Non-perturbative strings from automorphic forms

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Talk based on:

two papers with Ling Bao, Axel Kleinschmidt, Daniel Persson and Boris Pioline:

- *"Instanton Corrections to the Universal Hypermultiplet and Automorphic Forms on SU(2,1)"*
  `arXiv:0909.4299 [hep-th]` in CNTP (Comm. in Number Theory and Physics)
- *"Rigid Calabi-Yau threefolds, Picard Eisenstein series and instantons"*
  `arXiv:1005.4848 [hep-th]`

Introduction and Content

Introduction:

- In QFT perturbative and non-perturbative (instanton) corrections are calculated one by one.
- In string theory there are organizing principles giving such corrections summed up into closed functions so called automorphic functions! [Green, Gutperle]
- Such principles stem in string/M theory from the appearance of double cosets $G(\mathbf{Z})\backslash G/K$.

Here we discuss various implications for the string effective action:

- exact dependence on moduli of certain higher derivative terms (some comments)
- at the two-derivative level, an example of a sigma model with all possible quantum corrections included (our work)

Higher derivative terms

Moduli dependence of higher derivative terms:
in type IIB 10d with $\tau = \chi + ie^{-\phi}$ [Green, Gutperle, Russo, Vanhove]

- $\mathcal{E}_{3/2}(\tau)R^4$: two "constant terms" (i.e. perturbative)
- $\mathcal{E}_{5/2}(\tau)\partial^4 R^4$: two "constant terms"
- $\mathcal{E}^{(10)}_{(0,1)}(\tau)\partial^6 R^4$: inhomogeneous "Eisenstein" giving four "constant terms" plus other "constant" terms
- $\partial^n R^4$ with $n = 8$ and higher: no results like these!
- After comp to lower dimensions: limits to one higher spacetime dimension, pert string and M-theory limits can be obtained using a slight variation of the construction of these Eisenstein functions. [Pioline], [Green, Russo, Vanhove, Miller]

Our approach

An example of a non-trivial exact sigma model: UHM

- Consider 4d supergravity theories with 2 supersymmetries obtained from the type IIA superstring compactified on Calabi-Yau 3-folds (CY3).
- Restrict to rigid CY3 and derive the relevant function, i.e. the Eisenstein series, for the universal hypermultiplet.
- Fourier expand this function and compare the result to the structure of the various types of known perturbative and non-perturbative quantum corrections.
- This will involve NS5 brane instanton contributions which are complicated and poorly understood in this context (for some recent results in IIA see [Persson, Pioline]).

Type II supergravity on CY3: common sector

Compactification on a CY3 gives $N = 2$ supergravity in 4d!
Start from low-energy type IIA and IIB supergravity in 10d:
the bosonic sector

- NS-NS sector common to both cases: $g_{MN},\ B_{MN},\ \phi$
- The Hodge numbers $h_{0,0} = 1,\ h_{1,1} \geq 1,\ h_{2,1} \geq 0,\ h_{3,0} = 1$ imply the following massless fields in four dimensions:
  - $g_{\mu\nu},\ g_{ij}\,(2h_{2,1}\ scalars),\ g_{i\bar{j}}\,(h_{1,1}\ scalars)$
  - $B_{\mu\nu},\ B_{i\bar{j}}\,(h_{1,1}\ scalars)$: Note that $B_{\mu\nu}$ will be dualized to ψ.
  - ϕ
  - no isometries means $h_{1,0} = 0$ and thus no vectors in this sector
  - also $h_{2,0} = 0$ means no massless fields from B_{ij}

Summary: the NS-NS sector provides in 4d Minkowski

- one metric and two scalar (ϕ, ψ)
- plus $2h_{2,1} + 2h_{1,1}$ scalars, i.e. the geometric moduli of CY3 (with the Kahler ones complexified)

Type II supergravity on CY3: the IIA RR sector

Additional physical moduli from embedding CY3 in supergravity:

- R-R sector type IIA: $C_M, \ C_{MNP}$ give in 4d
  - C_μ $(one\ graviphoton\ from\ h_{0,0} = 1)$
  - $C_{\mu\, i \bar{j}}$ $(h_{1,1}\ vectors)$
  - $C_{i j \bar{k}}$ $(2h_{2,1}\ scalars)$
  - $C_{i j k}$ $(2\ scalars\ from\ h_{3,0} = 1)$–>denoted χ and $\tilde{\chi}$
- Summary of $\mathcal{N} = 2$ multiplets in type IIA:
  - one graviton multiplet
  - one tensor mult. = the univ. hypermult. (UHM): ϕ, ψ, χ and $\tilde{\chi}$
  - $h_{1,1}$ vector multiplets
  - $h_{2,1}$ hypermultiplets
- => Moduli space: $\mathcal{M}_A = \mathcal{M}_A^{VM}(2h_{1,1}) \times \mathcal{M}_A^{HM}(4h_{2,1} + 4)$
- which is a product up to discrete groups (follows from susy and holonomy arguments, see e.g. [Aspinwall])

Type II supergravity on CY3: the IIB RR sector

Similar statements true for type IIB:

- R-R sector type IIB: $C,\ C_{MN},\ C^{+}_{MNPQ}$ give in 4d
  - C (one scalar)
  - $C_{\mu\nu}\,(one\,tensor),\ C_{i\bar{j}}\,(h_{1,1}\,scalars)$
  - $C_{\mu\nu\,i\bar{j}}\,(h_{1,1}\,scalars),\ C_{\mu\,ij\bar{k}}\,(2h_{2,1}\,vectors),$ $C_{\mu\,ijk}\,(h_{3,0}=1\,vector,\ not\,2\,since\,selfdual),$ $C_{ij\bar{k}\bar{l}}\,(h_{2,2}\,scalars,\ but\,not\,counted\,due\,to\,selfduality)$
- Summary type IIB:
  - one graviton multiplet, one tensor multiplet (UHM)
  - $h_{2,1}$ vector multiplets, $h_{1,1}$ hypermultiplets
- => Moduli space: $\mathcal{M}_B = \mathcal{M}_B^{VM}(2h_{2,1}) \times \mathcal{M}_B^{HM}(4h_{1,1}+4)$

Comments

Comments:

- The moduli spaces are in general complicated: $N = 2$ susy not enough to make them coset spaces, but we know that
  - $\mathcal{M}^{VM}$ is a 2n-dimensional special Kahler (SK) space which is Kahler
  - $\mathcal{M}^{HM}$ is a 4n-dimensional quaternionic-Kahler (QK) space which is NOT Kahler in general
- torus compactifications give more susy and moduli spaces which are coset spaces (see e.g. [Aspinwall])

Simplifications arise if we consider rigid CY3's:

[Cecotti, Ferrara, Girardello]

- They have $h_{2,1} = 0$
- There are a number of examples of such CY3's (see e.g. [Yui])
- Type IIA case gives then $\mathcal{M}_A^{UHM} = SU(2,1)/(SU(2) \times U(1))$

This is the case we will study here:

- $\mathcal{M}_A^{UHM}$ is homogeneous and Kahler, a nice exception!

Effective actions

In the effective action the metric on $\mathcal{M}^{UHM}$

- in perturbation theory is known to have corrections only at one-loop
  [Antoniadis et al],[Strominger] [Anguelova, Rocek, Vandoren]
- non-perturbative corrections are due to D2- and NS5-brane instantons [Becker, Becker, Strominger]

Compare to the D(-1) instanton corrections to IIB in 10d [Green, Gutperle]

- the relevant double coset is $SL(2;\mathbf{Z})\backslash SL(2,\mathbf{R})/U(1)$
- the R^4 is multiplied by the non-holomorphic Eisenstein series $\mathcal{E}_s(\tau,\bar{\tau}) = \Sigma'_{(m,n)} \frac{(Im\tau)^s}{|m+\tau n|^{2s}}$, with $s = \frac{3}{2}$
  - the instanton action is $S^{p=mn}_{D(-1)} = 2\pi|mn|e^{-\phi} - 2\pi imnC_0$
  - it has two "constant terms" (tree and 1-loop)

Instanton corrected Type IIA supergravity on rigid CY3

The non-perturbative type IIA effective action in 4d on a rigid CY3 depends on

- the double coset $SU(2,1;\mathbf{Z}[i])\backslash SU(2,1)/(SU(2)\times U(1))$ where the gaussian integers $\mathbf{Z}[i]=m+in$, with m,n integers,
- and an associated Eisenstein function living on this double coset (the fundamental domain= moduli space).

Note:

- the assumption that $SU(2,1;\mathbf{Z}[i])$ is the correct discrete group is based on:
  - the non-abelian Heisenberg group of $SU(2,1)$ is known to be relevant [Becker, Becker]
  - electric-magnetic duality [Becker, Becker]
  - the assumption that $SL(2,\mathbf{Z})$ is a subgroup acting on $\chi+ie^{-\phi}$

Instanton corrected Type IIA supergravity on rigid CY3

A complication:
The quantum corrected σ-model metric is no longer homogeneous and Kahler!
How do we determine its exact metric when it is not Kahler but only quaternionic?

- Use the twistor space $\mathcal{Z}_{\mathcal{M}_{UHM}}$ which is Kahler and project back to $\mathcal{M}_{UHM}$ [Alexandrov, Pioline, et al]

Back to our problem:
Given the double coset the Eisenstein function can be obtained

- as a Poincare series,
- from an adelic construction and spherical vectors,
- or —>

Construction of the Eisenstein function

Given a double coset there is a standard way to get the Eisenstein function [Obers, Pioline] applied here to the UHM:

- $\mathcal{E}_s(\mathcal{K}) := \Sigma'_{\omega^\dagger \cdot \eta \cdot \omega = 0} (\omega^\dagger \cdot \mathcal{K} \cdot \omega)^{-s}$
  - ω is a non-zero 3-dim vector of gaussian integers $Z[i]$
  - $\mathcal{K}$ is the metric on the coset $SU(2,1)/(SU(2) \times U(1))$ obtained from the Iwasawa decomposition of $SU(2,1)$: $\tilde{\mathcal{K}} = \tilde{\nu}\tilde{\nu}^\dagger = \mathcal{K} - \eta$
  - the constraint on the summation is needed to make the Eisenstein series an eigenfunction of the Laplace operator on the coset space
  - the coset is similar to the upper half plane $SL(2,R)/U(1)$ with a metric involving $Im(\tau) > 0$: here we have R^3 x R^+ denoted $\mathbf{CH}^2$ with a metric involving $\mathcal{F}(z_1, z_2) = Im(z_1) - \frac{1}{2}|z_2|^2 > 0$
  - z_1, z_2 are related to the *UHM* fields $\phi, \psi, \chi, \tilde{\chi}$
  - it has also zero third Casimir => in the principle discrete series of $SU(2,1)$ (see [Bars, Teng])

Fourier expansion of the Eisenstein function: tree level

Dividing the lattice summation (over six integers) into sectors =>

- $\mathcal{E}_s = \mathcal{E}_s^{const} + \mathcal{E}_s^{abelian} + \mathcal{E}_s^{non-abelian}$ which for $s = 3/2$:
  - $\mathcal{E}_s^{const}(\phi)$ = the tree and 1-loop terms
  - $\mathcal{E}_s^{abelian}(\phi, \chi, \bar{\chi})$: encodes D2-brane instantons
  - $\mathcal{E}_s^{non-abelian}(\phi, \chi, \bar{\chi}, \psi)$: encodes NS5-brane instantons and bound states with D2-brane instantons
- This sum of terms is obtained by first doing the $\omega_3 = 0$ part of the sum over six integers not all zero, which means summing only over ω_1 since the constraint sets also $\omega_2 = 0$ =>
- the tree level term equals $4\zeta_{\mathbb{Q}[i]}(s)e^{-2s\phi}$, where ζ is the Dedekind zeta function $\frac{1}{4}\Sigma'_{(m,n)}\frac{1}{(m^2+n^2)^s}$
- the relevant value is $s = \frac{3}{2}$ so the corrections enter via $e^{\phi}\mathcal{E}_s$

Fourier expansion of the Eisenstein function: the perturbative term at one loop

The structure of the perturbative terms and Langlands functional relation: Turning to the terms with ω_3 non-zero

- we need to solve the constraint $n_1^2 + n_2^2 + 2m_1p_2 - 2m_2p_1 = 0$: use Bezout's identity $m_1p_2 - m_2p_1 = d$ (for which solutions m_i exist only if $d = gcd(p_1, p_2)$)
- solving the constraint expresses m_1, m_2 in terms of m, d
- after a Poisson resummation in m–> $\tilde{m}$ the non-abelian term comes from the sum with $\tilde{m}$ non-zero
- the $\tilde{m} = 0$ then needs another resummation involving n_1, n_2 which produces two new integers l_1, l_2 to sum over: First: l_1, l_2 both zero
  - provides the second constant term i.e. the one-loop term: $4\zeta_{\mathbb{Q}[i]}(s)\frac{\mathfrak{Z}(2-s)}{\mathfrak{Z}(s)}e^{-2(2-s)\Phi}$
  - with $s = \frac{3}{2}$ we get $e^{(-\phi)}$ and thus no ϕ dependence in $e^{\phi}\mathcal{E}_s$

Fourier expansion of the Eisenstein function: the non-perturbative terms

- Secondly: l_1, l_2 not both zero: the abelian non-perturbative terms:
  $$e^{-2\phi} \sum'_{(\ell_1,\ell_2)\in\mathbb{Z}^2} C^{(A)}_{\ell_1,\ell_2}(s)\, K_{2s-2}\left(2\pi e^{-\phi}\sqrt{\ell_1^2+\ell_2^2}\right) e^{-2\pi i(\ell_1\chi+\ell_2\tilde{\chi})}$$
  - encodes the effects of D2-brane instantons with charges (l_1, l_2)
  - expanding the Bessel function at weak coupling gives the D-brane instanton action
  - computing the coefficients C gives instanton measures involving double sums over gaussian divisors which generalize previously found D2 single sum instanton measures
- the non-abelian terms involve Hermite and Whittaker functions ([Ishikawa]) and are much more complicated due to the non-abelian structure of the Heisenberg group (ψ gives a twisted bundle over $\chi, \tilde{\chi}$ which can be seen from the σ-model metric)
- the NS5 brane instanton measures are not yet understood
- expanding these functions now gives NS5 brane instanton actions

Summary so far

- Langlands functional relation: supported by the constant and abelian terms

  $$\mathfrak{Z}(s)\mathcal{P}_s = \mathfrak{Z}(2-s)\mathcal{P}_{2-s}$$

  - where $\mathfrak{Z}(s) := \zeta_{\mathbb{Q}(i)*}(s)\beta_*(2s-1)$
    where the first factor is the completed Dedekind zeta function and the second is the completed Dirichlet beta function (like the Riemann zeta function but with alternating sum only over odd integers)
  - and $\mathcal{E}_s(\mathcal{K}) = 4\zeta_{\mathbb{Q}(i)}(s)\,\mathcal{P}_s(\mathcal{Z})$
- NS5 instanton action:
  $S_{k,q} = 2\pi(|k|e^{-2\phi} + 2|k|\left(\tilde{\chi} - n\right)^2 - iq\chi + 2ik(\psi + \chi\tilde{\chi}))$
- the abelian measure: $\mu_s(\ell_1, \ell_2) = \sum_{\omega_3'|\Lambda} |\omega_3'|^{2-2s} \sum_{z|\frac{\Lambda}{\omega_3'}} |z|^{4-4s}$

  where the first sum is over primitive gaussian divisors and the second over all gaussian divisors

Generalization to other kinds of integers

The results so far can be related via the c-map to the prepotential

- F(X)=$\tau X^2/2$

for $\tau = i$. This leads to gaussian integers which can be generalized to any quadratic imaginary integers in the Stark-Heegner sequence:

- d=1,2,3,7,11,19,43,67,163

which are the only cases having a unique factorization into integers. The related complex multiplication property stems from the metric

$$ds^2 = d\phi^2 + \tfrac{1}{2}e^{2\phi}\frac{|d\zeta + \omega_d d\tilde{\zeta}|}{Im(\omega_d)} + \frac{1}{4}e^{4\phi}(d\sigma - \tilde{\zeta}d\zeta + \zeta d\tilde{\zeta})^2$$

which leads to sums over numbers $m + \omega_d n$, with m, n ordinary integers.

L-series and Multiplicativity

To form L-functions one needs a series of numbers which are either "absolutely multiplicative" or just "multiplicative": multiplicative means that the series $N(n)$ satisfies

$$N(p)N(q) = N(pq) \qquad (1)$$

for all coprimes p and q. If valid for all integers p,q then N(n) is absolutely multiplicative. An example is the usual Dirichlet series, for N(n) absolutely multiplicative

$$\Phi(s) = \Sigma_{n=1}^{\infty} N(n)N^{-s} = \Pi_p (N(p^n)p^{-ns}) = \Pi_p \frac{1}{1 - N(n)p^{-s}} \qquad (2)$$

where the sum is over all primes p.

L-series and Multiplicativity: the second constant term

To get the second perturbative term one must compute the number N(d) of solutions (n_1, n_2) of $n_1^2 + n_2^2 = 0$ mod(2d) both positive and $n_1 < d, n_2 < 2d$: N(d) is multiplicative and the relevant sum gives

$$L(N,s) = \Sigma_{d=1}^{\infty} N(d) d^{-s} = \frac{1}{1-s^{1-s}} \Pi_{p_1} \frac{1-p^{-s}}{(1-p^{1-s})^2} \Pi_{p_2} \frac{1+p^{-s}}{(1-p^{1-s})(1+p^{1-} \quad (3)$$

where the products are over primes $p_1 = 1 mod 4$ and $p_2 = 3 mod 4$. Then

$$L(N,s) = \frac{\beta(s-1)\zeta(s-1)}{\beta(s)} \quad (4)$$

Functional equations for completed functions like

$$\beta_*(s) = \beta_*(1-s) \quad (5)$$

will then appear!

Conclusions

This has been an exercise in

- constructing Eisenstein functions
- Fourier expanding them
- extracting the physical information contained in them

For the UHM we are probably on the right track but not there yet!

Thanks for your attention!