6d (2,0) Superconformal Field Theories

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Ascona, July 22 2014
Recent Developments in String Theory

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Outline

Douglas 2014 KITP
Lambert 2014 Banff
Papageorgakis 2014 Simons
Moore 2014 Princeton

- 6d (2,0) Theories
- 5d Approach
- Junctions: Origin of 5d YM couplings, N^3 problem
- Highly Effective Action Schwarz 2013
- (2,0) Theory on Rx S^5 & R x CP^2
- Concluding Remarks

6d (2,0) Superconformal Theories

- A, D, E type: type IIB on $R^{1+5} \times C^2/\Gamma_{ADE}$ Witten 1996,
  - A_{N-1}, D_N type: N M5 branes, N M5 (+OM5)
- superconformal symmetry: $OSp(2,6|2) \supset O(2,8) \times USp(4)_R$
- fields: B, Φ_I (I=1,2,3,4,5), Ψ (3+5, 8)
  - selfdual strength H=dB=*H, purely quantum $\hbar=1$

- Nonabelian Version
  - covariant derivative?
  - interaction?
- N^3 degrees of freedom: Klebanov and Tseytlin 1996, Harvey-Minasian-Moore 1998…..

- Say something exact about (2,0) theories?

5-dim Approach

- Douglas (2010), Lambert-Papageorgakis-Schmidt-Sommerfeld (2010)
- $x_6 \sim x_6 + 2\pi R_6$
- Low energy dynamics: $F_{\mu\nu} \sim R_6 H_{\mu\nu 6}$ for abelian theory,
  - 5-dim N=2 U(1) Maxwell Theory with $g_5^2 = 8\pi R_6$
- N M5 branes on S1 = N D4 branes
  - 5-dim N=2 U(N) Yang-Mills Theory
  - Instantons = KK modes: $H_{0\mu\nu} H_{6\mu\nu} \sim \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$
  - No additional KK mode is necessary
- Instantons= Kaluza-Klein modes , $8\pi^2 / g_5^2 = 1/R_6$
  - threshold bound states of k instantons for Kaluza-Klein momentum k quantum state
  - massive tensor multiplet: Instanton (1,0), 4 (1/2,0), 5(0,0) ; anti-instanton (0,1), 4(0,1/2), 5(0,0) of $SU(2)_l \times SU(2)_r$ of SO(4) rotational symmetry.
  - the strong coupling limit $(R \to \infty)$ is the 6d (2,0) theory,
- perturbative approach: D^2F^4 term at 6-loop (Bern et.al. 1210):
  - incomplete? need charged instanton contribution? (Papageorgakis and Boyston 2014)

BPS Equations in Coulomb Phase

- Coulomb Phase for Group G of rank r_G and dimension d_G: SU(N)
  - Cartan: 1/2 BPS r_G massless tensor multiplets: N
  - # of positive roots = $(d_G-r_G)/2= h_G r_G$ 1/2 BPS selfdual strings: N(N-1)/2
  - h_G=Coxeter number = dual Coxeter number for simple laced group

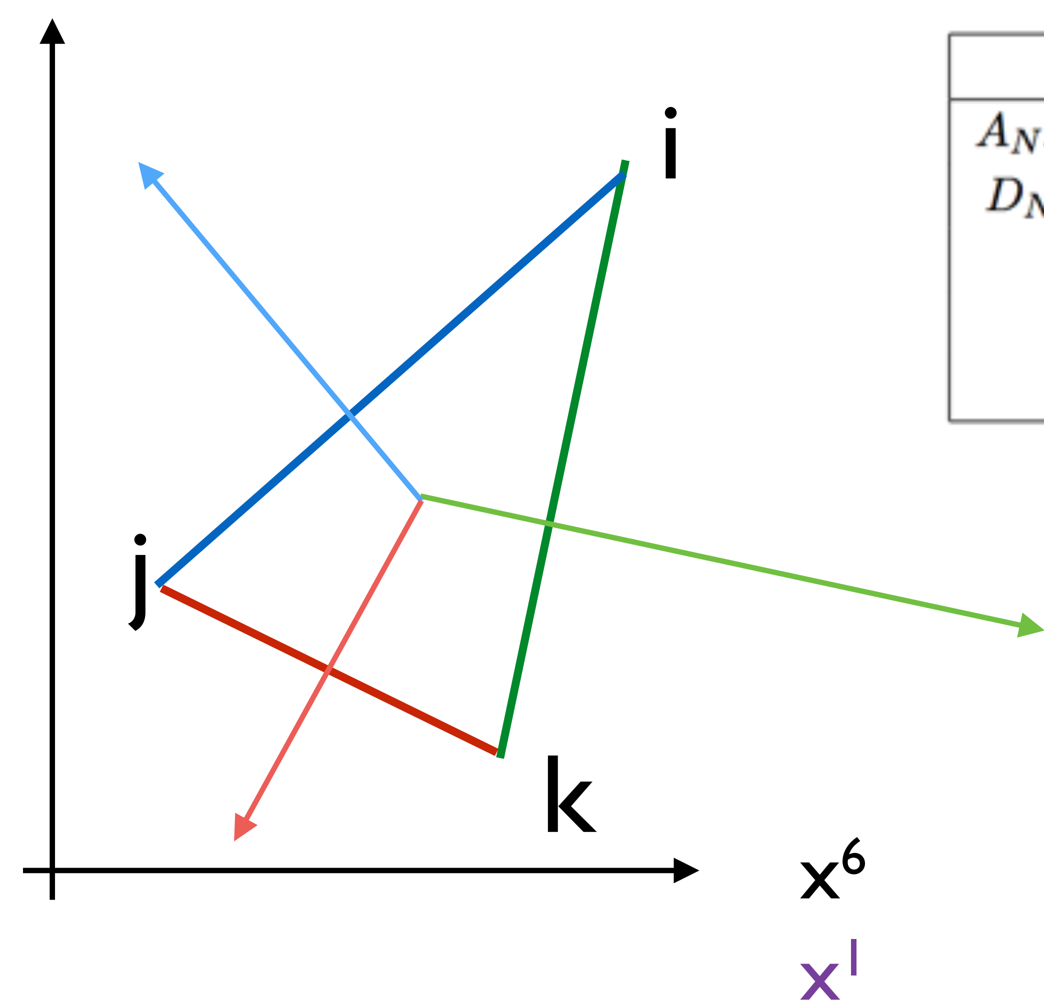
- Locking the spatial rotation $SO(5)_{rot}$ and the R-symmetry $SO(5)_R$
  - $H_{0\mu\nu}=\partial_\mu\varphi_\nu-\partial_\nu\varphi_\mu, \ \partial_\mu\varphi_\mu=0$ (11 eqs) ???? Webs of 1/16 BPS selfdual string junctions
  - 4d Kapustin-Witten equation, Knot invariance, …
  - 5d dyonic monopole string webs: Kapustin-Witten equation (Ho-Ung Yee, KL) lock $SO(4)_{rot}$ with $SO(4)_R$ of $SO(6)_R$
    - $F_{ab} = \epsilon_{abcd} D_c \Phi_d -i[\Phi_a, \Phi_b], \ D_a \Phi_a=0$ 7 equations = octonion
    - $F_{a0} = D_a \Phi_5$ 4 equations: 7+4=11 equations
    - Gauss law $D_a^2 \Phi_5 -[\Phi_a,[\Phi_a, \Phi_5]]=0$

1/4 BPS String Junctions

- Monopole strings, self-dual strings in Coulomb phase
- count 1/2 + wave BPS (Strominger 95)& 1/4 BPS objects:

$$\frac{N(N-1)}{2} + \frac{N(N-1)(N-2)}{6} = \frac{N(N^2-1)}{6} = h_G d_G /6$$

- Bolognesi-K.L. (2011): Counting 1/4 BPS objects



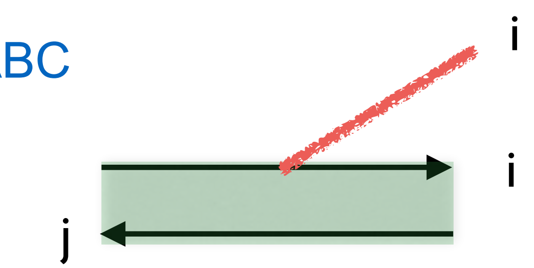
Group	r_G	d_G	h_G	$c_G/3$
$A_{N-1} = SU(N)$	$N-1$	N^2-1	N	$\frac{1}{3}N(N^2-1)$
$D_N = SO(2N)$	N	$N(2N-1)$	$2(N-1)$	$\frac{2}{3}N(2N-1)(N-1)$
E_6	6	78	12	312
E_7	7	133	18	798
E_8	8	248	30	2480

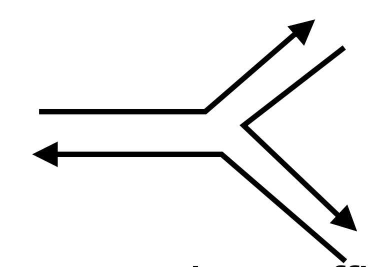
N-cube

- The reaction of M5 branes under the general background of $SO(5)_R$ gauge field F and gravitational curvature R :
  - Single M5 brane: $$I_8(1) = \frac{1}{48}\left[p_2(F) - p_2(R) - \frac{1}{4}(p_1(F) - p_1(R))^2\right]$$
    - p_k=k-th Pontryagin class
    - ADE type: $$I_8(G) = r_G I_8(1) + c_G \times \frac{p_2(F)}{24}$$ Harvey, Minasian, Moore (1998), Intriligator (2000), Yi (2001)
  - AGT reduction to 2-dim: Toda model with $Q=(\epsilon_1+\epsilon_2)^2 / \epsilon_1\epsilon_2$ Alday,Benini,Tachikawa 2010
    - central charge: $$c_{\mathrm{Toda}}[G] = r_G + c_G Q^2$$
- anomaly coefficient: $c_G=h_G\, d_G$??? 1/4 BPS objects Bolognesi & KL (2011)
- S^5 partition function: Casimir Energy (Kim & Kim 2012, Kallen et.al.2012),
  - $E= r_G/24 + c_G/6$ where c_G= 2-loop = $f_{ABC}f_{ABC}$ = 2nd Casimir

5d Yang-Mills Theory in Coulomb Phase: Revisited

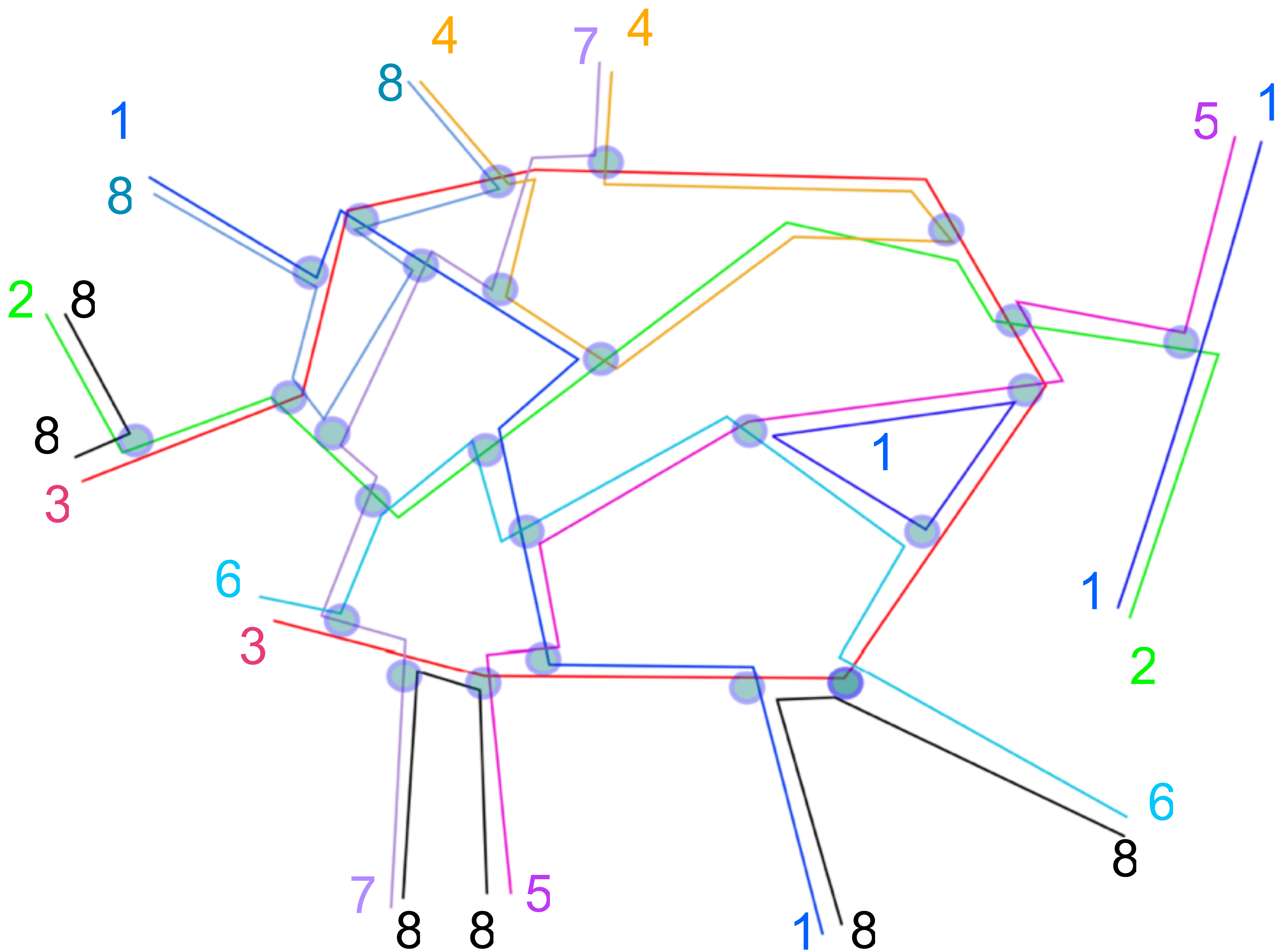
- 5d Massless Vector in Cartan = 6d Massless tensor: $F_{\mu\nu}=H_{\mu\nu 6}$
  - 1/2 BPS 6d massless modes
- 5d Massive W-bosons= wrapped selfdual strings for each roots
  - 1/2 BPS 6d selfdual strings
  - 1/2 BPS 5d magnetic monopole strings
- Interaction vertex: off-shell, structure constant f_{ABC}
  - 5d γWW $f_{\alpha\text{-}\alpha i}$: 6d selfdual string (α) + massless cartan i
    - 1/4 BPS self string + momentum
  - 5d WWW $f_{\alpha\beta\gamma}$: 6d wrapped selfdual string junction off-shell
    - 1/4 BPS selfdual string junction
  - for simple-raced group ADE: the second Casimir $f_{ABC}f_{ABC}$ is equal to anomaly coefficient c_G





- Weyl Vector $\rho= \Sigma\ \alpha>0\ /2$, (the sum of positive roots), $2\rho^2 = c_G/6$

High Temperature



Highly Effective Lagrangian

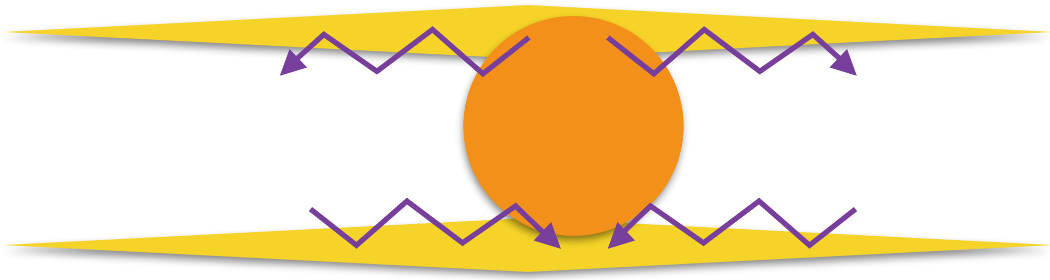
- M5 brane probe on near geometry of N M5 branes
- Dirac-Born-Infeld action+ Wess-Zumino action (Schwarz 2013)

$$L = -\frac{2\Phi^3}{N}\left[\sqrt{\det\left(\delta^M_N + \frac{N\partial^M\Phi_I\partial_N\phi_I}{4\pi\Phi^3}\right)} - 1\right]$$

$$L = -\frac{1}{4\pi}\partial^M\Phi_I\partial_M\Phi_I + \frac{N(\partial^M\Phi_I\partial_M\Phi_I)^2}{64\pi^2\Phi^3}$$

- 5d reduction: $x_6 \sim x_6 + 2\pi R_6$, $\varphi = 2\pi R_6\,\Phi$, $1/R_6 = 8\pi^2/g_5^2$, $L_5 = 2\pi R_6 L$

$$L_5 = -\frac{1}{g_5^2}(\partial\phi)^2 + \frac{N}{64\pi^2}\frac{(\partial\phi)^4}{\phi^3}$$



- 5d W-boson 1-loop => the second term

T^2-compactification

- D4 on a circle of radius R_5 : $x_5 \sim x_5 + 2\pi R_5$
- From 5-dim to 4-dim, $L_4 = 2\pi R_5 L_5$
  - periodic D3 on dual circle: $A_5 = \sigma \sim \sigma + 1/R_5$

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\phi^2 + (\sigma + n^2/R_4)^2)^2} = \oint \frac{dz}{2i \tan \pi z} \frac{1}{(\phi^2 + (\sigma + z/R_5)^2)^2} \approx \frac{2\pi R_5}{4} \frac{1}{\phi^3}$$

- $1/g_4^2 = 1/(4\pi g_s) = R_5/(4\pi R_6) = 2\pi R_5/g_5^2$,

$$L_5 = -\frac{1}{g_5^2}(\partial\phi)^2 + \frac{N}{64\pi^2}\frac{(\partial\phi)^4}{\phi^3} \longrightarrow L_4 = -\frac{1}{4\pi g_s}(\partial\phi)^2 + \frac{N}{16\pi^2}\frac{(\partial\phi)^4}{\phi^4}$$

- S-dual invariant ($\theta=0$), $g_s \rightarrow 1/g_s$, $\varphi^2/g_s \rightarrow \varphi^2/g_s$
- DBI action for D3 brane in the Coulomb phase (S-invariant)

$$L_4 = -\frac{1}{2g_s^2 N}\phi^4 \left[\sqrt{\det(\delta^\mu{}_\nu + \frac{g_s N}{\pi}\frac{\partial^\mu\phi\partial_\mu\phi}{\phi^4})} - 1 \right]$$

higher order

- ✳ 6d term $N^2(\partial\Phi)^6/\Phi^6$: junction-anti-junction??
- ✳ 4d term $N^2(\partial\varphi)^6/\varphi^8$: 2-loop exact: Buchbinder-Petrov-Tseytlin (2001)
- ✳ Large N, finite N correction: for 2-loop: $N(N+1)/2= N + N(N-1)/2$ for 2-loop

- ✳ BPS configuration? selfdual strings…
- ✳ Multi-centered: Harmonic function: $H=1/|\Phi-m_1|^3 + 1/|\Phi-m_2|^3+\ldots$.

- ✳ WZ terms~ $N(N+1)/2$, 2-loop…Intriligator 2000

New Minimal Principle (Old?) on R^{1+4} x T^2

- The Optical Theorem: S=1+ T, $S^\dagger S=1$
  - $T-T^\dagger = T^\dagger T$

$$2i\mathrm{Im} < i|T|i >= \sum_n | < n|T|i > |^2$$

  - Here the sum is only for electric sector or one of SL(2,Z) orbit
  - After the sum, the result is independent of the choice of the orbit
  - No double sum over KK modes of both directions: $|m+n\tau|$
- Questions:
  - instantons in 5d= KK mode along x^6 = magnetic dof
  - so how would they cure the divergence?

Dyonic Instantons in 5d N=2 SYM: $SO(4)_{rot}$ x $SP(4)_R$

$$Q = Q^{+}_{+} \qquad \left.\begin{array}{c} SU(2)_{2R} \\ SU(2)_{1R} \end{array}\right\} \Rightarrow SU(2)_R$$

- Index for BPS states with k instantons

$$I_k(\mu^i, \gamma_1, \gamma_2, \gamma_3) = \mathrm{Tr}_k\left[(-1)^F e^{-\beta Q^2} e^{-\mu^i \Pi_i} e^{-i\gamma_1(2J_{1L}) - i\gamma_2(2J_{2L}) - i\gamma_R(2J_R)}\right]$$

adjoint hyper flavor

  - μ_i : chemical potential for $U(1)^N \subset U(N)_{color}$
  - $\gamma_1, \gamma_2, \gamma_R$: chemical potential for $SU(2)_{1L}$, $SU(2)_{2L}$, $SU(2)_R$

- calculate the index by the localization: $I(q, \mu^i, \gamma_{1,2,3}) = \sum_{k=0}^{\infty} q^k I_k$

- 5d *N=2** instanton partition function on $R^4 \times S^1$: $t \sim t + \beta$

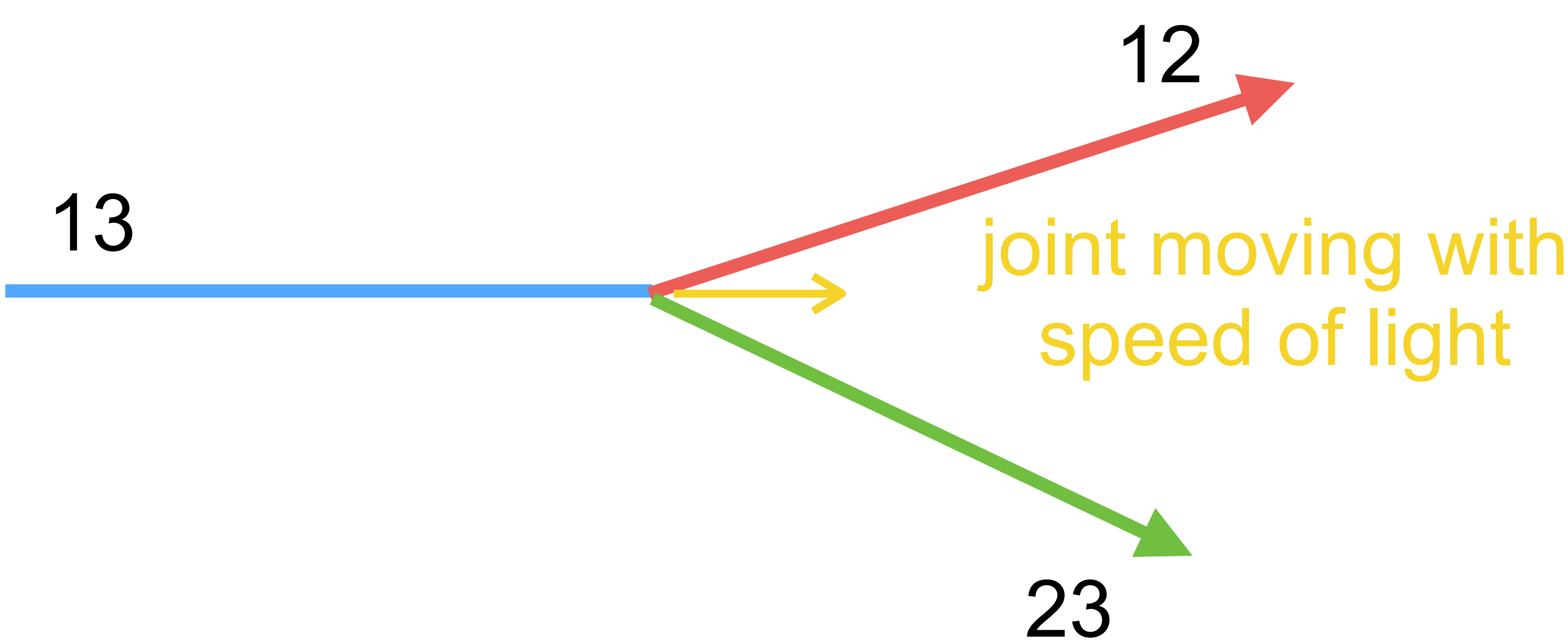
- In $\beta \to 0$ and small chemical potential limit, the index becomes 4d Nekrasov instanton partition function :

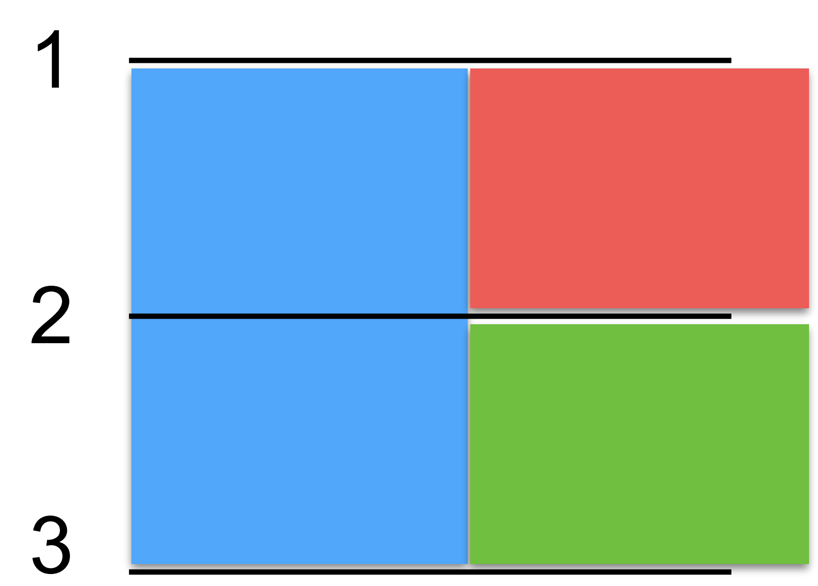
$$a_i = \frac{\mu_i}{2} \qquad -\epsilon_1 = i\frac{\gamma_1 - \gamma_R}{2} \qquad \epsilon_2 = i\frac{\gamma_1 + \gamma_R}{2}, \quad m = i\frac{\gamma_2}{2} \qquad q = e^{2\pi i \tau}$$

Scalar Vev — Omega deformation parameter — Adj hypermultiplet mass — instanton fugacity

Dyonic Instanton Index Function for U(N)

- The expansion is done in instanton numbers and some functions of the Coulomb parameters which can be regarded as the chemical potential for the electric charge.
- Instead, one can have a dual version of the index function which is expanded in the charge and a function of instantons.
- Nekrasov-Okunkov(2003), Kim-Kim Koh-Lee-Lee(2011), Haghighat-Iqbal-Kozcaz-Lockhart- Vafa (2013)
- Additional States appearing in dyonic instanton calculations
- Can be regarded as the degenerate limit of junctions
- There is a threshold bound states of W-boson for 13 and instantons on 2nd M5 branes





6d (2,0) Theories on $S^1 \times S^5$ Kim-KL(2013),Kim^3-KL(2013)

* Difficulties with Non-abelian B field and its strength H=dB
* Generalize ABJM to M5 brane theory
  * Mode out by R^8/Z_K
  * Weak coupling limit
  * No fixed point
* R^{1+5}/Z_K has a fixed point
* Consider the 6d (2,0) theory on $R \times S^5$: TheRadial Quantization
* S^5 = a circle fibration over CP^2
  * $ds^2_{S5} = ds^2_{CP2} + (dy + V)^2, \quad dV = 2J, \ J = -{*J} \ , \ y \sim y + 2\pi$
* $AdS_7 \times S^4/Z_K$ (Tomasiello, 2013): 6d theory with D6 and <u>D6</u> : still 6d theory with H=*H

Index Function on S^1 x S^5

- Supercharge $$Q^{R_1,R_2}_{j_1,j_2,j_3} \Rightarrow Q = Q^{\frac{1}{2},\frac{1}{2}}_{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}, S = Q^\dagger$$

- BPS bound: $$E = j_1 + j_2 + j_3 + 2(R_1 + R_2)$$

- 6-dim index function:

$$I = \mathrm{Tr}\left[(-1)^F e^{-\beta'\{Q,S\}} e^{-\beta\left(E - \frac{R_1+R_2}{2} - m(R_1-R_2) + aj_1 + bj_2 + cj_3\right)}\right], \ a + b + c = 0$$

- Euclidean Path Integral of (2,0) Theory on S^1x S^5
  - S^5= S^1 fiber over CP^2 : $-i\,\partial_y$ = KK modes

$$k \equiv j_1 + j_2 + j_3$$

- Z_K modding keeps only k/K=integer modes

6d Abelian Theory (Fermion+ Scalar)

- on R x S^5, ……. (Could Include H=dB)

$$-\frac{i}{2}\bar{\lambda}\Gamma^M\hat{\nabla}_M\lambda - \frac{1}{2}\partial_M\phi_I\partial^M\phi_I - \frac{2}{r^2}\phi_I\phi_I$$

- gamma matrices Γ^M, ρ^a
- Symplectic Majorana $\lambda = -BC\,\lambda^*$, $\varepsilon = BC\,\varepsilon^*$
- Weyl: $\Gamma^7\,\lambda=\lambda$, $\Gamma^7\,\varepsilon = -\varepsilon$
- 32 supersymmetry

$$\begin{aligned} \delta\phi_I &= -\bar{\lambda}\rho_I\epsilon = +\bar{\epsilon}\rho_I\lambda, \\ \delta\lambda &= +\frac{i}{6}H_{MNP}\Gamma^{MNP}\epsilon + i\partial_M\phi_I\Gamma^M\rho_I\epsilon - 2\phi_I\rho_I\tilde{\epsilon}, \\ \delta\bar{\lambda} &= -\frac{i}{6}H_{MNP}\bar{\epsilon}\Gamma^{MNP} + i\partial_M\phi_I\bar{\epsilon}\Gamma^M\rho_I - 2\bar{\tilde{\epsilon}}\rho_I\phi_I. \end{aligned}$$

- additional condition on Killing spinor:

$$\hat{\nabla}_M\epsilon = \frac{i}{2r}\Gamma_M\tilde{\epsilon}, \quad \Gamma^M\hat{\nabla}_M\tilde{\epsilon} = 2i\epsilon, \qquad \tilde{\epsilon} = \pm\Gamma_0\epsilon.$$

Twisting & Dimensional Reduction to R x CP^2

Killing spinor eq $\nabla_M \epsilon_\pm = \pm \frac{1}{2r} \Gamma_M \Gamma_\tau \epsilon_\pm$

* Killing spinors: SO(1,5)=SU(2,2) chiral spinor and 4-dim of Sp(2)=SO(5)$_R$
* 32 Killing spinors = 3x8 (SU(3) triplet) +1x 8 (SU(3) singlet) under SU(3) isometry of CP^2:
  * (I) $\varepsilon_+ \sim$ exp(-it/2 +3i y/2)… : singlet
  * (II) $\varepsilon_+ \sim$ exp(-it/2 - i y/2)… : triplet
* Twisting

$$\epsilon_{old} = e^{-\frac{y}{4} M_{IJ}\rho_{IJ}} \epsilon_{new},$$
$$\lambda_{old} = e^{-\frac{y}{4} M_{IJ}\rho_{IJ}} \lambda_{new},$$
$$(\phi_1 + i\phi_2)_{old} = e^{+(3+p)iy/2} (\phi_1 + \phi_2)_{new}$$
$$(\phi_4 + i\phi_5)_{old} = e^{+(3-p)iy/2} (\phi_4 + i\phi_5)_{new}.$$

$$M_{12} = -M_{21} = \frac{3+p}{2}, \; M_{45} = -M_{54} = \frac{3-p}{2}$$
$$p = \cdots, -5, -3, -1, 1, 3, 5, \cdots .$$

$$\partial_y \to \partial_y + \frac{3i}{2}(R_1 + R_2) + \frac{ip}{2}(R_1 - R_2)$$

$$k \equiv j_1 + j_2 + j_3 + \frac{3}{2}(R_1 + R_2) + \frac{p}{2}(R_1 - R_2), \; p = \text{odd integer}$$

Singlets ε_+, ε_- for $Q = Q^{++}_{---}, S = Q^{--}_{+++}$

5d Lagrangian

$Q = Q^{++}_{---},\ S = Q^{--}_{+++}$

- Lagrangian on R x CP^2 with 2 supersymmetries for any p:

$$
\begin{aligned}
S \;=\; & \frac{K}{4\pi^2}\int_{\mathrm{R}\times\mathrm{CP}^2} d^5x\sqrt{|g|}\ \mathrm{tr}\Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2\sqrt{|g|}}\epsilon^{\mu\nu\rho\sigma\eta}J_{\mu\nu}\Big(A_\rho\partial_\sigma A_\eta-\frac{2i}{3}A_\rho A_\sigma A_\eta\Big)\\
& -\frac{1}{2}D_\mu\phi_I D^\mu\phi_I+\frac{1}{4}[\phi_I,\phi_J]^2-2\phi_I^2-\frac{1}{2}(M_{IJ}\phi_J)^2-i(3-p)[\phi_1,\phi_2]\phi_3-i(3+p)[\phi_4,\phi_5]\phi_3\\
& -\frac{i}{2}\bar\lambda\gamma^\mu D_\mu\lambda-\frac{i}{2}\bar\lambda\rho_I[\phi_I,\lambda]-\frac{1}{8}\bar\lambda\gamma^{mn}\lambda J_{mn}+\frac{1}{8}\bar\lambda M_{IJ}\rho_{IJ}\lambda\Big], \qquad (2.27)
\end{aligned}
$$

- Supersymmetry Transformation

$$
\begin{aligned}
\delta A_\mu &= +i\bar\lambda\gamma_\mu\epsilon=-i\bar\epsilon\gamma_\mu\lambda,\quad \delta\phi_I=-\bar\lambda\rho_I\epsilon=\bar\epsilon\rho_I\lambda,\\
\delta\lambda &= +\frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\epsilon+iD_\mu\phi_I\rho_I\gamma^\mu\epsilon-\frac{i}{2}[\phi_I,\phi_J]\rho_{IJ}\epsilon-2\phi_I\rho_I\tilde\epsilon-M_{IJ}\phi_I\rho_J\epsilon.
\end{aligned}
$$

- p/2=-1/2 : $k = j_1+j_2+j_3+R_1+2R_2$
  - additional supersymmetries: Total 8 supersymmetries

$Q^{+-}_{-++},\ Q^{+-}_{+-+},\ Q^{+-}_{++-}$ conjugates

Coupling Constant Quantization

- Instanton number on CP2

$$\nu = \frac{1}{8\pi^2}\int_{\mathrm{CP}^2} \mathrm{Tr}(F \wedge F) = \frac{1}{16\pi^2}\int_{\mathrm{CP}^2} d^4x\sqrt{|g|}\;\mathrm{Tr}F_{\mu\nu}\tilde{F}^{\mu\nu}.$$

- Instantons represents the momentum K or energy K:

$$\frac{1}{g_{YM}^2} = \frac{K}{4\pi^2 r}$$

- Another approach to quantization: F=2J: 2π flux on a cycle, 1/2 instanton for abelian theory

$$\frac{K}{4\pi^2}\int_{\mathrm{R}\times\mathrm{CP}^2} d^5x\;\frac{1}{2}\epsilon^{\mu\nu\rho\sigma\eta}J_{\mu\nu}\partial_\rho A_\sigma A_\eta \;\Rightarrow K\int dt\;A_0$$

- ‘t Hooft coupling constant: λ= N/K
- Large K => Free Theory

Expected Enhanced Supersymmetries

- Killing spinors with $p/2=-1/2$, $k = j_1+j_2+j_3+ R_1+ 2R_2$
  - $k=0$: 8 kinds
  - $k= \pm 1$: 14 kinds
  - $k= \pm 2$: 8 kinds
  - $k= \pm 3$: 2 kinds
- # of supersymmetries
  - $K \geqq 4$: 8 supersymmetries
  - $K=3$: 10 supersymmetries
  - $K=2$: 16 supersymmetries
  - $K=1$: 32 supersymmetries

the index function on S^1 x S^5

Kim, Kim (2012), Kim, Kim, Kim (2012), Minahan-Nedelin-Zabzine,(2012)

- 5d SYM on S^5
  - S-dual version of the index
  - Vacuum energy:

$$(\epsilon_0)_{index} = \lim_{\beta' \to 0} \mathrm{Tr}\left[(-1)^F \frac{E-R}{2} e^{-\beta'(E-R_1)} \right] = \frac{N(N^2-1)}{6} + \frac{N}{24}$$

- S^1 x CP^2 path integral off-shell
- Stationary phase: $D^1=D^2=0$, $F= 2s\, J$, $\varphi + D^3=4s$, $s=\mathrm{diag}(s_1,s_2,\ldots,s_N)$
  - analogous to 3-dim Monopole operator
- Path Integral: Off-shell, localization (K=1 case_

$$\sum_{s_1,s_2,\cdots s_N=-\infty}^{\infty} \frac{1}{|W_s|} \oint \left[\frac{d\lambda_i}{2\pi}\right] e^{\frac{\beta}{2}\sum_{i=1}^N s_i^2 - i\sum_i s_i\lambda_i} Z^{(1)}_{\rm pert} Z^{(1)}_{\rm inst} \cdot Z^{(2)}_{\rm pert} Z^{(2)}_{\rm inst} \cdot Z^{(3)}_{\rm pert} Z^{(3)}_{\rm inst} \ .$$

- For K=1, well-confirmed for k ≤ N with N=1,2,3 with the AdS/CFT calculation

Strange Vacua

- K=1, F=2sJ background

$$U(2)\ (1,-1)$$
$$U(3)\ (2,0,-2),(2,-1,-1),(1,1,-2),(1,0,-1)$$
$$U(4)\ (3,1,-1,-3),(3,1,-2,-2),(2,2,-1,-3),(3,0,-1,-2),$$
$$(2,1,0,-3),(2,0,0,-2),(2,0,-1,-1),(1,1,0,-2),(1,0,0,-1)$$

- the Lowest one $s_G = 2\rho \cdot H$ with negative energy $-2\rho^2 = cG/6$, where ρ = Weyl vector
- Ground State for Index: K ≤ N (Strong ‘t Hooft coupling λ=N/K)

K	$U(2)$	$U(3)$	$U(4)$	$U(5)$	$U(6)$	$U(7)$	$U(8)$	$U(9)$	$U(N)$
1	-1	-4	-10	-20	-35	-56	-84	-120	$-\frac{N(N^2-1)}{6}$
2	0	-1	-2	-5	-8	-14	-20	-30	
3		0	-1	-2	-3	-6	-9	-12	
4			0	-1	-2	-3	-4	-7	
5				0	-1	-2	-3	-4	
6					0	-1	-2	-3	
7						0	-1	-2	
8							0	-1	
9								0	

Table 1: Vacuum energies divided by K, at general $\mathbb{Z}_K$ (and fluxes)

Check with AdS/CFT

- E.g. k = N = 3: (all results multiplied by vacuum energy factor & $e^{-3\beta}$) $y_i = e^{-\beta a_i}\ ,\quad y = e^{\beta(m-\frac{1}{2})}$

$$
\begin{aligned}
Z_{(2,0,-2)} &= 3\left[y^2(y_1+y_2+y_3)+y(y_1^2+y_2^2+y_3^2)+y^{-1}(y_1+y_2+y_3)-(1+\frac{y_1}{y_2}+\frac{y_2}{y_1}+\cdots)+y^3\right] \\
&\quad +6y\left[y(y_1+y_2+y_3)-(y_1^{-1}+y_2^{-1}+y_3^{-1})+y^{-1}+y^2\right]+y^3 \\
Z_{(2,-1,-1)}+Z_{(1,1,-2)} &= -2y\left[y(y_1+y_2+y_3)-(y_1^{-1}+y_2^{-1}+y_3^{-1})+y^{-1}+y^2\right] \\
&\quad -2y\left[y(y_1+y_2+y_3)-(y_1^{-1}+y_2^{-1}+y_3^{-1})+y^{-1}+y^2\right] \\
&\quad -4y^3-4y^2(y_1+y_2+y_3)-2y\left(y_1^2+y_2^2+y_3^2-\frac{1}{y_1}-\frac{1}{y_2}-\frac{1}{y_3}\right)+2\left(\frac{y_1}{y_2}+\frac{y_2}{y_3}+\cdots\right)-2y^{-1}(y_1+y_2+y_3) \\
Z_{(1,0,-1)} &= y^3+y^2(y_1+y_2+y_3)-y(y_1^{-1}+y_2^{-1}+y_3^{-1})+1
\end{aligned}
$$

add all

$$
Z_{SUGRA} = 3y^3+2y^2(y_1+y_2+y_3)+y\left(y_1^2+y_2^2+y_3^3-\frac{1}{y_1}-\frac{1}{y_2}-\frac{1}{y_3}\right)-\left(\frac{y_1}{y_2}+\frac{y_2}{y_1}+\cdots\right)+y^{-1}(y_1+y_2+y_3)
$$

- Non-zero flux states contributing to the index
  - s=(N-1,N-3,…,-(N-1)) = ρ : SU(N) Weyl vector
  - index vacuum energy: $2\rho^2 = c_G/6$

$$
E_0 = -\frac{N(N^2-1)}{6}
$$

SU(2) Case

* BPS Eq. for Homogeneous Configuration with instanton number n^2

$$A = V\mathrm{diag}(n, -n), \;\; F = 2J\mathrm{diag}(n, -n),$$

* homogeneous bosonic solutions possible only with n=+1,-1
* but gauss law is violated
* for one of the constant bps solutions, the homogeneous fermionic zero mode is possible.
* gauss law can be satisfied with fermionic contribution for K=1 but not for K>1.
* energetic is more complicated to due to zero-point contribution to the classical one,...

Conclusion

- We understand more of M5 branes but want a lot more..

- Interesting Avenue to Explore
  - AGT, Toda, Instanton Partition function in Coulomb Phase,….. DOF
  - 6d Theory on R x CP^2 : Near BPS objects and their conformal dimension (perturbative approach, …..)
  - Partition functions on S^6
  - Higher order terms in effective action, WZ terms,…. Intriligator…
  - Understand the six-loop log divergence of 5d theory and its relation to instantons??
  - Rastelli,… Bootstrap
  - Curved manifolds, lower dimensional theories
  - 6d (1,0) Theories