Rényi entropies in the Chern-Simons/CFT_2 correspondence

Juan Jottar

work in progress
with Alejandra Castro and Jan de Boer

Motivation: Entanglement

- There are interesting systems (e.g. FQH) whose ground states do not break symmetries of the Hamiltonian. Phases can be distinguished by their entanglement instead $\Rightarrow$ a "quantum order parameter" that measures non-local correlations
- From a different perspective, there has been renewed interest on entanglement and its relation to the emergence of classical spacetime (geometry)
- Also an important concept in quantum information theory

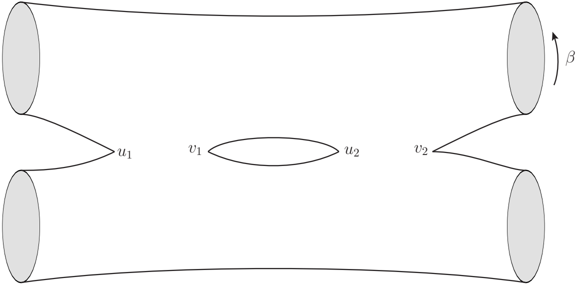
Motivation: Rényi entropies in 2d CFTs

- Consider a quantum system in a pure (or mixed) state, with density operator $\rho = |\Psi\rangle\langle\Psi|$ (or $\rho = e^{-\beta H}$).
- Partition the Hilbert space as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ ($B = A^c$), the reduced density matrix for subsystem A is defined as $\rho_A = \mathrm{Tr}_B \rho$.
- The entanglement entropy S_A associated with A is given by the Von Neumann entropy of ρ_A: $S_A = -\mathrm{Tr}_A\, \rho_A \log \rho_A$.
- Usually computed from analytic continuation of Rényi entropies:

$$S_A = \lim_{n\to 1} S_A^{(n)} = -\lim_{n\to 1} \frac{1}{n-1} \ln \mathrm{Tr} \left[\rho_A^n\right]$$

Motivation: Rényi entropies in 2d CFTs

- Imagine 2d theory at finite temperature $T = \beta^{-1}$, on the infinite line. Let A consist of two disjoint spatial intervals $[u_1, v_1] \cup [u_2, v_2]$. $\mathrm{Tr}[\rho_A^2]$ is computed by the path integral on



- For N disjoint intervals and n replicas, $g(\mathcal{R}_{N,n}) = (N-1)(n-1)$

$$\mathrm{Tr}\left[\rho_A^n\right] = \frac{Z(\mathcal{R}_{n,N})}{Z_1^n}$$

Rényi's via twist operators

- Orbifold the theory as $\mathcal{C}^n/\mathbb{Z}_n$. Twist operators σ_n enact the $\mathbb{Z}_n$ replica symmetry.

$$\mathrm{Tr}\left[\rho_A^n\right] = \Big\langle \sigma_n(u_1,0)\tilde{\sigma}_n(v_1,0)\ldots\sigma_n(u_N,0)\tilde{\sigma}_n(v_N,0)\Big\rangle_{\mathbb{C}}$$

- More generally (Cardy, Castro-Alvaredo, Doyon 2007)

$$\langle\mathcal{O}(x,\tau;\text{sheet } i)\ldots\rangle_{\mathcal{R}_{n,1}} = \frac{\langle\sigma_n(u,0)\tilde{\sigma}_n(v,0)\mathcal{O}_i(x,\tau)\ldots\rangle_{\mathbb{C}}}{\langle\sigma_n(u,0)\tilde{\sigma}_n(v,0)\rangle_{\mathbb{C}}}$$

- Single-interval result is universal $S^{(n)} = \frac{c}{6}(1+n^{-1})\ln(\frac{v-u}{\epsilon})$ (Holzhey, Larsen, Wilczek 1994; Calabrese, Cardy 2004)

- For N intervals, result becomes universal at large-c: assuming a sparse spectrum of light operators, result determined by the Virasoro vacuum block at large central charge (Hartman 2013)

Rényi's via AdS_3/CFT_2

- At large-c one can alternatively evaluate $Z(\mathcal{R}_{n,N})$ using holography
- Represent $\mathcal{R}_{n,N} = \mathbb{C}/\Gamma$, Γ a discrete subgroup of $\subset PSL(2,\mathbb{C})$ (Schottky uniformization)
- Descends from quotient of AdS_3 by element of isometry group $\Rightarrow$ we fill in the branched cover with Euclidean AdS_3 space, with suitable choice of contractible cycles, obtaining a handlebody geometry
- Evaluate properly regularized gravitational action on the handlebody solution (Takhtajan, Zograf 1988; Krasnov 2000) to obtain the large-c partition function and the Rényi entropies (Faulkner 2013)

The Chern-Simons perspective

- Standard AdS$_3$ gravity can be written as an $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ Chern-Simons theory. (Achucarro, Townsend 1986; Witten 1988)
- Take $3d$ gravity with a negative cosmological constant $\Lambda = -1/\ell^2$. Combine dreibein e^a and (dual) spin connection $\omega^a = \epsilon^{abc}\omega_{bc}$ into

$$A = \omega + \frac{e}{\ell}\,, \qquad \bar{A} = \omega - \frac{e}{\ell}$$

- Defining $CS(A) = A \wedge dA + \frac{2}{3} A \wedge A \wedge A$ one finds

$$I_{CS} \equiv \frac{k}{4\pi}\int_M \mathrm{Tr}\Big[CS(A) - CS(\bar{A})\Big]$$

$$= \frac{1}{16\pi G_3}\left[\int_M d^3x\sqrt{|g|}\left(\mathcal{R} + \frac{2}{\ell^2}\right) - \int_{\partial M}\omega^a \wedge e_a\right]$$

$k = \frac{\ell}{4G_3}$

Boundary conditions in Chern-Simons theory

- Consider a radial coordinate ρ (boundary at $\rho \to \infty$) and boundary coordinates $x^{\pm} = \frac{t}{\ell} \pm \sigma$. Fix radial gauge as

$$A = b^{-1} a(x^+, x^-)\, b + b^{-1} db\,, \quad \bar{A} = b\, \bar{a}(x^+, x^-)\, b^{-1} + b\, db^{-1}$$

- In the Chern-Simons formulation, the Brown-Henneaux b.c. amount to (Coussaert, Henneaux, van Driel 1995):
  1. Impose $A_-|_{\partial M} \to 0$, $\bar{A}_+|_{\partial M} \to 0 \Rightarrow a_- = \bar{a}_+ = 0$. The asymptotic symmetries are generated by two copies of a Kac-Moody algebra.
  2. Further demand $A - A_{AdS_3} \xrightarrow[\rho\to\infty]{} \mathcal{O}(1)$, which implements

$$a_+ = \begin{pmatrix} 0 & T(x^+)/k \\ 1 & 0 \end{pmatrix}$$

Drinfeld-Sokolov reduction! The asymptotic symmetries reduce to two copies of the Virasoro algebra with central charge $c = 6k = 3\ell/(2G_3)$

- Under residual gauge transformations that preserve the D-S boundary conditions

$$\delta T = 2T\partial_+\epsilon + \epsilon\,\partial_+ T + \frac{k}{2}\partial_+^3\epsilon$$

- The space of asymptotically anti-de Sitter solutions with a flat boundary metric can be then parameterized as (Bañados 1999)

$$a = \left(L_1 - \frac{T(x^+)}{k}L_{-1}\right)dx^+\,, \quad \bar{a} = \left(-L_{-1} + \frac{\bar{T}(x^-)}{k}L_1\right)dx^-$$

with corresponding metrics

$$\frac{ds^2}{\ell^2} = d\rho^2 + \frac{1}{k}\left(T(x^+)\,dx^{+\,2} + \bar{T}(x^-)\,dx^{-\,2}\right) - \left(e^{2\rho} + \frac{T(x^+)\,\bar{T}(x^-)}{k^2}\,e^{-2\rho}\right)dx^+dx^-$$

Examples: global AdS_3 has $T_{AdS_3} = \bar{T}_{AdS_3} = -k/4$ and BTZ has

$$T_{BTZ} = \frac{1}{2}(M\ell - J) = k\frac{\pi^2\ell^2}{\beta_-^2} \qquad \bar{T}_{BTZ} = \frac{1}{2}(M\ell + J) = k\frac{\pi^2\ell^2}{\beta_+^2}$$

Flat connections on the branched cover

- Strategy: use monodromy conditions to constrain $a_z = \begin{pmatrix} 0 & T(z)/k \\ 1 & 0 \end{pmatrix}$
- Example: vacuum state. Let $M_i = \mathcal{P}e^{\oint_{z_i} A}$. Well-behaved monodromy around branch points z_i imply

$$T(z) = \sum_i \frac{\Delta_i}{(z - z_i)^2} + \frac{p_i}{z - z_i}$$

- Then, $(M_i)^n = \mathbb{1}$ fixes $\Delta_i = \Delta = (c/24)(1 - n^{-2})\ \forall i$
- Trivial monodromy at infinity + remaining cycles fixes p_i. E.g. $p_1 = -p_2 = 2\Delta/(z_2 - z_1)$ for single interval.
- Evaluate variation of the action under $z \to z + \delta z$

$$\delta \ln Z(\mathcal{R}_{n,N}) = -\frac{n}{\pi}\int_{\mathbb{C}} d^2 z \left[T(z)\bar{\partial}\delta z\right] = 2n\sum_{i=1}^{2N} p_i \delta z_i$$

- Up to this point everything parallels calculation in metric formalism (Faulkner 2013)

What does the CS formulation buy us? (work in progress)

- Extends straightforwardly to higher spin AdS_3/CFT_2 dualities (Gaberdiel, Gopakumar 2010)
- E.g.: $sl(3,\mathbb{R}) \oplus sl(3,\mathbb{R})$ CS theory $\Rightarrow$ Gravity nonlinearly coupled to rank 3 symmetric tensor $\phi_{\mu\nu\rho} = \mathrm{Tr}\left[e_{(\mu}e_{\nu}e_{\rho)}\right]$. Boundary: CFT with W_3 symmetry (stress tensor and weight $(3,0)$, $(0,3)$ currents).

$$a_z = \begin{pmatrix} 0 & T & W \\ 1 & 0 & T \\ 0 & 1 & 0 \end{pmatrix}$$

Now one uses monodromies of flat $sl(3)$ connections

- Extends to recently defined "generalized higher spin entropy" also (Hijano, Kraus 2014)

What does the CS formulation buy us? (work in progress)

- Extends to excited states dual to e.g. conical defects. Involves monodromies in the presence of insertions of operators creating the sate.

- Allows to incorporate sources, e.g.

$$a_z = \begin{pmatrix} 0 & T/k \\ 1 & 0 \end{pmatrix}, \qquad a_{\bar{z}} = \begin{pmatrix} * & * \\ \mu & * \end{pmatrix}$$

Flatness $\leftrightarrow$ Ward identity: $\bar{\partial} T = \mu \partial T + 2T\partial\mu - \frac{k}{2}\partial^3\mu$

dual to CFT deformation $S \to S_{CFT} + \int d^2z\, \mu T$. Involves monodromy problem for non-holomorphic connection.

Entanglement entropy for higher spin theories

- A proposal for entanglement entropy in CFTs dual to higher spin theories in AdS_3 is (de Boer, J.I.J. 2013, Ammon, Castro, Iqbal 2013)

$$S_{ent} = k_{cs} \log \mathrm{Tr}_{\mathcal{R}} \left[\mathcal{P} \exp\left(\int_Q^P \bar{A} \right) \mathcal{P} \exp\left(\int_P^Q A \right) \right] \Big|_{\rho_P = \rho_Q = \rho_0 \to \infty}$$

- Evaluating for e.g. higher spin black holes, provides analytic results for entanglement entropy in CFTs deformed by chemical potentials μ for higher spin currents, non-perturbatively in the sources.
- Recently verified perturbatively in CFT, to $\mathcal{O}(\mu^2)$ (Datta, David, Ferlaino, Prem Kumar 2014)
- Hope to use Rényi entropies to prove the proposal.