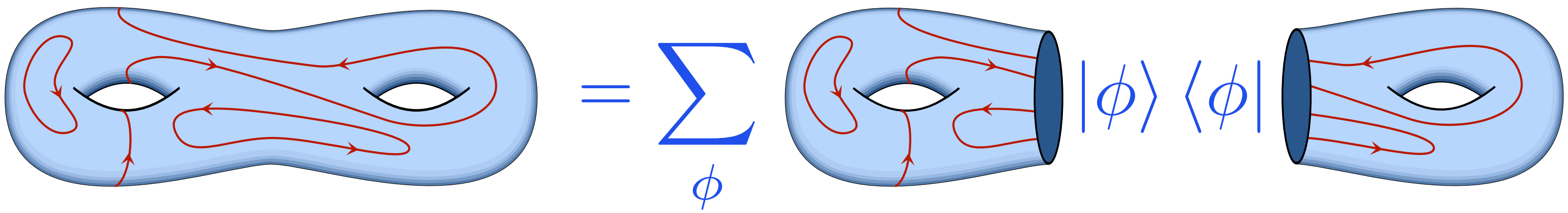
Topological defects preserving $\widehat{g}_k \oplus \widehat{g}_k$

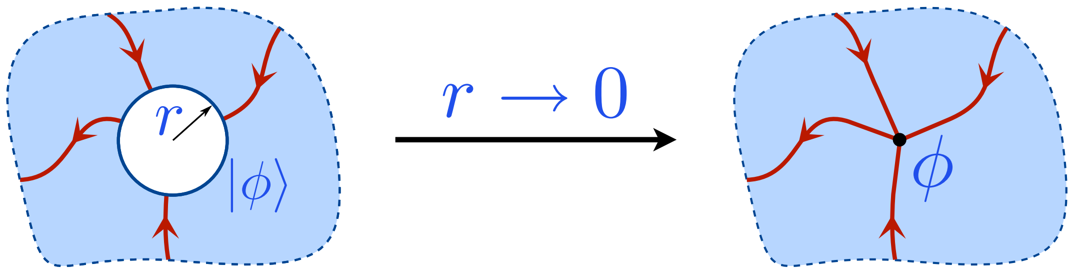
Petkova, Zuber '00
Fröhlich, Fuchs, Schweigert, IR '06

all known

$\equiv$ integrable highest weight reps. of $\widehat{g}_k$

Defect junctions



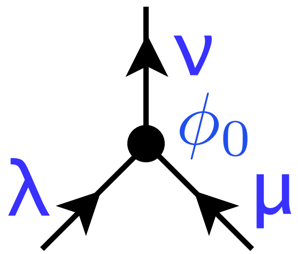


Twisted state space

$$\mathcal{H}_{\lambda_1\cdots\lambda_n} = \bigoplus_{\alpha,\beta\in P_k^+} \left(\widehat{V}_\alpha \otimes_{\mathbb{C}} \widehat{V}_\beta\right)^{\oplus N_{\lambda_1\cdots\lambda_n\,\alpha\beta}{}^0}$$

. . . defect junctions

Remarks:

1) 

Φ₀ with left/right conformal weight 0

parametrised by $\phi_0 \in \mathrm{Hom}(\,\widehat{V}_\lambda \hat{\otimes} \widehat{V}_\mu\,,\, \widehat{V}_\nu\,)$

2) jump defects ≡ simple currents

$$Z(G) \longrightarrow P_k^+ \qquad \widehat{V}_{\lambda_x} \hat{\otimes} \widehat{V}_{\lambda_y} \;\cong\; \widehat{V}_{\lambda_{xy}}$$
$$z \longmapsto \lambda_z$$

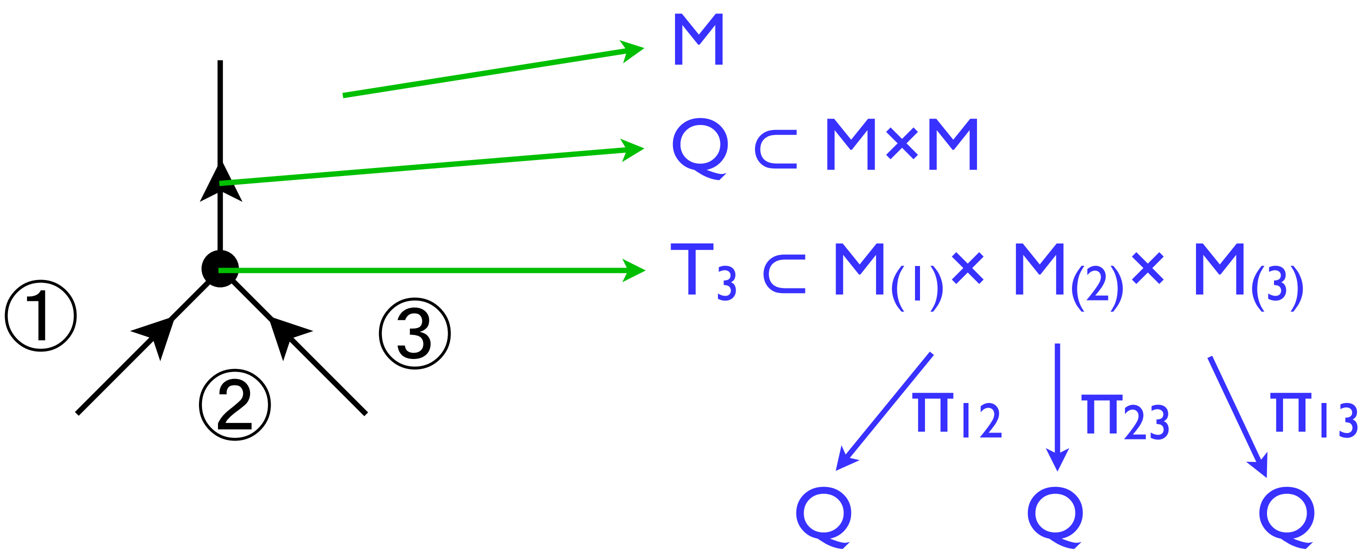
Fuchs '91

Z(G) iso to simple current group (except $\widehat{e}(8)_2$)

. . . defect junctions

3) in the classical theory

Suszek, IR '08



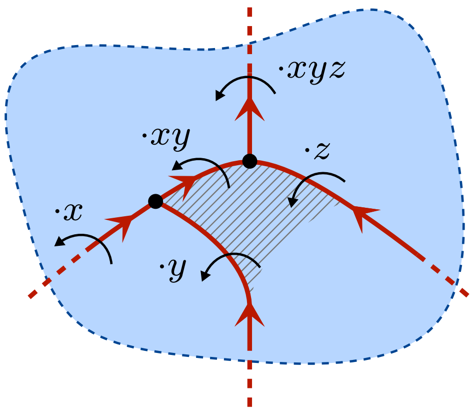
such that $\pi_{12}^*\omega + \pi_{23}^*\omega = \pi_{13}^*\omega$ on T_3

twisted scalar field φ with values in U(1)

≡ 2-morphism $\varphi : (\pi_{23}^*\Phi \star id) \circ \pi_{12}^*\Phi \Longrightarrow \pi_{13}^*\Phi$

Classical and quantum jump defects

G compact, simple, connected and simply connected Lie group, centre Z(G)

Σ_L = 

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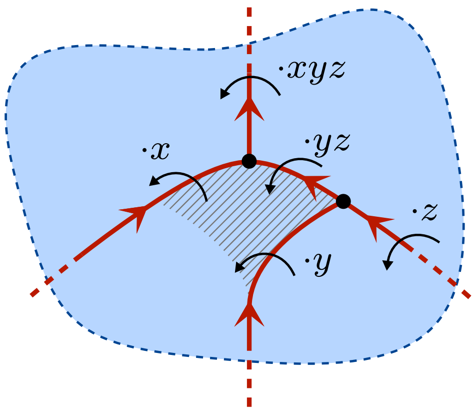
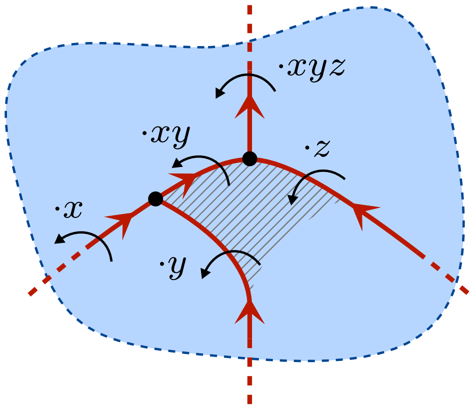
Σ_R = 

Pick $X_L : \Sigma_L \to G$,

fix $X_R : \Sigma_R \to G$ as X_L outside shaded region and

$y \cdot X_L$ inside shaded region

. . . classical and quantum jump defects

$\Sigma_L =$ 

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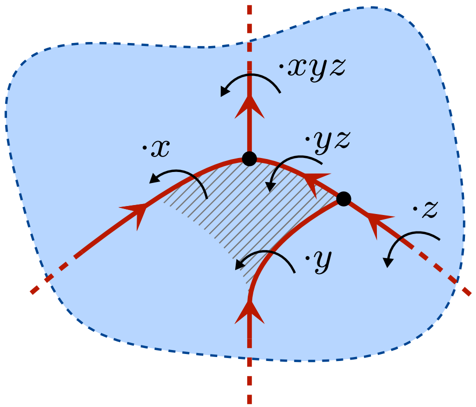
$\Sigma_R =$ 

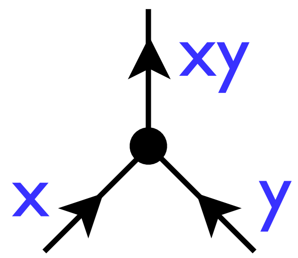
classical $\quad e^{-S[\Sigma_L, X_L]} = \psi_{\mathcal{G}^{\star k}}(x,y,z)\, e^{-S[\Sigma_R, X_R]}$

(geometric calculation using gerbe data from Gawedzki, Reis '03)

quantum $\quad \mathrm{Corr}_{\Sigma_L} = \psi_{\widehat{g}_k}(x,y,z)\, \mathrm{Corr}_{\Sigma_R}$

(representation theory of $\widehat{g}_k$, quantum 6j symbols restricted to simple current sector)

. . . classical and quantum jump defects



$\psi_{\mathcal{G}^{\star k}}$ and $\psi_{\widehat{g}_k}$

- are defined up to $\lambda_{x,y} \in U(1)$

$$\psi(x,y,z) \;\mapsto\; \psi(x,y,z) \cdot \frac{\lambda_{y,z}\lambda_{x,yz}}{\lambda_{xy,z}\lambda_{x,y}}$$

- obey

$$\frac{\psi(y,z,w)\,\psi(x,yz,w)\,\psi(x,y,z)}{\psi(xy,z,w)\,\psi(x,y,zw)} \;=\; 1$$

- contain information independent of choices

$$[\psi] \in H^3(\,Z(G)\,,\,U(1)\,)$$

. . . classical and quantum jump defects

- find that, for all (compact, simple, connected and simply connected) Lie groups G and levels $k \in \mathbb{Z}_{>0}$

$$[\psi_{\mathcal{G}^{\star k}}] \;=\; [\psi_{\widehat{g}_k}]$$

. . . classical and quantum jump defects

Comments:

1) discrete symmetry group S of CFT implemented by defects $\rightarrow$ $[\psi] \in H^3(S, U(1))$

2) $[\psi]$ gives obstruction to orbifolding by S
(classically : equivariant structure on gerbe
quantum : consistent 3-string interactions)

Can orbifold by S if and only if $[\psi] = 1$.

. . . classical and quantum jump defects

Gawedzki, Reis '02 '03

3) Relation $[\psi_{\mathcal{G}\star k}] \;=\; [\psi_{\widehat{g}_k}]$ already partially known
from orbifolding obstruction: Fix S ⊂ Z(G),
then the values k for which $[\psi_{\mathcal{G}\star k}|_S] = 1$
are precisely those for which $[\psi_{\widehat{g}_k}|_S] = 1$.

Orbifolds

Take superposition of symmetry generating defects

e.g.

(classical)

$$Q = \bigcup_{z \in Z(G)} Q_z \subset G \times G$$

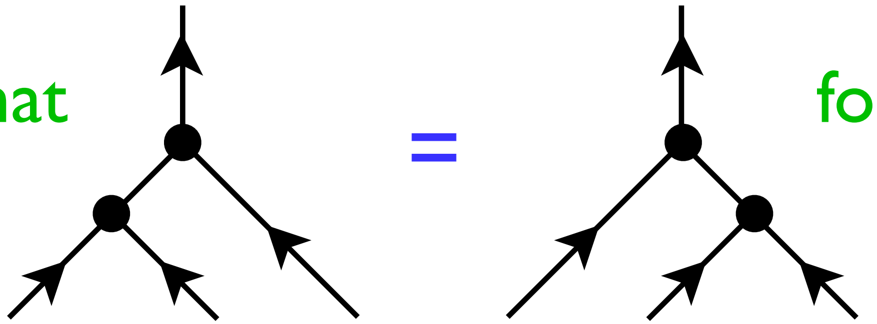
(quantum)

$$B = \bigoplus_{z \in Z(G)} \widehat{V}_{\lambda_z}$$

if [ψ] = 1 choose

- twisted scalar φ on T_3 (classical)
- $\varphi \in \mathrm{Hom}(B \hat{\otimes} B, B)$ (quantum)

such that



for e^{-S} (classical)

Corr (quantum)

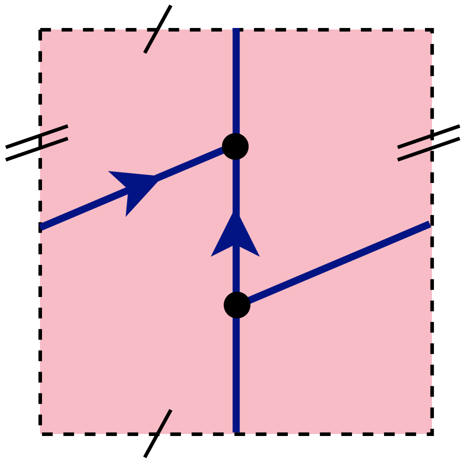
+ non-degeneracy condition

. . . orbifolds

orbifold amplitude:

embed fine enough defect network

e.g. torus:



. . . orbifolds

In CFT:

- $B = \bigoplus_{z \in Z(G)} \widehat{V}_{\lambda_z}$ is special case

- in general: **B** integrable highest weight rep of $\widehat{g}_k$
  with associative $\varphi \in \mathrm{Hom}(B \hat{\otimes} B, B)$
  + non-degeneray condition

Fuchs, Schweigert, IR '02

- get 'generalised' orbifold
  e.g. for E7 invariant of $\widehat{su}(2)_{16}$ take
  $$B = \widehat{V}_{(0)} \oplus \widehat{V}_{(8)} \oplus \widehat{V}_{(16)}$$

. . . orbifolds

In fact:

Kong, IR '08

- all CFTs well-defined at genus 0 and 1, with
  - $\widehat{g}_k \oplus \widehat{g}_k$ symmetry
  - unique vacuum state
  - non-degenerate two-point function

  are orbifolds of the charge-conjugate theory in above sense.

- holds for rational vertex operator algebras

Summary

- defects in classical sigma models
- classical and quantum defect junctions
- 3-cocycle from symmetry implemented by defects
- 3-cocycle agrees for jump defects in classical and quantum WZW models
- relation to orbifolds