Exact results for supersymmetric cusps in ABJM theory

based on 1705.10780 and 1402.4128

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Outline

Introduction

Supersymmetric Wilson loops

Let's use our favourite theory as an example: $\mathcal{N} = 4$ SYM

one can define *locally* supersymmetric Wilson loops!

$$W_{1/2} = \frac{1}{N} \mathrm{Tr}\, \mathcal{P} \exp\left[-i\, g \int_\Gamma d\tau \, \left(A_\mu \, \dot{x}^\mu(\tau) + i\, n_I(\tau)\, |\dot{x}|\, \Phi^I\right)\right] \qquad I = 1, \ldots 6$$

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- preserves $1/2$ of supercharges (locally), i.e. **1/2-BPS**

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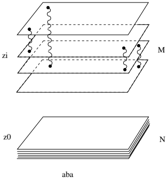
- natural object in $\mathcal{N} = 4$: massive quarks via *Higgsing*
- strong coupling **AdS/CFT dual:** [Maldacena; Rey, Yee 98] **vev computed via minimal surfaces**



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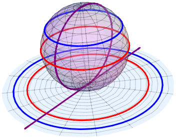

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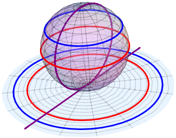

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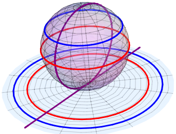

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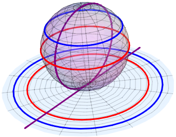
- *conformal map* from sphere to plane
- anomalous transformations [Drukker, Gross 01]



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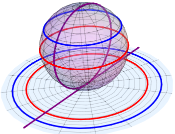

they are amenable of exact results via *localization*

[Erickson, Semenoff, Zarembo 00; Pestun 07]

Localization

[Pestun 05]

- theories on a *compact* manifold
- invariance under *fermionic symmetry* (conserved supercharge)
- ∞-dim path integral reduces to a finite-dim matrix model (MM)
- can compute expectation values of operators preserving supercharge as MM average
- often this simplifies in the planar limit (saddle point approx)

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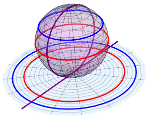
Example:

can compute the exact vev of **supersymmetric Wilson loops** in $\mathcal{N}=4$ SYM from a (*gaussian*) matrix model ($\lambda = g^2 N$)

$$\langle W_{1/2}{}^{|} \rangle = 1$$



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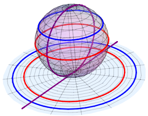
$$\langle W_{1/2}{}^{\circ} \rangle = \frac{1}{N} L^1_{N-1}\left(-\frac{\lambda}{4N}\right) e^{\frac{\lambda}{8N}} = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) + \mathcal{O}\left(\frac{1}{N^2}\right)$$



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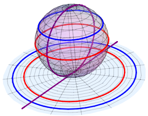

[Drukker, Giombi, Ricci, Trancanelli 07; Pestun 09]

Cusps

cusp anomalous dimension

- even supersymmetric WL develop UV divergences for *cusped* contours
- renormalization: **cusp anomalous dimension** $\Gamma_{cusp}(\phi)$

[Polyakov 80; Korchemsky and Radyushkin 87]

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If the rays are light-like, $\phi \to i\,\infty$, additional divergence: $\Gamma_{cusp}(\phi) \sim \varphi\,\Gamma^{\infty}_{cusp}$ where $\phi = i\,\varphi$:

- controls **IR** divergence of **amplitudes** of massless particles
  [Magnea, Sterman, 90; Bern, Dixon, Smirnov, 05]
- anomalous dimension of **twist-2** operators
  [Korchemsky 89; Korchemsky and Marchesini 93]

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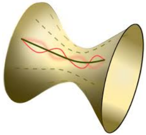
- anomalous dimension of **twist-2** operators [Korchemsky 89; Korchemsky and Marchesini 93]
- in planar $\mathcal{N} = 4$ ('t Hooft coupling $\lambda = g^2 N$):
  - $sl(2)$ sector anomalous dimensions (large spin)
  - AdS/CFT description **GKP string** (folded string rotating in AdS) [Gubser, Klebanov, Polyakov 02]



  - computed up to four loops at weak coupling (λ^4) and up to two loops at strong coupling order $\left(\frac{1}{\sqrt{\lambda}}\right)$ [Bern, Carrasco, Czakon, Dixon, Johanson, Kosower, Smirnov... Kruczenski; Giombi, Ricci, Roiban, Tirziu, Tseytlin, Vergu...]
  - *integrability*...

Integrability

$\mathcal{N} = 4$ SYM integrability

spectral problem of *planar* $\mathcal{N} = 4$ SYM is described by an *integrable spin chain*

[Minahan, Zarembo 02; Beisert, Staudacher 03 + many more]

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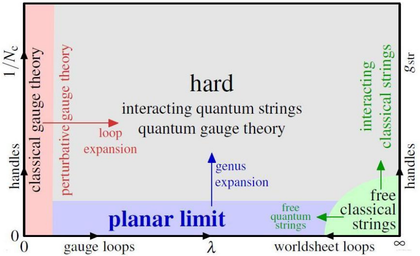

[from Beisert *et al.* review]

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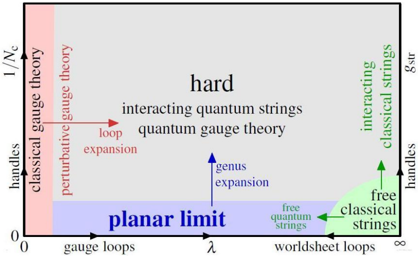

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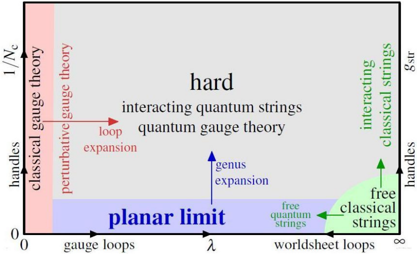

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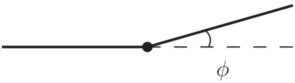
For space-like (supersymmetric) cusps?

?

Supersymmetric cusps in $\mathcal{N} = 4$ SYM

Two Wilson lines at an angle ϕ:

- IR behaviour of **scattering** of massive colored particles [Korchemsky, Radyushkin 92]
- **Regge limit** of 4-pt massive amplitudes [Henn, Naculich, Schnitzer, Spradlin 10]
- **quark-antiquark** potential on sphere at angle $\pi - \phi$



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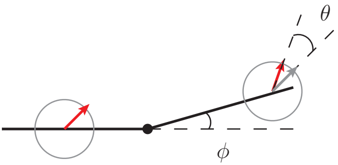

locally 1/2-BPS cusp

constructed with two 1/2-BPS rays at a (geometric) angle ϕ

- change of angle in the internal space as well: $\frac{n_1 \cdot n_2}{|n_1||n_2|} = \cos\theta$
  [Drukker, Gross, Ooguri 99]
- BPS condition for $\phi^2 = \theta^2$ the cusp is supersymmetric (1/4-BPS): $\Gamma_{cusp}(\phi = \pm\theta) = 0$
- computation at *weak* and *strong* coupling

[Correa, Henn, Maldacena, Sever 12
Henn, Huber 13
Grozin, Henn, Korchemsky, Marquard 15
Drukker, Forini 09-10]

The Bremsstrahlung function

Bremsstrahlung function

We define B as the **small angle limit** of the cusp anomalous dimension:

$$\Gamma_{cusp} = -B(g, N)\, \phi^2 + \dots$$

- in a CFT energy emitted by a heavy quark: $\Delta E = 2\pi\, B \int dt\, \dot{v}^2$

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exact computation by both integrability and localization !

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1. localization based
   - exploits BPS condition $\Gamma_{cusp}(\phi^2 = \theta^2) = 0$
   - derivation passes through relation of B with *displacement operator* on line defect and **latitude 1/4-BPS WLs**
   - exact expression: $\boxed{B = \dfrac{1}{2\pi^2}\,\lambda\,\partial_\lambda \langle W^{\circ}_{1/2}\rangle = \dfrac{1}{4\pi^2}\,\dfrac{\sqrt{\lambda}\, I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} + \mathcal{O}(N^{-2})}$

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[Correa, Maldacena, Sever; Drukker; Gromov, Sever 12
Gromov, Levkovich-Maslyuk, Sizov 13]

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Comment: comparing the two exact results

one can determine potential **finite renormalization** of coupling constants

To summarize

$\mathcal{N} = 4$ SYM is cool

- spectral problem is integrable
- exact susy Wilson loop vev via localization
- Bremsstrahlung function can be computed exactly by BOTH

can this program be extended to *other theories*?

- localization
- integrability

?

Natural candidates

- **ABJM theory** in 3 dimensions
- $\mathcal{N} = 2$ SCFTs in 4 dimensions

[Fiol, Gerchkovits, Komargodski 15]

Supersymmetric Wilson loops in ABJM theory

ABJM theory

[Aharony, Bergman, Jafferis, Maldacena 08]

- $d = 3$, $\mathcal{N} = 6$ SCFT $OSp(4|6)$
- Chern–Simons with gauge group $U(N)_k \times U(N)_{-k}$, $k \in \mathbb{Z} \rightarrow (A, \hat{A})$
- Bifundamental matter fields $(Y^I, \bar{\psi}_I)$, $I = 1, 2, 3, 4$, $(\mathbf{N}, \bar{\mathbf{N}})$

ABJM theory

[Aharony, Bergman, Jafferis, Maldacena 08]

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- Chern–Simons with gauge group $U(N)_k \times U(N)_{-k}$, $k \in \mathbb{Z} \to (A, \hat{A})$
- Bifundamental matter fields $(Y^I, \bar{\psi}_I)$, $I = 1, 2, 3, 4$, $(\mathbf{N}, \bar{\mathbf{N}})$
- low energy theory on N $M2$ branes probing a $\mathbb{C}^4/\mathbb{Z}_k$
- AdS_4/CFT_3 correspondence: M–th on $AdS_4 \times S^7/\mathbb{Z}_k$ $N \gg k^5$
IIA string-th on $AdS_4 \times \mathbb{CP}^3$ $k \ll N \ll k^5$

$\mathcal{N} = 6$ ABJM SCFT in 3d $k \gg N$	Type IIA ST on $AdS_4 \times CP^3$ $k \ll N \ll k^5$	M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$ $N \gg k^5$

$$\lambda = \frac{N}{k}$$

ABJM theory

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- integrable in the planar limit

[Minahan, Zarembo; Gromov and Vieira 08 + many more]

$\mathcal{N} = 6$ ABJM SCFT in 3d $k \gg N$	INTEGRABILITY $(N \to \infty)$	Type IIA ST on $AdS_4 \times CP^3$ $k \ll N \ll k^5$	M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$ $N \gg k^5$

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The interpolating h-function

The $h(\lambda)$ function

ABJM (or $AdS_4 \times CP^3$) integrability features a nontrivial interpolating function of the 't Hooft coupling $\lambda = \frac{N}{k}$

[Gaiotto, Giombi, Yin; Grignani, Harmark, Orselli; Nishioka, Takayanagi; Gromov, Vieira 08]

- magnon dispersion relation for ABJM spin chain

$$E = \frac{1}{2}\sqrt{1 + 16\, h^2(\lambda) \sin^2 \frac{p}{2}}$$

- scaling function (twist-2 / *light-like cusp anomalous dimension*)

[Gromov and Vieira 08]

$$\Gamma^{\infty}_{\mathrm{ABJM}}(\lambda) = \frac{1}{2}\,\Gamma^{\infty}_{\mathcal{N}=4}(\lambda_{\mathrm{YM}})\Big|_{\frac{\sqrt{\lambda_{\mathrm{YM}}}}{4\pi}\to h(\lambda)} \qquad \lambda_{YM} = g^2 N$$

- $h(\lambda)$ appears in all integrability based predictions, it is needed for comparisons with results obtained by other methods

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- $h(\lambda)$ appears in all integrability based predictions, it is needed for comparisons with results obtained by other methods
- **conjecture** for its value to all orders, inspired by analogy with *localization*

[Gromov, Sizov 14]

$$\lambda = \frac{\sinh 2\pi h(\lambda)}{2\pi}\, {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; -\sinh^2 2\pi h(\lambda)\right)$$

The interpolating h-function: expansions

- *weak* and *strong* coupling expansions from **conjecture**:

$$h(\lambda) = \begin{cases} \lambda - \frac{\pi^2}{3}\lambda^3 + \frac{5\pi^4}{12}\lambda^5 - \frac{893\pi^6}{1260}\lambda^7 + \mathcal{O}(\lambda^9) & \lambda \ll 1 \\ \sqrt{\frac{1}{2}\left(\lambda - \frac{1}{24}\right)} - \frac{\log 2}{2\pi} + \mathcal{O}\left(e^{-2\pi\sqrt{2\lambda}}\right) & \lambda \gg 1 \end{cases}$$

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  $SU(2) \times SU(2)$ sector dispersion relation to 2 loops
  [Minahan, Ohlsson-Sax, Sieg 09]
  IR divergence of 2-loop scattering amplitudes/ UV of cusped WL
  [MSB, Leoni, Mauri, Penati, Santambrogio; Chen, Huang 12; Henn, Plefka, Wiegandt 10]

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- tests against **perturbative** results:
  $SU(2) \times SU(2)$ sector dispersion relation to 4 loops
  [Leoni, Mauri, Minahan, Ohlsson Sax, Sieg, Tartaglino-Mazzuchelli 10]

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  $AdS_4 \times CP^3$ spinning string computation
  [Mc Loughlin, Roiban; Alday, Arutyunov, Bykov; Krishnan 08 + more]

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  confirms predicted *anomalous radius shift*

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- mutual consistency of several computations and ingredients: conjecture must be correct, but **proof** desirable

A **proof** of the exact $h(\lambda)$ could be derived computing the same observable exactly by two independent methods, e.g. integrability *and* localization: Bremsstrahlung is a candidate

!

Supersymmetric Wilson loops

- 1/2 *locally* supersymmetric WL in $\mathcal{N} = 4$ SYM : coupling to a scalars

$$W_{1/2}[\Gamma] = \frac{1}{N} \operatorname{Tr} \mathcal{P} \exp \left[-i \int_\Gamma d\tau \, \left(A \cdot \dot{x} - i \, |\dot{x}| \, n_I \, \Phi^I \right) (\tau) \right]$$

Supersymmetric Wilson loops

$1/6$ *locally* supersymmetric WL in $\mathcal{N}=6$ **ABJM**: coupling to a scalars

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$$M = \mathrm{diag}\,(-1,-1,1,1)$$

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- $1/2$ *locally* supersymmetric WLs in ABJM require coupling to both scalars and fermions
- $U(N_1|N_2)$ *supermatrix* structure

$$\mathcal{L}_{1/2} = \left(\begin{array}{c|c} \text{bosonic } N_1\times N_1 & \text{fermionic } N_1\times N_2 \\ \hline \text{fermionic } N_2\times N_1 & \text{bosonic } N_2\times N_2 \end{array}\right)$$

- 1/2-BPS WLs are *cohomologically equivalent* to comb of 1/6-BPS WLs (under supercharge Q)

[Drukker, Trancanelli 10]

$$W_{1/2} - \begin{pmatrix} W_{1/6} & 0 \\ 0 & \hat{W}_{1/6} \end{pmatrix} = Q\,V \qquad \text{difference is } Q\text{-exact}$$

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- these WLs were given an interpretation via *Higgsing* [Lee, Lee 10]

1/2 BPS Wilson loop in details

[Drukker, Trancanelli 10]

$$W[\Gamma] = \frac{1}{2N} \mathrm{Tr} \left[\mathrm{P} \exp \left(-i \int_\Gamma d\tau \ \mathcal{L}(\tau) \right) \right]$$

$$\mathcal{L} = \begin{pmatrix} \mathcal{A} & i\sqrt{\frac{2\pi}{k}} |\dot{x}| \eta_I^\alpha \bar{\psi}_\alpha^I \\ -i\sqrt{\frac{2\pi}{k}} |\dot{x}| \psi_I^\alpha \bar{\eta}_\alpha^I & \hat{\mathcal{A}} \end{pmatrix}$$

$$\begin{cases} \mathcal{A} \equiv A_\mu \dot{x}^\mu - \frac{2\pi i}{k} |\dot{x}| \mathcal{M}_J{}^I Y_I \bar{Y}^J \\ \hat{\mathcal{A}} \equiv \hat{A}_\mu \dot{x}^\mu - \frac{2\pi i}{k} |\dot{x}| \mathcal{M}_J{}^I \bar{Y}^J Y_I \end{cases} , \quad \mathcal{M} = \mathrm{diag}\,(-1, +1, +1, +1)$$

- Tr denotes the standard matrix trace (and not the super-trace)
- $\mathcal{M}$ are matrices in R-symmetry space controlling coupling to bi-scalars
- η, $\bar{\eta}$ are *commuting* spinors controlling coupling to fermions

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global susy

- contours exist where charges are preserved *globally*, e.g. **line** and **circle**
- amenable of an **exact computation** via *localization*

Exact results for ABJM WL

$\mathcal{N} = 2$ CS-matter theories on S^3 can be **localized**

!

[Kapustin, Willet, Yaakov 09; Drukker, Marino, Putrov 10 + many more]

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$$\mathcal{Z} = \int \prod_{a=1}^{N} d\lambda_a \, e^{i\pi k \lambda_a^2} \prod_{b=1}^{N} d\hat{\lambda}_b \, e^{-i\pi k \hat{\lambda}_b^2} \frac{\prod_{a<b}^{N} \sinh^2(\pi(\lambda_a - \lambda_b)) \prod_{a<b}^{N} \sinh^2(\pi(\hat{\lambda}_a - \hat{\lambda}_b))}{\prod_{a=1}^{N} \prod_{b=1}^{N} \cosh^2(\pi(\lambda_a - \hat{\lambda}_b))}$$

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$$\times \frac{1}{N}\sum_{a=1}^{N} e^{2\pi\lambda_a}$$

- expectation value of 1/6 -BPS **circular** WLs

$$\langle W_{1/6}^{\circ}\rangle = 1+i\pi\frac{N}{k}+\frac{1}{6}\left(1+2N^2\right)\pi^2\frac{1}{k^2}+\frac{1}{6}\,i\,N\left(4+N^2\right)\pi^3\frac{1}{k^3}+\ldots$$

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- *imaginary part* in localization, due to *framing* of WL: removing it

[Calugareanu 59; Witten 89]

[Kapustin, Willet, Yaakov 09; Closset, Dumitrescu, Festuccia, Komargodski, Seiberg 12]

$$\left\langle W_{1/6}^{\circ}\right\rangle_0 = 1 + \frac{\pi^2}{6\,k^2}\left(5N^2+1\right) + \ldots\,, \qquad \left\langle W_{1/2}\right\rangle_0 = 1 + \frac{\pi^2\left(2N^2+1\right)}{6\,k^2} + \ldots$$

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- perturbative check of 1/6-BPS up to two loops at framing 0

[Drukker, Plefka, Young; Chen, Wu; Rey, Suyama, Tamaguchi 09]

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$$\langle W_{1/6}^{\circ} \rangle_1 = 1 + i\pi \frac{N}{k} + \frac{1}{6}\left(1 + 2N^2\right) \pi^2 \frac{1}{k^2} + \boxed{\frac{1}{6} \, i \, N \left(4 + N^2\right) \pi^3 \frac{1}{k^3}} + \dots , \quad \langle W_{1/2}^{\circ} \rangle_1 = \frac{\langle W_{1/6}^{\circ} \rangle_1 + \langle \hat{W}_{1/6}^{\circ} \rangle_1}{2}$$

- *imaginary part* in localization, due to *framing* of WL: removing it

[Calugareanu 59; Witten 89]

[Kapustin, Willet, Yaakov 09; Closset, Dumitrescu, Festuccia, Komargodski, Seiberg 12]

$$\left\langle W_{1/6}^{\circ} \right\rangle_0 = 1 + \frac{\pi^2}{6\,k^2} \left(5N^2 + 1\right) + \dots , \quad \left\langle W_{1/2} \right\rangle_0 = 1 + \frac{\pi^2 \left(2N^2 + 1\right)}{6\,k^2} + \dots$$

- perturbative check of 1/6-BPS at 3 loops at framing 1

[MSB, Griguolo, Leoni, Mauri, Penati, Seminara 16]

Exact results for ABJM WL

$\mathcal{N} = 2$ CS-matter theories on S^3 can be **localized** !

[Kapustin, Willet, Yaakov 09; Drukker, Marino, Putrov 10 + many more]

- **ABJM** matrix model (*not gaussian*)

$$\langle W_{1/2}^{\circ}\rangle_1 = \int \prod_{a=1}^{N} d\lambda_a \, e^{i\pi k \lambda_a^2} \prod_{b=1}^{N} d\hat{\lambda}_b \, e^{-i\pi k \hat{\lambda}_b^2} \frac{\prod_{a<b}^{N} \sinh^2(\pi(\lambda_a - \lambda_b)) \prod_{a<b}^{N} \sinh^2(\pi(\hat{\lambda}_a - \hat{\lambda}_b))}{\prod_{a=1}^{N}\prod_{b=1}^{N} \cosh^2(\pi(\lambda_a - \hat{\lambda}_b))} \times$$

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- perturbative check of *multiple winding* at 2, 3 loops (recursively)

[MSB 16]

- *multiple winding* n

[Klemm, Marino, Soroush 12]

Latitude contour

Latitude

one can deform contour of 1/6- and 1/2-BPS WLs from the equator to a "*latitude*" on S^2 **partly** preserving supersymmetry

[Cardinali, Griguolo, Martelloni, Seminara 12; MSB, Griguolo, Leoni, Penati, Seminara 14]

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  - vev is **NOT** given by a simple rescaling of the coupling (unlike $\mathcal{N}=4$ SYM)
  - vev depends on a **single parameter** combination $\nu = \sin 2\alpha \, \cos\theta$

$$\langle W_B(\theta,\alpha)\rangle = w_B(\nu) \qquad \langle W_F(\theta,\alpha)\rangle = w_F(\nu)$$

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- strong coupling dual description [Aguilera-Damia, Correa, Silva 14]
- an exact vev via *localization* is missing, but might be possible ⚠

Extremal limit

[MSB, Leoni 16]

The ABJ theory

Take the ABJ model, with gauge group $U(N_1)_k \times U(N_2)_{-k}$ $|N_1 - N_2| < k$

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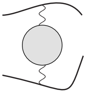
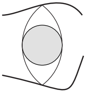
- in the $N_1 \gg N_2$ limit the WL reduces to pure CS $\longrightarrow$ exact vev [Witten 89]
- in the opposite limit $\boxed{N_2 \gg N_1}$ the leading approximation in N_2 is given by two classes of diagrams to **all orders** in k ($\lambda_i \equiv N_i/k$)



$$= \lambda_1\, f_{YM}(\lambda_2) \frac{-\dot{x}_1 \cdot \dot{x}_2}{|x_1 - x_2|^2}$$



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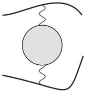

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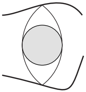

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- by requiring SUSY (finiteness on the line) and comparing to localization results for the circular 1/6-BPS WL fixes the exact value of f_{YM} and f_O

$$f_O(\lambda_2) = f_{YM}(\lambda_2) = \frac{1}{\pi} \sin \pi\lambda_2$$

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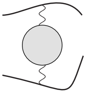

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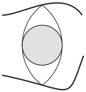

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In the extremal limit $N_2 \gg N_1$

- can compute vev of 1/6-BPS WL on **all** *smooth* contours
- exact cusp anomalous dimension, Bremsstrahlung, h-function†
- can prove integrability of $SU(2)$ sector

!

† conjecture for ABJ exact h-function

[Cavagliá, Gromov, Levkovich-Maslyuk 16]

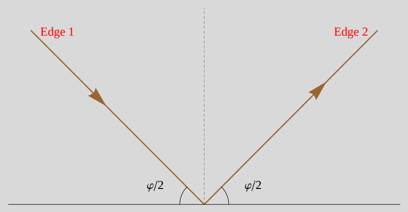
Supersymmetric cusps in ABJM model

Supersymmetric cusps

Construct locally BPS cusps in **ABJM** with:

$$W[\Gamma] = \frac{1}{\mathcal{N}} \mathrm{Tr} \left[\mathrm{P} \exp \left(-i \int_\Gamma d\tau \; \mathcal{L}_{1/6 \text{ or } 1/2}(\tau) \right) \right]$$

evaluated along the contour Γ: two rays intersecting at an angle $\pi - \phi$



$$\Gamma : \quad x^0 = 0 \quad x^1 = \tau \cos \frac{\phi}{2} \quad x^2 = |\tau| \sin \frac{\phi}{2} \quad -\infty \leq \tau \leq \infty$$

- $\mathcal{L}_{1/6}$ is the $U(N)$ Lie-algebra connection for $1/6$-BPS WL
- $\mathcal{L}_{1/2}$ is the $U(N|N)$ Lie-*super*algebra *super*connection for $1/2$-BPS WL
- for the $1/6$-BPS we could not find a *globally* preserved supercharge
- for the $1/2$-BPS...

The 1/2-BPS cusp

SUSY preserving cusp configuration

One can choose the parameters in such a way that 4/24 of supercharges are **preserved** on the cusped contour

[Griguolo, Martelloni, Poggi, Seminara 12]

The fermionic couplings on each straight-line factorize

$$\eta_{iM}^{\alpha} = n_{iM}\eta_i^{\alpha} \qquad \text{and} \qquad \bar{\eta}_{i\alpha}^{M} = \bar{n}_i^{M}\bar{\eta}_{i\alpha}$$

- On the first edge $\quad n_{1M} = \begin{pmatrix} \cos\frac{\theta}{4} & \sin\frac{\theta}{4} & 0 & 0 \end{pmatrix} \quad \eta_1^{\alpha} = \begin{pmatrix} e^{-i\frac{\phi}{4}} & e^{i\frac{\phi}{4}} \end{pmatrix}$

$$\bar{n}_1^M = \begin{pmatrix} \cos\frac{\theta}{4} \\ \sin\frac{\theta}{4} \\ 0 \\ 0 \end{pmatrix} \quad \bar{\eta}_{1\alpha} = i\begin{pmatrix} e^{i\frac{\phi}{4}} \\ e^{-i\frac{\phi}{4}} \end{pmatrix} \quad M_1 = \widehat{M}_1 = \begin{pmatrix} -\cos\frac{\theta}{2} & -\sin\frac{\theta}{2} & 0 & 0 \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- while on the second edge

$$n_{2M} = \begin{pmatrix} \cos\frac{\theta}{4} & -\sin\frac{\theta}{4} & 0 & 0 \end{pmatrix} \quad \eta_2^{\alpha} = \begin{pmatrix} e^{i\frac{\phi}{4}} & e^{-i\frac{\phi}{4}} \end{pmatrix}$$

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The 1/2-BPS cusp

A second angle

The angle θ is the counterpart of ϕ in $R-$symmetry space: angular separation of the two edges in the internal space $\mathbf{CP}^3$

[Griguolo, Martelloni, Poggi, Seminara 12]

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Exact Bremsstrahlung for 1/6-BPS

[Lewkowycz, Maldacena 13]

Consider small ϕ limit of cusp constructed from 2 locally 1/6-BPS rays (not a supersymmetric configuration *globally*, though):

- connect **Bremsstrahlung** to *stress tensor 1pt-function*
- connect latter to *entanglement entropy* of sphere with Wilson line insertion
- in 3d *branched* sphere $\rightarrow$ *squashed* sphere S_b^3

$$S_W = (1 - n\,\partial_n)\left|\log\langle W(S^3_{b=\sqrt{n}})\rangle\right|\Big|_{n=1}$$

- trade WL on S_b^3 with multiply wound W_n on S^3

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exact formula for 1/6-BPS lines Bremsstrahlung $B_{1/6}$

$$B_{1/6} = \frac{1}{4\pi^2}\,\partial_n \left|\langle W^{\circ}_{1/6,n}\rangle_1\right| \Big|_{n=1}$$

- tests:

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$$B_{1/6} = \boxed{\frac{\lambda^2}{2}} - \frac{\pi^2\lambda^4}{2} + \frac{47\pi^4\lambda^6}{72} + \dots \qquad \lambda \ll 1$$

- tests: weak coupling 2-loop

[MSB, Griguolo, Leoni, Penati, Seminara 14]

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- trade WL on S^3_b with multiply wound W_n on S^3

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- tests: strong coupling: up to subleading order

[Aguilera-Damia, Correa, Silva 14]

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✓

postulate exact Bremsstrahlung for 1/2-BPS cusp

$$B_{1/2} = \frac{1}{4\pi^2}\,\partial_n \left| \langle W^{\circ}_{1/2,n} \rangle_1 \right| \Big|_{n=1}$$

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postulate exact Bremsstrahlung for 1/2-BPS cusp

$$B_{1/2} = \frac{1}{4\pi^2}\,\partial_n\,\big|\langle W^{\circ}_{1/2,n}\rangle_1\big|\Big|_{n=1} = \big|\langle W^{\circ}_{1/6,n}\rangle + (-1)^n\,\langle \bar{W}^{\circ}_{1/6,n}\rangle_1\big|\Big|_{n=1}$$

- use cohomological equivalence

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- connect **Bremsstrahlung** to *stress tensor 1pt-function*
- connect latter to *entanglement entropy* of sphere with Wilson line insertion
- in 3d *branched* sphere $\to$ *squashed* sphere S_b^3

$$S_W = (1 - n\,\partial_n) \big| \log \langle W(S^3_{b=\sqrt{n}}) \rangle \big| \Big|_{n=1}$$

- trade WL on S_b^3 with multiply wound W_n on S^3

exact formula for 1/6-BPS lines Bremsstrahlung $B_{1/6}$

$$B_{1/6} = \frac{1}{4\pi^2}\,\partial_n\, \big| \langle W^{\circ}_{1/6,n} \rangle_1 \big| \Big|_{n=1}$$

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$$B_{1/2} = \frac{1}{4\pi^2}\,\partial_n\, \big| \langle W^{\circ}_{1/2,n} \rangle_1 \big| \Big|_{n=1} = \big| \langle W^{\circ}_{1/6,n} \rangle + (-1)^n \langle \bar{W}^{\circ}_{1/6,n} \rangle_1 \big| \Big|_{n=1}$$

- use cohomological equivalence
- $B_{1/2}$ is an odd function of k ($B_{1/2} < 0$ for $k < 0$??)

Exact Bremsstrahlung for 1/6-BPS

[Lewkowycz, Maldacena 13]

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In summary, the situation with the 1/2 BPS Wilson loop is confusing and requires more thought.

exact 1/2-BPS Bremsstrahlung via latitudes?

[MSB, Griguolo, Leoni, Penati and Seminara 14]

1) Let's conjecture that the Bremsstrahlung is obtained by deriving the **latitude** WL wrt ν

$$B_{1/2}(k,N) = \frac{1}{4\pi^2}\,\partial_\nu\,\log\langle W_F^\circ(\nu,k,N)\rangle_0\Big|_{\nu=1}$$

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and at strong coupling (up to subleading order!) [Aguilera-Damia, Correa, Silva 14; Forini, Giangreco Puletti, Ohlsson Sax 12]

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agrees with weak coupling calculation, which is effectively only 1 loop and at strong coupling (up to subleading order!)

[Aguilera-Damia, Correa, Silva 14; Forini, Giangreco Puletti, Ohlsson Sax 12]

However . . .

Three-loop prediction

Still, assuming the conjecture holds true

we can derive a **three-loop** prediction at weak coupling

- using the *localization* expectation value for the 1/6-BPS Wilson loops

$$\langle W^{\circ}_{1/6}\rangle_1 = 1 + i\,\pi\,\frac{N}{k} + \frac{1}{6}\left(1+2N^2\right)\pi^2\,\frac{1}{k^2} + \frac{1}{6}\,i\,N\left(4+N^2\right)\pi^3\,\frac{1}{k^3} + O(k^{-5})$$
$$\langle \hat{W}^{\circ}_{1/6}\rangle_1 = \langle W^{\circ}_{1/6}\rangle_1^*$$

and plugging them in the above conjecture we find

$$B_{1/2}(k,N) = \frac{N}{8\,k}\boxed{-\frac{\pi^2\,N\,(N^2-3)}{48\,k^3}} + \mathcal{O}\left(k^{-4}\right)$$

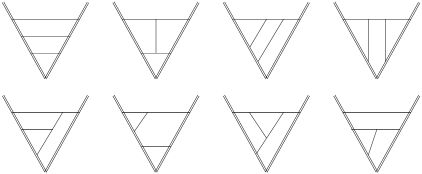
- 1-loop and (vanishing) 2-loop contributions agree with direct computation [Griguolo, Martelloni, Poggi and Seminara 12]

- first *non-planar* correction at three-loops
- $B_{1/2}$ is an *odd* function of k ($B < 0$ for $k < 0$??)
- from the conjecture $B_{1/2}$ comes entirely from the imaginary part of $W^{\circ}_{1/6}$ i.e. from the **framing** of the Wilson loop

Three-loop test

Computing a three-loop cusp

- *Eight* topologies
- Fill them in with any possible particle (here also *fermions* can attach to Wilson line)



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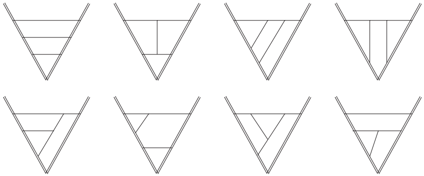

In CS theories:

- CS theory contains *ubiquitous* ε tensors (e.g. gauge propagator and vertex, γ-algebra)
- cusp lies on a *plane*, only contribution with **even number** of ε tensors can contribute
- from the Feynman rules at odd loops: completely gluonic graphs + many more are ruled out

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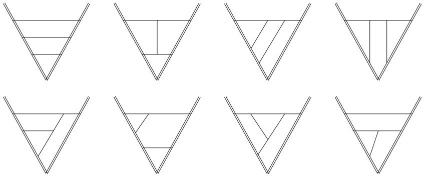

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Still...

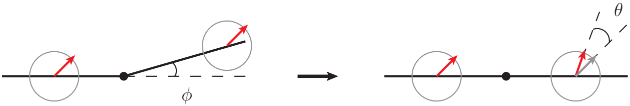
it is a *massive* computation (if done just bruteforce)

Computing the cusp at 0 angle

Use BPS condition: $\Gamma_{1/2} = -B_{1/2}\left(\phi^2 - \theta^2\right) \qquad \phi, \theta \ll 1$

$$B_{1/2} = \boxed{\frac{1}{2}\frac{\partial^2}{\partial\theta^2}\Gamma_{1/2}\Big|_{\phi=\theta=0}} = -\frac{1}{2}\frac{\partial^2}{\partial\phi^2}\Gamma_{1/2}\Big|_{\phi=\theta=0}$$

to extract Bremsstrahlung from cusp at $\phi = 0$.



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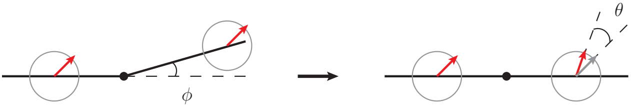

Advantages:

- reduced **number** of diagrams
- diagram **algebra** is easier
- **integrals** are much easier: propagator-type (GPXT) [Chetyrkin, Kataev, Tkachov 80]

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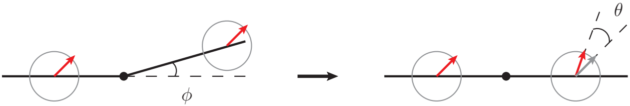

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Can use the *Korchemsky-Radyushkin* prescription [Korchemsky, Radyushkin 87]

$$\log\langle W(\theta)\rangle\Big|_{\phi=0} = \log V(\theta) - \log V(0)\Big|_{\phi=0}$$

to compute only the **1PI part**.

Renormalization of the cusp

divergences

the expectation value of the cusped WL suffers from both IR and UV divergences

- regulate IR divergences with cutoff L
- use dimensional regularization for UV divergence $d = 3 - 2\epsilon$ (DRED)
- *renormalize* the UV div multiplicatively $\langle W_R(\theta)\rangle = Z_{cusp}^{-1} \langle W(\theta)\rangle$
- extract the **cusp anomalous dimension**

$$\boxed{\Gamma_{cusp}(k,N)\Big|_{\phi=0} = \frac{d\log Z_{cusp}}{d\log\mu}\Big|_{\phi=0}}$$

simple poles in ϵ
$\mu =$ renormalization scale

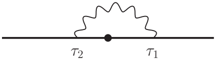
- in our case this is restricted to the $\phi = 0$ limit
- extract Bremsstrahlung function, taking the double derivative

$$B_{1/2}(k,N) = \frac{1}{2}\,\partial_\theta^2\,\Gamma_{1/2}(k,N,\phi=0,\theta)\Big|_{\theta=0}$$

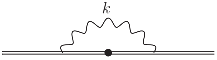
HQET picture

[Gervais, Neveu 80; Korchemsky, Radyushkin 87]

- Fourier transforming the Wilson line wrt the parameters, one obtains a *heavy quark effective theory* **HQET** propagator



$\longrightarrow$



$$\int_0^{+\infty} d\tau_1 \int_{-\infty}^{0} d\tau_2 \frac{1}{[(x_1 - x_2)^2]^{1/2-\epsilon}} \qquad \int \frac{d^{3-2\epsilon}k}{(2\pi)^{3-2\epsilon}} \frac{1}{k^2\,(i\,k\cdot v)^2}$$

$$\frac{1}{-i\,k\cdot v}$$

v: *velocity* of quark

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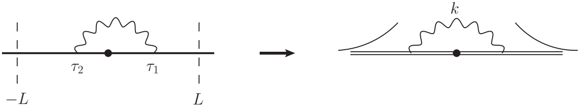

$$\int_0^{+L} d\tau_1 \int_{-L}^{0} d\tau_2 \frac{1}{\left[(x_1-x_2)^2\right]^{1/2-\epsilon}} \qquad\qquad \int \frac{d^{3-2\epsilon}k}{(2\pi)^{3-2\epsilon}} \frac{1}{k^2\,(i\,k\cdot v+\delta)^2}$$

- IR regulator: *residual energy* to massive quark

$$\frac{1}{-i\,k\cdot v} \longrightarrow \frac{1}{-i\,k\cdot v-\delta}$$

v: *velocity* of quark
δ: *residual energy*

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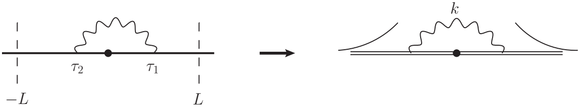
[Gervais, Neveu 80; Korchemsky, Radyushkin 87]

- Fourier transforming the Wilson line wrt the parameters, one obtains a *heavy quark effective theory* **HQET** propagator



$$\int_0^{+L} d\tau_1 \int_{-L}^{0} d\tau_2 \, \frac{1}{[(x_1 - x_2)^2]^{1/2-\epsilon}} \qquad \int \frac{d^{3-2\epsilon} k}{(2\pi)^{3-2\epsilon}} \frac{1}{k^2 \, (i\, k \cdot v + \delta)^2}$$

- IR regulator: *residual energy* to massive quark

$$\frac{1}{-i\, k \cdot v} \longrightarrow \frac{1}{-i\, k \cdot v - \delta}$$

v: *velocity* of quark
δ: *residual energy*

- cusp: heavy-quark form factor, velocities at an angle ϕ

Advantage of the HQET picture

can use powerful *integration-by-parts* (**IBP**) identities for reduction to *master integrals* (**MI**)

Reduction to master integrals

- IBP identities, e.g. $\int d^d k \, \frac{\partial}{\partial k} \cdot k \, \frac{1}{k^2(k-p)^2} = 0$ [Chetyrkin, Tkachov 81]
- can reduce all integrals of a given topology to a set of master integrals;
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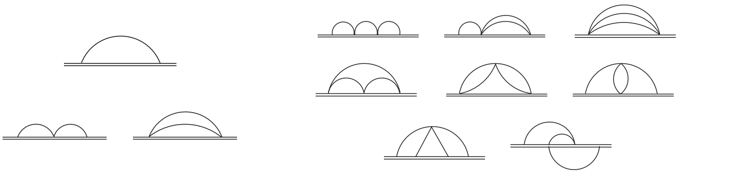
- nonplanar integrals have *linearly dependent* denominators (peculiarity of linear HQET propagators): reduce by *partial fractioning*

Reduction to master integrals

- in particular there are 1, 2 and 7+1 master integrals at 1, 2 and 3 loops



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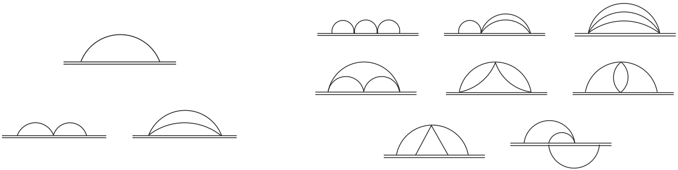

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write down diagram → perform tensor algebra → scalar products to inverse propagators → reduce nonplanar by partial fractioning → reduce to masters → substitute expansions of master integrals

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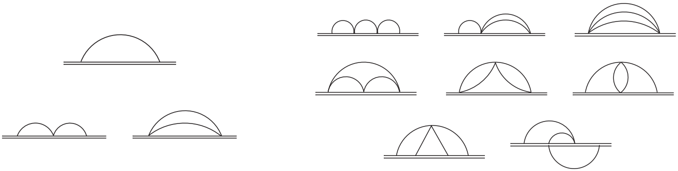

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Performance

The C++ reduction of FIRE5 reduces the needed 1259 planar + 584 nonplanar needed integrals in a few minutes!

The results

$$\log\langle W(\phi=0,\theta)\rangle = \frac{N(C_\theta-1)}{2\,k\,\epsilon} + \frac{\pi^2\,N\,(C_\theta-1)\left(N^2\,(C_\theta-2)+C_\theta^2+2\right)}{36\,k^3\,\epsilon} + \dots$$

Properties: $C_\theta = \cos\frac{\theta}{2}$

- **exponentiates**,
- *uniform* degree of *transcendentality*,
- BPS condition, i.e. for $\theta = 0$, (a bit trivial at $\phi = 0$, but checked up to 2 loops for whole cusp)
- can extract cusp anomalous dimension

$$\Gamma_{cusp}(k,N,\phi=0) = \frac{N\,(1-C_\theta)}{k} - \frac{\pi^2\,N\,(C_\theta-1)\left(C_\theta^2+N^2\,(C_\theta-2)+2\right)}{6\,k^3} + \dots$$

Bremsstrahlung function

$$B_{1/2}(k,N) = \frac{N}{8\,k} \boxed{-\frac{\pi^2\,N\,(N^2-3)}{48\,k^3}} + \dots$$

This result is in **agreement** with the conjecture (including nonplanar)

!

Conclusions

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proposal

a conjecture exists for the exact 1/2-BPS Bremsstrahlung function in **ABJM** theory in terms of supersymmetric circular Wilson loops

test

we have performed a successful **3-loop** test of it at weak coupling, including color subleading corrections

Outlook

Wishlist:

- compute the **whole 3-loop cusp**  elliptic functions

- *integrability* based computation of Bremsstrahlung and proof of exact $h(\lambda)$ progress. . .
  [Bombardelli, Cavagliá, Fioravanti, Gromov, Tateo 17]

- The generalized cusp should be captured by some scattering of *massive* particles in *Higgsed* ABJM. Partial results for massive amplitudes in **ABJM**
  [Caron-Huot, Huang 12; MSB 15]

- computation of 1/2-BPS Bremsstrahlung can be easily extended to the **ABJ** case, namely different ranks N_1, N_2.
  The result, though, looks *bizarre*: no sign of exponentiation.
  How to interpret the divergence and extract cusp anomalous dimension Γ_{cusp}? ?

Thank you!

A look at the whole cusp

Opening the angle

Complications:

- algebra of diagram is more involved, but doable
- reduction to master integrals is more intensive, but doable and here gives 2, 8 and 71 master integrals at 1, 2 and 3 loops
- master integrals are *functions* of ϕ rather than numbers

Strategy

Try to solve MI's by **differential equations**

[Grozin, Henn, Korchemsky, Marquard 15]

- 1 and 2 loop result computed, up to finite part 
- 3 loops is being computed 
- difficulty: *elliptic sectors*

Differential equations in canonical form

- set of MI's obey closed *systems of differential equations*

$$\partial_x \vec{f}(x) = A(x,\epsilon)\,\vec{f}(x) = \left(A_0(x) + A_1(x)\,\epsilon + \ldots\right)\,\vec{f}(x)$$

- can be used to solve integrals efficiently [Gehrman, Remiddi 01]
- have to provide boundary conditions
- a special case: *Fuchsian* system in ϵ-form [Henn 13]

$$\partial_x \vec{f}(x) = \epsilon\,\tilde{A}(x)\,\vec{f}(x) \qquad \tilde{A} = \sum_{x_i \in alphabet} \frac{1}{x - x_i}$$

- iterative (and **simple**) expansion in ϵ in terms of *generalised polylogarithms* (HPL, Goncharov and more general *dlog* forms)

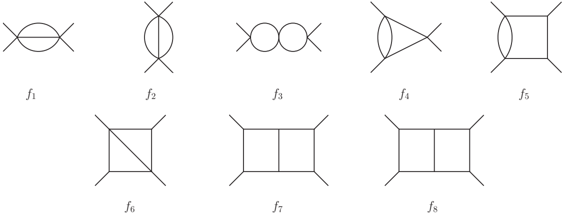
$$G_{a_1 \ldots a_n}(x) = \int_0^x \frac{dt_1}{t_1 - a_1} \ldots \int_0^{t_{n-1}} \frac{dt_n}{t_n - a_n}$$

Existence of ϵ-form guarantees **uniform degree** of **transcendentality** by construction!

An example in 3d

The problem: 2-loop 4-point massless **scattering** in 3d

- solved for ABJM and $\mathcal{N} = 8$ SYM by direct computation via MB
  [Chen, Huang; MSB, Leoni, Mauri, Penati, Santambrogio 10; MSB, Leoni 12; Bianchi, MSB 13]
- the problem can be reduced in general to a set of 8 MI's



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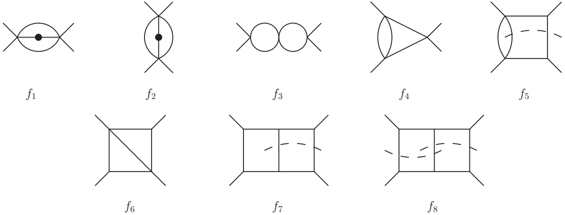

- $\exists$ canonical basis with variable $\dfrac{s}{t} \equiv \dfrac{(1-q^2)^2}{4q^2}$ and alphabet $\{0, \pm 1, \pm i\}$
- MIs can be solved in terms of *Goncharov polylogarithms* to **all** orders in ϵ

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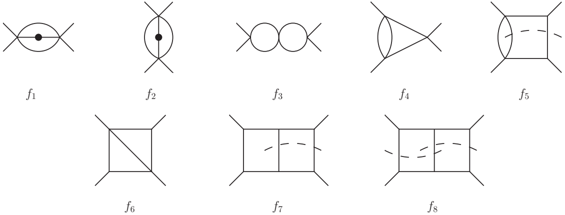

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- one of the basis element can be chosen to be the complete planar 4-pt 2-loop amplitude of **ABJM**
- proves uniform transcendentality of whole planar 4-pt amplitude of **ABJM** by *BDS exponentiation*

Elliptic sectors

- not all integrals can be solved in terms of polylogarithms
- in particular *elliptic sectors* may arise
- coupled, irreducible, *homogeneous* differential equations: NO ϵ-form
- this happens for 3-loop cusp in 3d (unlike 4d!)

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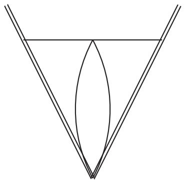

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- e.g. 2 MI's sector: $(f_1(x), f_2(x))$
    - each MI at a given order in ϵ expansion obeys 2^{nd} order DE:

$$f_1''(x) + \frac{(3x^4+1)\, f_1'(x)}{x(x^4-1)} + \frac{f_1(x)}{x^2} = r(x)$$

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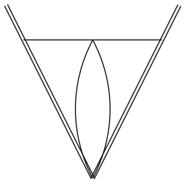

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  - *homogeneous* solution in terms of complete elliptic integrals:

$$f_1(x) = c_1 \underbrace{x\, K\left(x^4\right)}_{f_1^{(1)}} + c_2 \underbrace{x\, K\left(1-x^4\right)}_{f_1^{(2)}}$$

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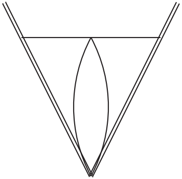

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  - inhomogeneous problem by *Euler's variation of constants*:

$$f_1(x) = c_1\, f_1^{(1)}(x) + c_2\, f_1^{(2)}(x) + \int_{x_0}^{x} dy\, \frac{-f_1^{(1)}(x)\, f_1^{(2)}(y) + f_1^{(2)}(x)\, f_1^{(1)}(y)}{\underset{\substack{\uparrow \\ \textit{Wronskian}}}{W(y)}}\, r(y)$$

  - this sector **can** be solved this way