SUSY gauge theory, Liouville theory and quantization of the Hitchin moduli spaces

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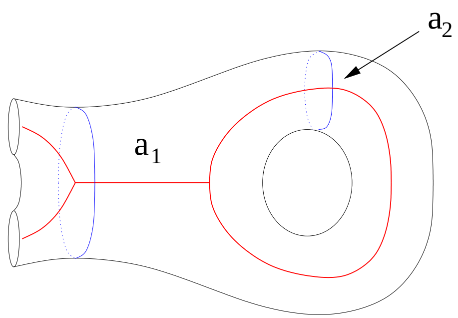
Gaiotto theories I — Lagrangian description

Gaiotto: Riemann surface $\mathcal{C} \longrightarrow$ N=2 gauge theory $\mathcal{G}_{\mathcal{C}}$.

Pants decomposition of $\mathcal{C}$,



$\mapsto$ Lagrangian description of $\mathcal{G}_{\mathcal{C}}$ such that:

- Curve c_r separating pair of pants $\mapsto$ Gauge group $SU(2)_r$, multiplet $(A_r, \lambda_r, \psi_r, a_r)$

- Gluing parameters $q_r \mapsto$ Gauge coupling for $SU(2)_r$

- Boundary β with length $l_\beta \mapsto$ hypermultiplet $(q_\beta, \tilde{q}_\beta, \chi_\beta, \tilde{\chi}_\beta)$ with mass $m_\beta \propto l_\beta$.

$r = 1, \ldots, 3g - 3 + n$ if $\mathcal{C}$ has genus g, n boundary components.

Gaiotto theories II — Remarks

- Gaiotto theories UV finite $\Rightarrow$ Couplings q_c not renormalized.

- **Asymptotically free** theories from limits of Gaiotto theories.

  Example: $N = 2^*$ theory $\Leftrightarrow \mathcal{C}$ has genus $g = 1$, $n = 1$ boundary components.

  Mass of hypermultiplet: m. Single UV gauge coupling q.

  Limit $m \to \infty$, $q \to 0$ with certain combination fixed:

  $\Rightarrow$ Pure $N = 2$ SUSY-Yang Mills (Seiberg-Witten).

Gaiotto theories III — S-duality

S-duality conjecture:

- Space of UV coupling constants $q_c \equiv$ Moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces.

- Lagrangian description exists near the components of the boundary $\partial\mathcal{M}_{g,n}$ which correspond to max. degenerate Riemann surfaces $\mathcal{C}$ (pants decomposition).

  Lagrangian description follows the rules summarized on the previous slide.

Example: $N = 2^*$-theory.

For $q = e^{2\pi i\tau}$ large there exists weakly coupled description with coupling $\tilde{q} = e^{-2\pi i/\tau}$.

Generalizes **Montonen-Olive** duality conjecture...

Gaiotto theories IV — Solution I

Solution à la Seiberg-Witten:

- Low-energy Lagrangian from prepotential $\mathcal{F}(a)$, where $a : c \mapsto a_c$: Special coordinates for $\mathcal{B}$: manifold of vacua.

- For Gaiotto theories: $\mathcal{B}$: Base of Hitchin fibration.

  The **Hitchin moduli space** $\mathcal{M}_{\mathrm{H}}(C)$ on a Riemann surface C is the space of solutions (A, θ) of the SU(2) self-duality equations

$$F_A + R^2 \, [\, \theta \, , \, \bar{\theta} \,] \, = 0 \, , \qquad \begin{aligned} \bar{\partial}_A \theta + \theta \, \bar{\partial}_A \, &= \, 0 \, , \\ \partial_A \bar{\theta} + \bar{\theta} \, \partial_A \, &= \, 0 \, , \end{aligned}$$

  where $d_A = d + A$ is an SU(2)-connection on a vector bundle V, and θ is a holomorphic one-form with values in $\mathrm{End}(V)$, modulo SU(2) gauge transformations. $\mathcal{M}_{\mathrm{H}}(C)$ is a space of complex dimension $6g - 6 + 2n$.

Gaiotto theories V — Solution II

To (A, θ) associate quadratic differential

$$\vartheta = \mathrm{tr}(\theta^2)\,.$$

Expanding ϑ with respect to a basis $\{\vartheta_1, \dots, \vartheta_{3g-3+n}\}$ of the $3g-3+n$-dimensional space of quadratic differentials,

$$\vartheta = \sum_{r=1}^{3g-3+n} H_r\,\vartheta_r\,,$$

defines functions H_r, $r = 1, \dots, 3g-3+n$ on $\mathcal{M}_{\mathrm{H}}(C)$.

H_r : Hamiltonians of **Hitchin's integrable system**.

The subspaces $\Theta_E \subset \mathcal{M}_{\mathrm{H}}(C)$ defined by the equations $H_r = E_r$ for $E = (E_1, \dots, E_{3g-3+n})$ are abelian varieties (complex tori) for generic E. This means that $\mathcal{M}_{\mathrm{H}}(C)$ can be described as a **torus fibration** with **base** $\mathcal{B}$ which can be identified with the space $\mathcal{Q}(C)$ of quadratic differentials on the underlying Riemann surface C.

Gaiotto theories VI — Solution III

Define the spectral curve

$$\Sigma = \{ (v,y) \,|\, \det(v-\theta(y)) = 0 \},$$

a double cover Σ of the surface C. On Σ introduce the differential $dS = vdy$. We then get two systems of coordinates a_r and a_s^D for $\mathcal{B}$ as the periods of S along the homology cycles α_r, β_s, $r,s = 1,\ldots,h$,

$$a_r = \int_{\alpha_r} dS\,, \qquad a_r^D = \int_{\beta_r} dS\,.$$

Both $a = (a_1,\ldots,a_h)$ and $a^{\mathrm{D}} = (a_1^D,\ldots,a_h^D)$ represent systems of coordinates for the base $\mathcal{B}$. The relation $a^{\mathrm{D}} = a^{\mathrm{D}}(a)$ **defines** a holomorphic function $\mathcal{F}(a)$ called prepotential such that

$$a_r^D = \frac{\partial \mathcal{F}}{\partial a_r}\,.$$

There are coordinates $\tau = (\tau_1,\ldots,\tau_h)$ on the torus fibres $\Theta_{E(a)}$ which are Poisson-conjugate to the variables a. The coordinates (a,τ) are **action-angle variables** for the Hitchin system.

Relation to Liouville theory I

Based on work of V. Pestun, `AGT` (Alday, Gaiotto, Tachikawa) have shown that partition function on S^4_R can be evaluated by localization techniques, and found

$$\mathcal{Z}_{\mathcal{G}_C}(S^4_R) \propto \left\langle e^{2\mu_n\varphi(z_n,\bar{z}_n)} \dots e^{2\mu_1\varphi(z_1,\bar{z}_1)} \right\rangle_{\mathcal{C},1}$$

(up to less interesting factors) where

- $\langle e^{2\mu_n\varphi(z_n,\bar{z}_n)} \dots e^{2\mu_1\varphi(z_1,\bar{z}_1)} \rangle_{\mathcal{C},b}$: Correlation function in Liouville theory on Riemann surface $\mathcal{C}$, formally defined as

$$\left\langle \prod_{r=1}^{n} e^{2\mu_r\varphi(z_r,\bar{z}_r)} \right\rangle_{\mathcal{C},b} = \int_{\varphi:C\to\mathbb{R}} [\mathcal{D}\varphi]\, e^{-S_{\mathrm{L}}[\varphi]} \prod_{r=1}^{n} e^{2\mu_r\varphi(z_r,\bar{z}_r)},$$

  where $S[\varphi] = \displaystyle\int_\Sigma \frac{d^2z}{4\pi} \left(\partial_z\varphi\partial_{\bar{z}}\varphi + 4\pi\, M\, e^{2b\varphi}\right)$.

- Parameters μ_r related to hypermultiplet mass parameters m_r as

$$\mu_r = 1 + i\frac{m_r}{R}\,.$$

Relation to Liouville theory II — Holomorphic factorization (i)

Both sides of the correspondence naturally come in holomorphically factorized form:

$$\mathcal{Z}_{\mathcal{G}_{\mathcal{C}}}(S^4_R) \;=\; \int d\nu(a)\,\big|\,\mathcal{Z}\big(\tfrac{1}{R},\tfrac{1}{R};a,m;q\big)\,\big|^2\,,$$

where

- $\mathcal{Z}(\epsilon_1,\epsilon_2;a,m;q)$: Moore-Nekrasov-Shatashvili instanton partition function for $\mathcal{G}_{\mathcal{C}}$ on $\mathbb{R}^4_{\epsilon_2,\epsilon_1}$ with scalar vevs given by a, hypermultiplet mass parameters m.

  $$\mathcal{Z}(\epsilon_1,\epsilon_2;a,m;q) = \left\langle \exp\left(\frac{1}{(2\pi i)^2}\int_{\mathbb{R}^4}\left[\omega\wedge\mathrm{Tr}(\phi F+\tfrac{1}{2}\psi\psi)-H\,\mathrm{Tr}(F\wedge F)\right]\right)\right\rangle_{\mathcal{G}_{\mathcal{C}}(a)}$$

  $$\omega = dx_1\wedge dx_2+dx_3\wedge dx_4,\; H(x)=\epsilon_1(x_1^2+x_2^2)+\epsilon_2(x_3^2+x_4^2).$$

- **Integral over a:** Integral over zero modes of scalars — not fixed on compact manifold S^4_R.

Relation to Liouville theory III — Holomorphic factorization (ii)

Liouville theory:

$$\left\langle \prod_{r=1}^{n} e^{2\mu_r(z_r,\bar{z}_r)} \right\rangle_C = \int d\nu(\alpha) \left| \mathcal{G}(b\,;\alpha,\mu\,;q) \right|^2 .$$

where

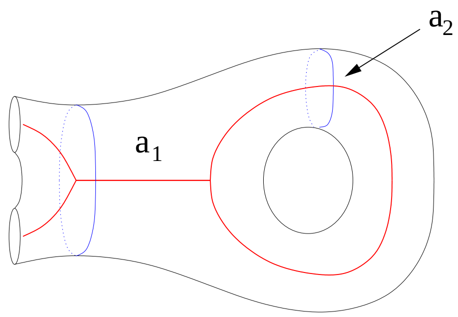
- $\mathcal{G}(b\,;\alpha,\mu\,;q)$: Liouville conformal block:
  - Defined from representation theory of the Virasoro algebra

    $$[\,L_n\,,\,L_m\,] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}\,, \qquad c = 1 + 6(b+b^{-1})^2\,.$$

  - Dependence on $\alpha = (\alpha_1,\dots,\alpha_{3g-3+n}$ from choice of representation $\mathcal{V}_{c,\Delta_r}$ (highest weight $\Delta_r = \alpha_r(b+b^{-1}-\alpha_r)$) associated to curve c_r in gluing construction from conformal blocks corresponding to pairs of pants.
- Integration over α: Integration over Virasoro-representations in $\mathcal{H}_{\mathrm{Liou}}$.

Relation to Liouville theory IV — Full dictionary

The dictionary: To a Riemann surface with pants decomposition associate



parameters $m = (m_1, m_2, \dots)$
associated to the boundary components,

parameters $a = (a_1, a_2, \dots)$
associated to cutting curves

- $\mathcal{Z}(\epsilon_1, \epsilon_2; a, m; q)$: Moore-Nekrasov-Shatashvili instanton partition function for $\mathcal{G}_\mathcal{C}$ on $\mathbb{R}^4_{\epsilon_2,\epsilon_1}$ with scalar vevs given by a_r, hypermultiplet mass parameters m_r.

- $\mathcal{G}(b\,;\alpha,\mu\,;q)$: Liouville conformal block $\langle\, V_{\mu_n}(z_n)\dots V_{\mu_1}(z_1)\,\rangle_\mathcal{C}$ defined by gluing three-punctured spheres according to given pants decomposition and Virasoro representations $\mathcal{V}_{c,\Delta_r}$ associated to curves γ_r.

Then (AGT)

$$\boxed{\mathcal{Z}(\epsilon_1, \epsilon_2; a, m; q) \;=\; \mathcal{G}(b\,;\alpha,\mu\,;q)}$$

if $\alpha_r = \frac{Q}{2} + a_r/\hbar$, $\epsilon_1 = \hbar b$, $\epsilon_2 = \hbar/b$, $c = 1 + 6Q^2$, $Q = b + b^{-1}$.

S-duality vs. Crossing symmetry / modular invariance

Note that holomorphically factorized representation depends on choice of pants decomposition of $\mathcal{C}$ — Lagrangian representation of $\mathcal{G}_{\mathcal{C}}$.

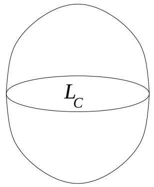
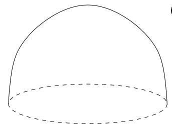
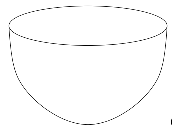
S-duality invariance $\Leftrightarrow$ Independence of pants decomposition.

This is highly nontrivial from the gauge theory point of view (instanton partition functions...), but **known** (`J.T. '01`) property of the Liouville correlation functions !

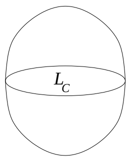
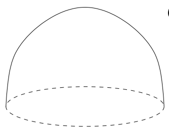
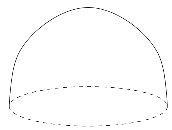
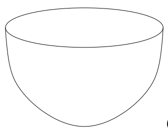
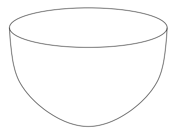
Remark: True in any "decent" CFT. Liouville theory is indeed decent despite the fact that $|0\rangle \notin \mathcal{H}_{\text{Liou}}$.

Towards explaining the AGT-relation I

Generalize $Z_{\mathcal{G}_C}(S^4)$ to expectation value of Wilson loop:

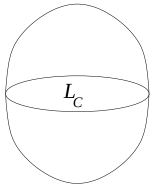
$\mathcal{Z}_{\mathcal{G}_{\mathcal{C}}}\Bigg($  $\Bigg) = \Bigg\langle$  $^{q}\ \Bigg|\ \mathsf{L}_c\ \Bigg|$  $_{q}\ \Bigg\rangle$

where

- $\mathcal{Z}_{\mathcal{G}_{\mathcal{C}}}\Bigg($  $\Bigg)$ Partition function on S_4 with Wilson loop in $SU(2)_c$.
- $\Bigg\langle$  $^{q}\ \Bigg|$ state created by performing path-integral over 
- $\Bigg|$  $_{q}\ \Bigg\rangle$ state created by performing path-integral over 
- L_c: operator on $\mathcal{H}_{\mathcal{G}_{\mathcal{C}}}(S_3)$ representing Wilson loop.

Towards explaining the AGT-relation II

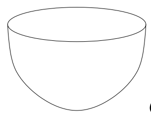
Note that indeed (Pestun; `AGT; AGGTV; DGOT`)

$\mathcal{Z}_{\mathcal{G}_{\mathcal{C}}}\Bigg($ 

$\Bigg) = \int d\nu(a)\,\big(\mathcal{G}(b\,;\alpha,\mu\,;q)\big)^{*}\,\big[2\cos\pi a\big]\,\mathcal{G}(b\,;\alpha,\mu\,;q)$

where $\mathcal{G}(b\,;\alpha,\mu\,;q)$: Liouville conformal block.

This suggests that

- Evaluation of partition function reduces to overlap in BPS-subspace $\mathcal{H}_{\mathrm{BPS}}$ of gauge theory on S_3, defined by condition $\mathsf{Q}|\,\psi\rangle = 0$ for Q: one of the supercharges.
- $\mathcal{G}(b\,;\alpha,\mu\,;q)$ is wave-function of projection $\Pi_{\mathrm{BPS}}\Big|$  $_{q}\Big\rangle$
- $\Pi_{\mathrm{BPS}}\mathsf{L}_c = \big[2\cos\pi a\big]$ in case of Wilson loop.

Towards explaining the AGT-relation III

Indeed, it is known (`J.T. '03`) that Liouville conformal block $\mathcal{G}(b\,;\alpha,\mu\,;q)$ is wave-function of a distinguished state $|\,q\,\rangle$ in **quantum Teichmüller theory**,

$$\boxed{\mathcal{G}(b\,;\alpha,\mu\,;q)\;=\;\langle\,\alpha\,|\,q\,\rangle\,.}$$

Quantum Teichmüller theory:

Basic coordinate functions on Teichmüller spaces $\mathcal{T}_{g,n}$: Length functions $L_c = 2\cosh\frac{l_c}{2}$.

There exists natural Poisson structure on $\mathcal{T}_{g,n}$, $\{\,L_c\,,\,L_{c'}\,\} = \text{known}$.

Quantization: Algebra $\mathcal{A}_b$ generated from operators L_c such that

$$\big[\,\mathsf{L}_c\,,\,\mathsf{L}_{c'}\,\big]\;=\;b^2\,\{\,L_c\,,\,L_{c'}\,\}+\mathcal{O}(b^2)\,.$$

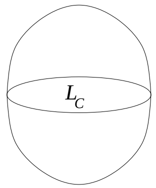
Realized on Hilbert spaces $\mathcal{H}_{\rm T}(C)\simeq L^2(\mathbb{R}_+^{3g-3+n};d\nu)$.

Towards explaining the AGT-relation IV

Bear in mind that $\mathcal{Z}(\epsilon_1, \epsilon_2; a, m; q) \equiv \mathcal{G}_m(\alpha; q) \equiv \langle\, \alpha \,|\, q \,\rangle$: Wave-function of a distinguished state $|\, q \,\rangle$ in $\mathcal{H}_{\mathrm{T}}(C)$.

We may thereby rewrite (`DGOT`)

$$\mathcal{Z}_{\mathcal{G}_{\mathcal{C}}}\Bigg($$



$$\Bigg) = \langle\, q \,|\, \mathsf{L}_c \,|\, q \,\rangle \,.$$

Natural conjecture:

$$\mathcal{H}_{\mathrm{BPS}} \simeq \mathcal{H}_{\mathrm{T}}(C)\text{: space of states obtained by quantizing } \mathcal{T}_{g,n}.$$

Towards explaining the AGT-relation V

On the other hand note that the Ω background $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$ effectively "compactifies" $\mathbb{R}^4$. Projection to Q-cohomology makes theory topological and allows to play with metric.

With origins in the work of Nekrasov-Shatashvili, Nekrasov-Witten proposed to

$$\text{replace} \quad \mathbb{R}^4_{\epsilon_1,\epsilon_2} \quad \text{by circle fibration} \quad \mathbb{R} \times [0,R] \times S_1 \times \tilde{S}_1,$$

such that S_1 shrinks to a point at 0, $\tilde{S}_1$ shrinks to a point at R.

Compactification of $\mathcal{G}_{\mathcal{C}}$ on $S_1 \times \tilde{S}_1$ $\longrightarrow$ open $2d$ sigma model on $R \times I$ with target $\mathcal{M}_{\mathrm{H}}(C)$.

Low energy theory: Q-invariant degrees of freedom $(\epsilon_2, \epsilon_1 \longrightarrow 0)$

- Zero modes of scalars a_γ and

- (constant) spatial parts of gauge fields $A_{x,\gamma} = \varphi_\gamma / R$.

Canonically conjugate action-angle variables for **Hitchin system** !!!

Towards explaining the AGT-relation VI

Boundary conditions ("branes") $\mathfrak{B}_1$, $\mathfrak{B}_2$ of open $2d$ sigma model at ends of $[0, R]$ determined by $4d$ gauge theory. Q-cohomology $\mathcal{H}_{\epsilon_2\epsilon_1} \simeq \mathcal{H}_{\mathrm{BPS}}$.

Nekrasov-Witten show that

a) $\mathcal{H}_{\epsilon_2\epsilon_1}$ carries commuting actions of $\mathcal{A}_b$, $\mathcal{A}_{1/b}$ —

— Quantized algebra of algebraic functions on $M_{\mathrm{H}}(\mathcal{C})$

($\mathcal{A}_b$: open strings from $\mathfrak{B}_1$ to itself, $\mathcal{A}_{1/b}$: open strings from $\mathfrak{B}_2$ to itself.)

b) $\mathcal{H}_{\epsilon_2\epsilon_1}$ can be realized as space of section of line bundle $K_N^{\frac{1}{2}}$ on $N = \mathrm{Op}_{\mathfrak{sl}_2}(\mathcal{C})$.

a) supports the identification

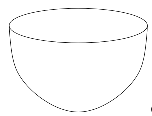
$$\boxed{\mathcal{H}_{\mathrm{T}}(\mathcal{C}) \simeq \mathcal{H}_{\epsilon_1\epsilon_2}\,,}$$

since operators L_c: Generators of $\mathcal{A}_b$.

That $\mathcal{H}_{\mathrm{T}}(\mathcal{C})$ satisfies b) has recently been argued in [arXiv:1005.2846] (J.T.).

Towards explaining the AGT-relation VII

Open question:

Is $\mathcal{Z}(\epsilon_1, \epsilon_2; a, m; q)$ **the wave-function of** $\Pi_{\mathrm{BPS}} \Big|$  $_q \Big\rangle$ **?**

Related **basic question:**

How is the quantization of the Hitchin moduli spaces related to quantum Teichmüller/Liouville theory?

Answer to basic question: **The main diagram** [arXiv:1005.2846] (J.T.)

$$
\begin{tikzcd}
 & \begin{array}{c}\text{Hitchin system}\\ \big(\,\mathcal{M}_{\rm H}(\mathcal{C})\,,\,I\,\big)\end{array} \arrow[dl, "(\mathrm{A})_{\epsilon_2}"'] \arrow[dr, "(\mathrm{B})_{\epsilon_1}"] & \\
\begin{array}{c}\text{Flat connections}\\ \big(\,\mathcal{M}_{\rm H}(\mathcal{C})\,,\,J\,\big)\end{array} \arrow[dr, "(\mathrm{C})_{\epsilon_1}"'] & & \begin{array}{c}\text{quantized Hitchin}\\ \text{systems}\end{array} \arrow[dl, "(\mathrm{D})_{\epsilon_2}"] \\
 & \begin{array}{c}\text{Liouville theory}\\ \text{q-Teichmüller theory}\end{array} &
\end{tikzcd}
$$

where the arrows may be schematically characterized as follows:

(A) Hyperkähler rotation within the Hitchin moduli space $\mathcal{M}_{\rm H}(C)$.

(B) Quantization

(C) Quantization

(D) This arrow will be referred to as *quantum* hyperkähler rotation.