Quantum mechanics and the holomorphic anomaly

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Based on

S.C., M. Mariño, 1612.07687

S.C., M. Mariño, R. Schiappa, to appear

1. Introduction
   - QM and topological strings
2. The double-well oscillator
   - WKB periods
   - Holomorphic anomaly
3. Beyond perturbation
   - An application
4. Conclusions and outlook

Long history of relations between QM and strings:

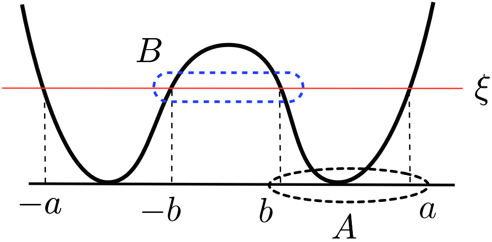
- Gauge theories & integrable systems
  [Nekrasov-Shatashvili]
- Topological strings & integrable hierarchies
  [Aganagic-Dijkgraaf-Klemm-Mariño-Vafa]
- Topological strings & spectral determinants
  [Grassi-Hatsuda-Mariño]

Our question:

Can the stringy machinery be applied even in QM problems without a stringy counterpart?

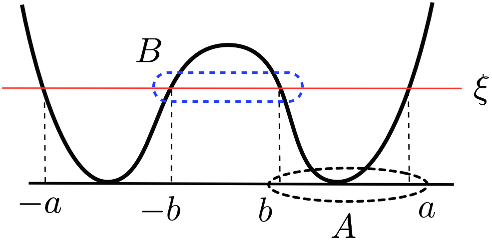
An example: consider the QM problem

$$-\frac{\hbar^2}{2}\psi'' + V\,\psi = \xi\psi, \qquad V(x) = \frac{x^2}{2}\,(1+gx)^2$$



WKB quantization: period around classically allowed (A) region

$$\int_b^a p(x;\xi_n)\,\mathrm{d}x \;\sim\; \pi\hbar\, n, \qquad p(x;\xi) := \sqrt{2}\sqrt{\xi - V(x)}.$$



All orders WKB: The Schroedinger equation defines a quantum differential on the curve $y^2 = p^2(x)$

$$P(x;\xi,\hbar) = p(x;\xi) + \sum_{n=1}^{\infty} \hbar^{2n} p_n(x;\xi)$$

through the WKB wavefunction

$$\psi(x;\xi) = P(x;\xi,\hbar)^{-1/2}\, e^{\frac{i}{\hbar}\int^x P(x';\xi,\hbar)\,\mathrm{d}x}$$

This defines a quantum period giving the all orders Bohr-Sommerfeld condition

$$\nu(\xi,\hbar) := \frac{1}{2\pi i}\oint_A P(x;\xi,\hbar)\,\mathrm{d}x \;\rightarrow\; \nu(\xi_n,\hbar) = n$$

yielding the ($\hbar$-)perturbative energy levels

Periods as an $\hbar$ series: use the Ricatti equation

$$\left.\begin{array}{c} Q := P + \frac{i\hbar\,\partial_x P}{2P} \\ \& \\ \text{Schroedinger eq.} \end{array}\right\} \Longrightarrow \; Q^2 - i\hbar\partial_x Q = p^2$$

and calculate recursively $p_n(x;\xi,\hbar)$

After integration, one gets the period as an $\hbar$ expansion

$$\nu(\xi,\hbar) = \frac{1}{2\pi i}\oint_A P(x;\xi,\hbar)\,\mathrm{d}x = t(\xi) + \sum_{n=1}^{\infty} t_n(\xi)\,\hbar^{2n}$$

Not perturbation theory!

WKB is a two parameter problem

- Perturbative in $\hbar$
- Exact in ξ
- Coefficients have $(2n)!$ divergence (as in string theory)

One *can* integrate by brute-force,

$$t(\xi) = \frac{\sqrt{\sqrt{32\xi}+1}}{12\pi}\left[\mathbf{E}\left(2-\frac{2}{\sqrt{32\xi}+1}\right)- \right. \\ \left. -\left(\sqrt{32\xi}-1\right)\mathbf{K}\left(2-\frac{2}{\sqrt{32\xi}+1}\right)\right]$$

but the higher $t_n(\xi)$ become unmanageable quickly!

A/B periods: we can also compute the B-cycle period corresponding to tunnelling effects

$$\frac{\partial F}{\partial \nu} := -i \oint_B P(x;\xi,\hbar)\,\mathrm{d}x, \quad \nu := \frac{1}{2\pi i}\oint_A P(x;\xi,\hbar)\,\mathrm{d}x$$

and define a quantum free energy F as in SW theory

$$F(\nu,\hbar) = \sum_{n=0}^{\infty} F_n(\nu)\hbar^{2n} = F_0(\nu) + F_1(\nu)\hbar^2 + O(\hbar^4)$$

The modulus of the elliptic curve $y^2 = p^2(x)$ is related to the prepotential $F_0(\nu)$ by

$$\tau(\nu) = \partial_{\nu\nu} F_0(\nu)$$

Can we use this structure to say more about $F(\tau,\hbar)$?

Upgrade $F_n(\tau)$ to modular forms $F_n(\tau, \bar{\tau})$ such that

$$\lim_{\bar{\tau} \to \infty} F_n(\tau, \bar{\tau}) = F_n(\tau)$$

They satisfy the (refined) holomorphic anomaly equations [BCOV]

$$\frac{\partial F_n(\tau, \bar{\tau})}{\partial \hat{E}_2} = -\frac{Y^2}{192} \sum_{r=1}^{n-1} D_\tau F_r \; D_\tau F_{n-r}, \qquad n \geq 1$$

in the NS limit [Huang-Klemm, Krefl-Walcher]

- Exact in τ (and hence ν)
- F_0 enters as the Yukawa $Y = \partial_{\nu\nu\nu} F_0$
- All the anholomorphic dependence is captured in $\hat{E}_2(\tau, \bar{\tau})$
- S-duality relates different problems (DW $\leftrightarrow$ quartic oscillator)

Solving the recursion:

- There is a holomorphic ambiguity at every order
  $\rightarrow$ Fix it with universal behaviour near singular points
- For $n \geq 2$ the expressions are nicely algebraic

$$F_1(\tau,\bar{\tau}) = -\frac{1}{24}\log\frac{\left(K_2^2 - K_4\right)K_4^2}{16K_2^6}$$

$$F_2(\tau,\bar{\tau}) = -Y^2\,\frac{5\hat{E}_2\left(2K_2^2 - 3K_4\right)^2 + 158K_2^5 - 330K_4K_2^3 + 135K_4^2K_2}{2211840K_2^2}$$

$$\ldots$$

with

$$K_2(\tau) = \vartheta_3^4(\tau) + \vartheta_4^4(\tau),\;\; K_4(\tau) = \vartheta_2^8(\tau),\;\; \hat{E}_2(\tau,\bar{\tau}) = E_2(\tau) - \frac{3}{\pi \mathrm{Im}\tau}$$

- **Most efficient way** so far to compute all orders WKB

Some motivation:

- QM interpretation of holomorphic anomaly [Witten]
- Invert reasoning to get HA for QM problems
  - Expected in problems related to topological strings (modified Mathieu potential, quantized mirror curves...)
  - **Tested in other genus one examples** (cubic, quartic oscillator)
- Proof for the HA in the NS limit of top. strings in [Grassi]

Beyond perturbation: transseries ansatz

$$F = F^{(0)} + e^{-\frac{A}{\hbar}} F^{(1)} + e^{-\frac{2A}{\hbar}} F^{(2)} + \dots$$

- Option A: Old way, use exact quantization [Zinn-Justin, Voros]

$$1 + \exp\left(\frac{i}{\hbar}\oint_A P(x;\xi,\hbar)\,\mathrm{d}x\right) = i\,\epsilon_{\text{parity}} \exp\left(\frac{i}{2\hbar}\oint_B P(x;\xi,\hbar)\,\mathrm{d}x\right)$$

- Option B: Holomorphic anomaly

  Upgrade recursion to full differential equation
  (as in [BCOV, Couso-Edelstein-Schiappa-Vonk])

$$\partial_{\hat{E}_2}\left(F - F_0^{(0)}\right) = -\frac{Y^2}{192}\left[D_\tau\left(F - F_0^{(0)}\right)\right]^2$$

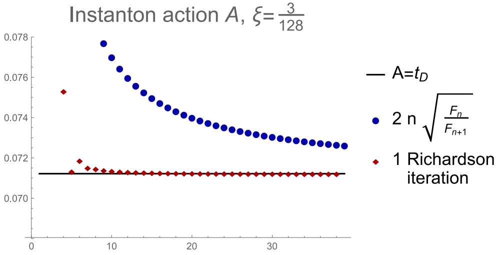
  and solve order by order in $e^{-\frac{A}{\hbar}}$... **Work in progress!**

An application: precision tests of resurgence

- HA: get $F_n^{(0)}$ up to very high n
- Resurgence: large order should be controlled by transseries

$$F_n^{(0)} = \frac{1}{i\pi}\frac{\Gamma(2n+2)}{A^{2n+2}}\left[F_0^{(1)} + \frac{F_1^{(1)}}{n} + O\left(\frac{1}{n^2}\right)\right],$$

Example: the action A is a (classical) period



Again: not perturbation theory, but all orders WKB!

Conclusions

- ☑ Periods of QM potentials with genus one spectral curves can be computed efficiently with the RHA
- ☑ DW, quartic, cubic, Mathieu, q. mirror curves...
- ☑ Access to high order $\hbar$ corrections $\rightarrow$ tests of resurgence
- ☐ Get a better understanding of the transseries
  In particular, go beyond the first instanton correction
- ☐ Borel resummation + transseries = exact quantization?
- ☐ Higher genus spectral curves