COVARIANT ENTANGLEMENT CONSTRUCTS

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based on earlier works w/ {M. Headrick, A. Lawrence, H. Maxfield, M. Rangamani, T. Takayanagi, E. Tonni}
& on work in progress w/ M. Headrick

Motivation

- Elucidate holography
  - Fundamental nature of spacetime & its relation to entanglement
  - Structure/characterization of CFTs (& states) w/ gravity dual
- Start w/ situations with large amount of symmetry (e.g. pure AdS)
  - Explicit calculations possible, can obtain analytical expressions
  - Use these to guess duality relations → entry in gauge/gravity dictionary
- But this has limitations
  - How to generalize? (e.g. time dependence)
  - Often symmetry brings degeneracy between logically distinct concepts
- Need to "covariantize"
  - Define a quantity which is purely geometrical (e.g. independent of any choice of coordinate systems) and fully general

Utility of covariant constructs

- Gives a general prescription
  - Definition of a quantity is equally robust on both sides of duality
  - Once beyond analytically tractable cases, might as well go for full generality (within the class of systems we want to consider, i.e. $N = \infty$)

We can learn a lot from classical bulk geometry!

Utility of covariant constructs

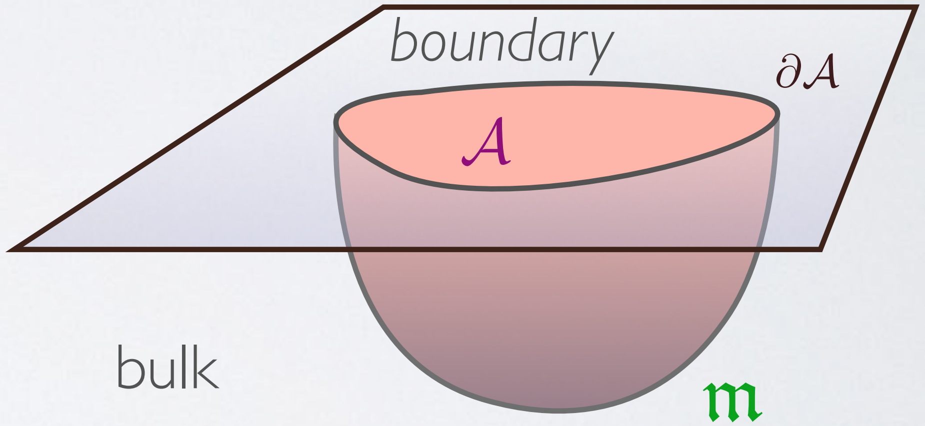
- Gives a general prescription
  - Definition of a quantity is equally robust on both sides of duality
  - Once beyond analytically tractable cases, might as well go for full generality (within the class of systems we want to consider, i.e. $N = \infty$)
- Time dependence interesting in its own right
  - Novel phenomena in out-of-equilibrium systems
  - New insight into the structure of the theory
- Breaks degeneracy between distinct constructs
  - Allows us to identify the true dual → underlying nature of the map
- Natural covariant constructs motivate new relations
  - Even if a given construct is not the sought dual, it eventually finds its use

Example: Holographic EE

Proposal [RT=Ryu & Takayanagi, '06] for *static* configurations:

In the bulk, entanglement entropy $S_{\mathcal{A}}$ for a boundary region $\mathcal{A}$ is captured by the area of a minimal co-dimension-2 bulk surface $\mathfrak{m}$ *at constant t* anchored on entangling surface $\partial\mathcal{A}$ & homologous to $\mathcal{A}$

$$S_{\mathcal{A}} = \min_{\partial\mathfrak{m}=\partial\mathcal{A}} \frac{\text{Area}(\mathfrak{m})}{4\,G_N}$$

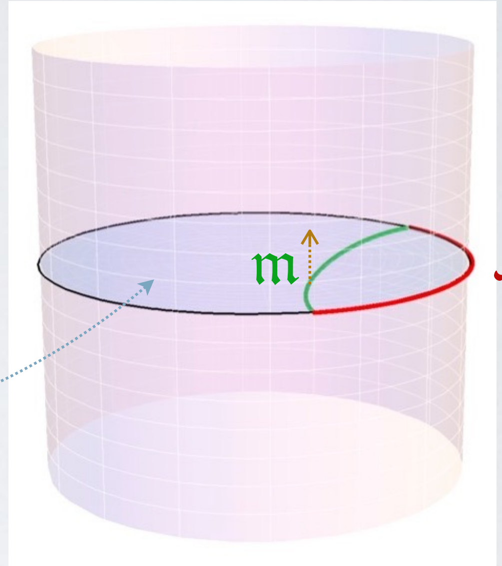


'Pf' in [Lewkowycz, Maldacena '13]

Covariant Holographic EE

But the RT prescription is not well-defined outside the context of static configurations:

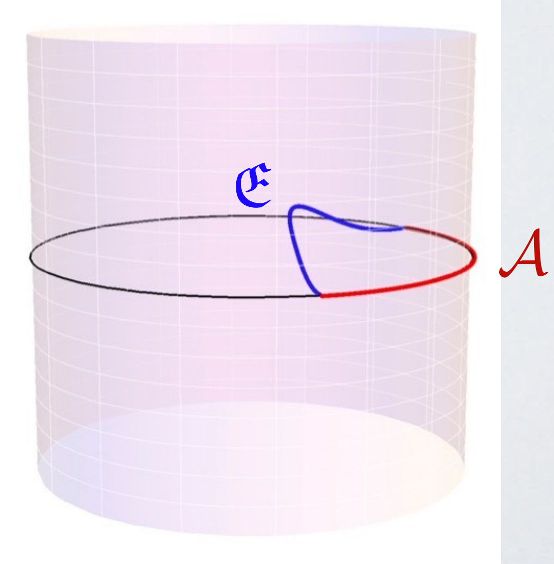
- In Lorentzian geometry, we can decrease the area arbitrarily by timelike deformations
- In time-dependent context, no natural notion of "const. t" slice…



In *time-dependent* situations, RT prescription must be covariantized:

Simplest candidate: [HRT = VH, Rangamani, Takayanagi '07]

minimal surface $\mathfrak{m}$ at constant time → extremal surface $\mathfrak{E}$ in the full bulk



This gives a well-defined quantity in any (arbitrarily time-dependent asymptotically AdS) spacetime
⇒ equally robust as in CFT

"Pf" in [Dong, Lewkowycz, Rangamani '16]

Curious features of EE:

- Extremal surfaces can have intricate behavior:
  - $\mathfrak{E}$ can have discontinuous jumps under smooth variations of $\mathcal{A}$

    $\rightarrow$ phase transitions in EE
  - $\mathfrak{E}$ can be topologically nontrivial even for simply-connected regions $\mathcal{A}$
- Holographic EE seems too local:
  - sharply-specified both on boundary and in bulk
  - but: $\rightarrow$ we can reconstruct the bulk metric (modulo caveats) solely from the set $\{S_{\mathcal{A}}\}$ for a suitable set of $\{\mathcal{A}\}$
- Holographic EE seems too non-local:
  - global minimization condition + homology constraint makes $S_{\mathcal{A}}$ sensitive to arbitrarily distant regions in the bulk…

Covariant Holographic EE

In fact, [HRT] identified 4 natural candidates:

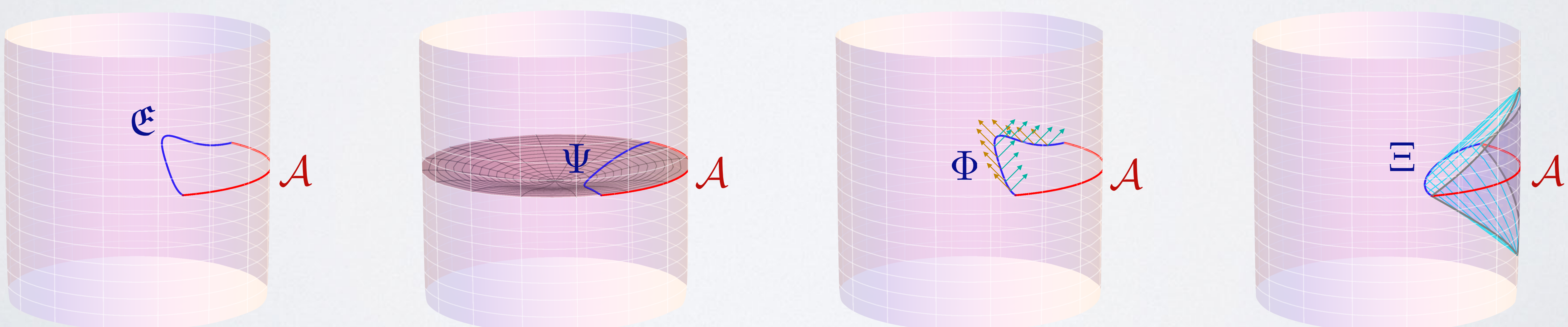
(all co-dim.2 surfaces ending on $\partial\mathcal{A}$, and coincident for ball regions $\mathcal{A}$ in pure AdS)

- $\mathfrak{E}$ = Extremal surface
- Ψ = Minimal-area surface on maximal-volume slice
- Φ = Surface with zero null expansions
- Ξ = Causal wedge rim

$\mathfrak{E} = \Phi$ is correct
= 'HRT prescription'

Later known as Causal Information Surface;
w/ area = causal holographic information χ

[VH, Rangamani '12]



Power of covariant constructs

- 'Natural' geometrical constructs (defined for general bulk spacetimes, independent of coordinates) provide useful candidates for dual of 'natural' quantities in CFT
- e.g. dual of $\rho_{\mathcal{A}}$? [Bousso, Leichenauer, Rosenhaus; Czech, Karczmarek, Nogueira, Van Raamsdonk;…]
- In generic Lorentzian spacetime, null congruences which define a causal set provide useful characterization of 'natural' bulk regions.

2 options:

…starting from bdy:

$D[\mathcal{A}] \rightsquigarrow$ Causal Wedge:

= future and past causally-separated from bdy region determined by $\rho_{\mathcal{A}}$

[VH & Rangamani]

…starting from bulk:

$\mathfrak{E} \rightsquigarrow$ Entanglement Wedge:

= spacelike-separated (toward $\mathcal{A}$) from $\mathfrak{E}$

[Headrick, VH, Lawrence, Rangamani]

also cf. [Wall],[CKNvR]

Argued to be the correct one:

[Dong, Harlow, Wall '16] via QEC & operator algebra

[Cotler, Hayden, Salton, Swingle, Walter '17] via recovery channels

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[VH & Rangamani]

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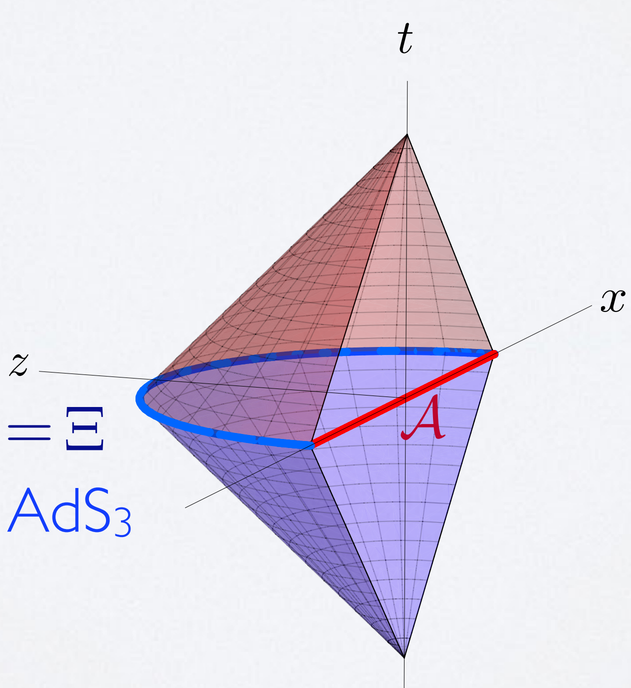
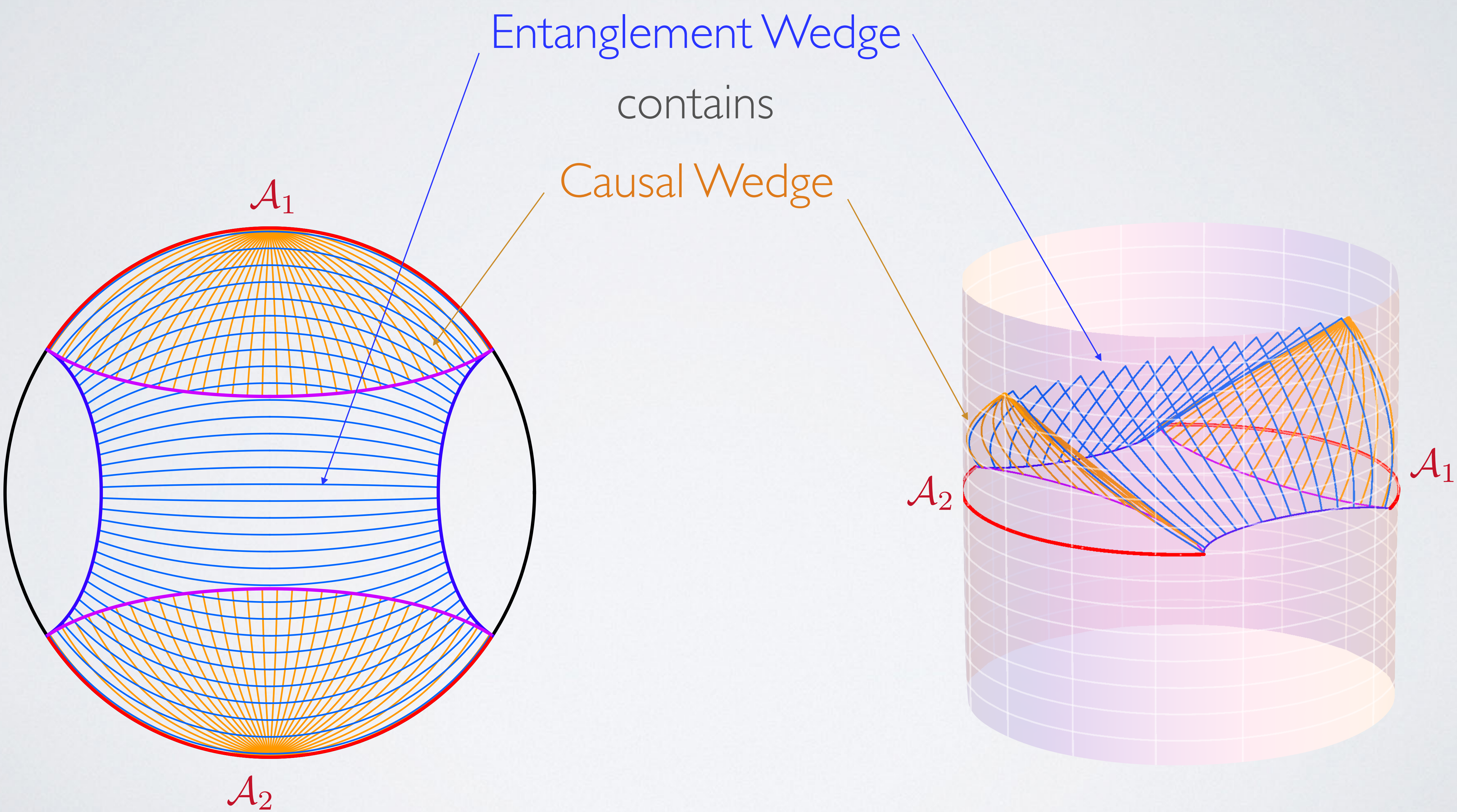


$\mathfrak{E} = \Xi$ in AdS_3

NB: in pure AdS, & for spherical $\mathcal{A}$, these coincide, but not in general.

Causal wedge vs. Entanglement wedge

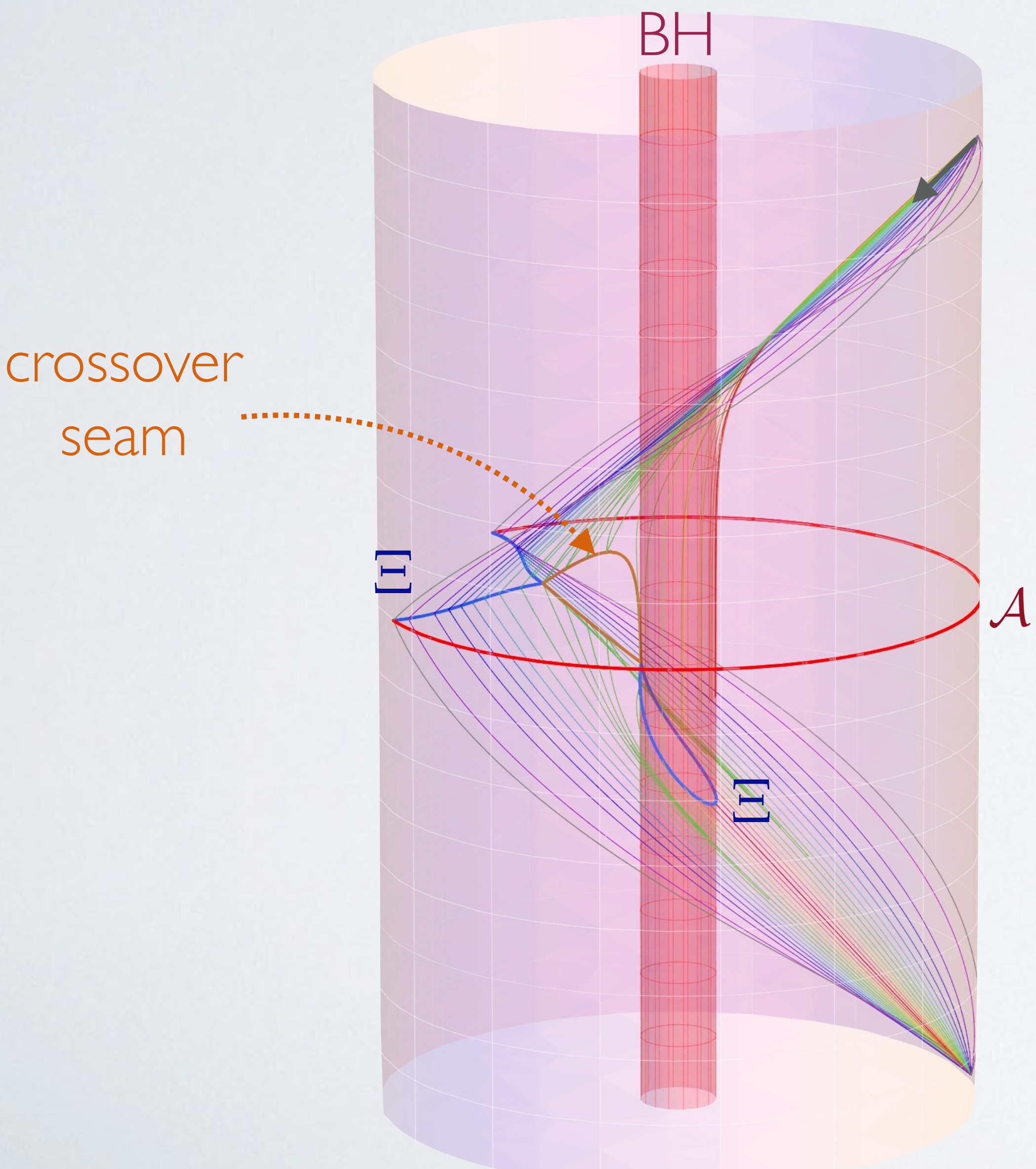
- Even in pure AdS_3, these can differ for composite regions $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$



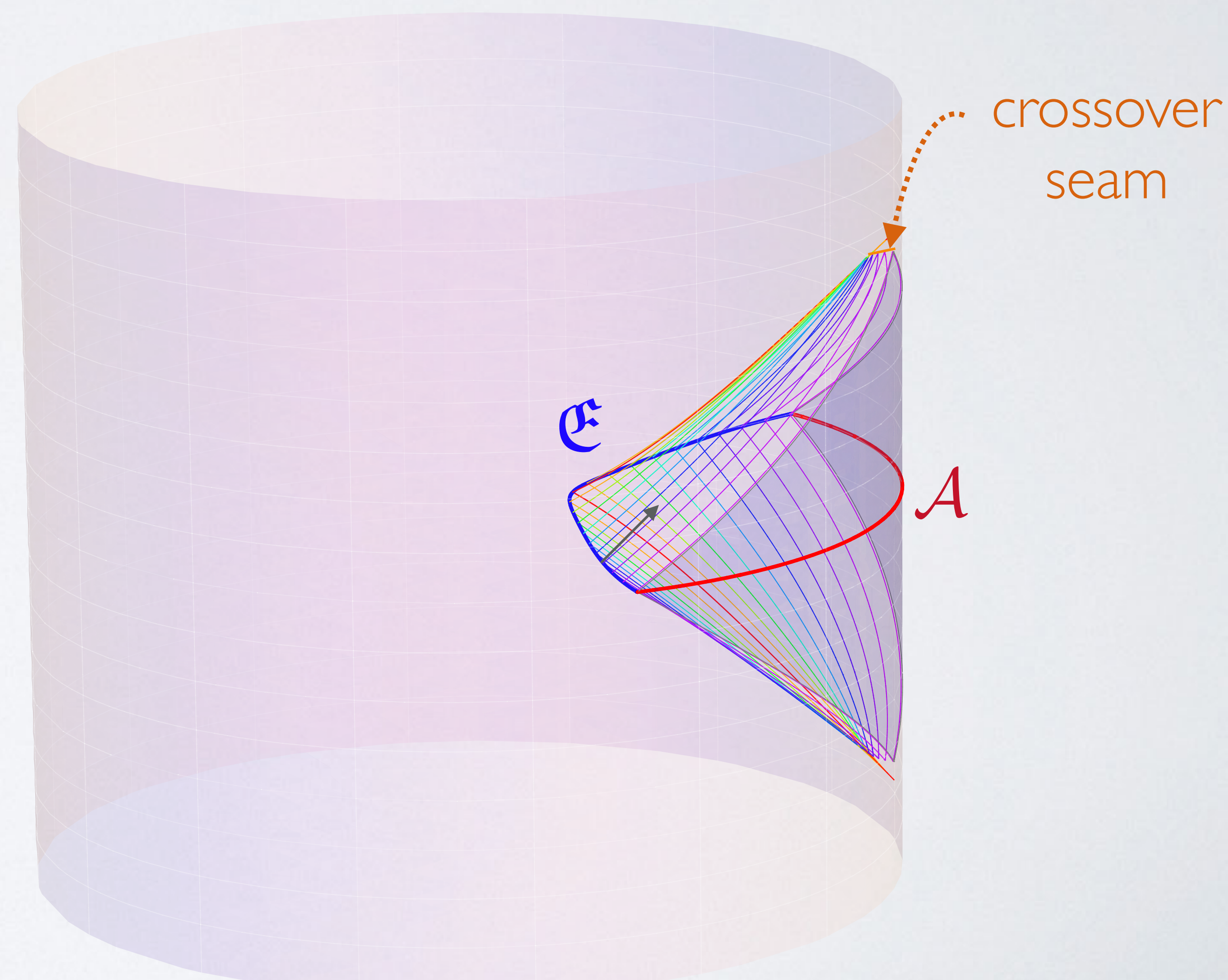
Causal wedge vs. Entanglement wedge

- crucial difference: in which direction can null generators cross…

$D[\mathcal{A}] \rightsquigarrow$ Causal Wedge:



$\mathfrak{E} \rightsquigarrow$ Entanglement Wedge:



Power of covariant constructs

$D[\mathcal{A}] \rightsquigarrow$ Causal Wedge:

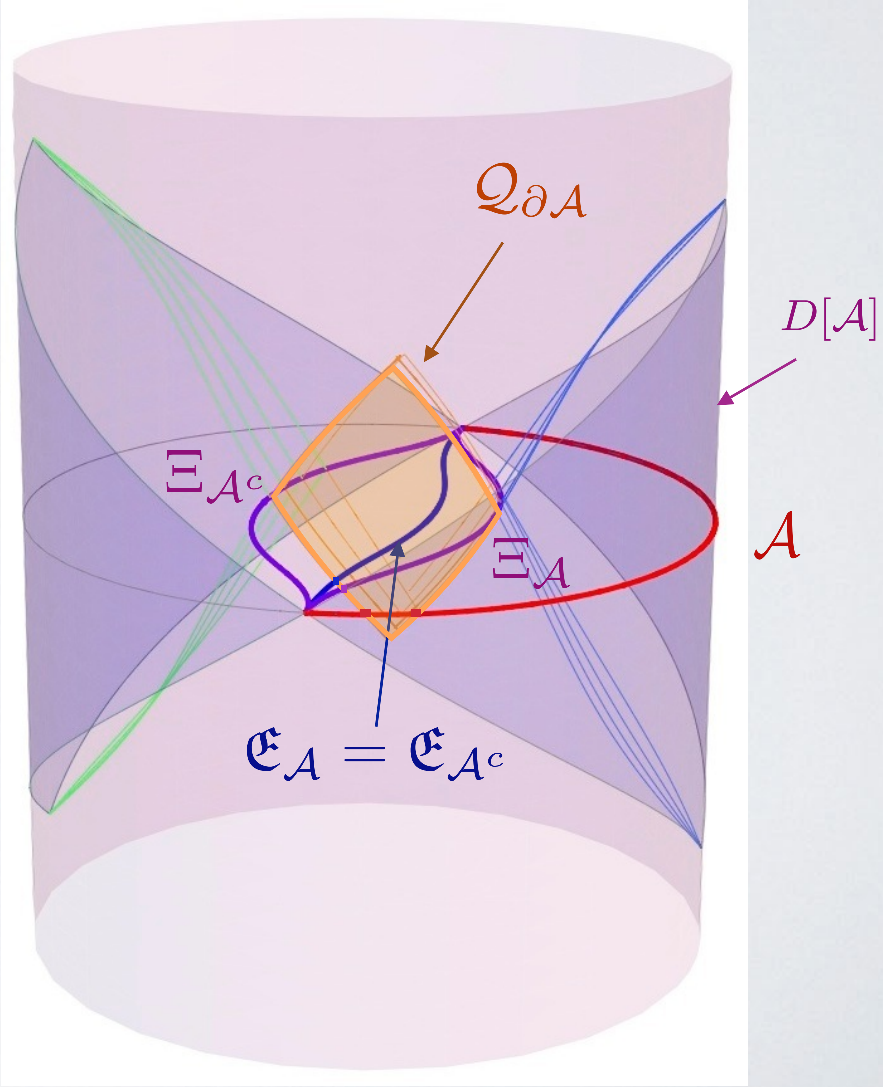
$\mathfrak{E} \rightsquigarrow$ Entanglement Wedge:

…continued past Ξ: $\rightsquigarrow$ Causal Shadow $\mathcal{Q}_{\partial\mathcal{A}}$

- We can prove the inclusion property [Headrick, VH, Lawrence, Rangamani; Wall]

CW ⊂ EW

or equivalently, $\mathfrak{E} \subset \mathcal{Q}_{\partial\mathcal{A}}$



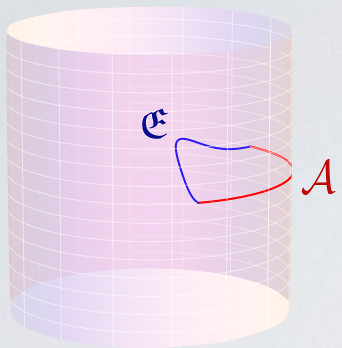
- Consequences:
  - HRT is consistent with CFT causality (= non-trivial check of HRT)
  - Entanglement plateaux
  - Entanglement wedge can reach deep inside a black hole!

Covariant re-formulations

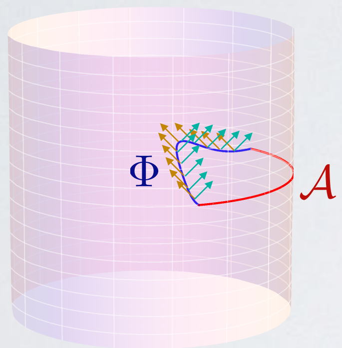
- Covariance is pre-requisite to construct being physically meaningful, but it need not be unique
    - Distinct geometrical formulations can turn out equivalent (cf. $\mathfrak{E} = \Phi$)

- This redundancy is useful
    - Each formulation can have its own advantages
    - e.g. different properties may be manifest in different formulations (cf. gauge / coordinate choice)
    - Re-formulation can reveal deeper relations (cf. ER=EPR [Maldacena, Susskind])

Covariant re-formulations of HEE



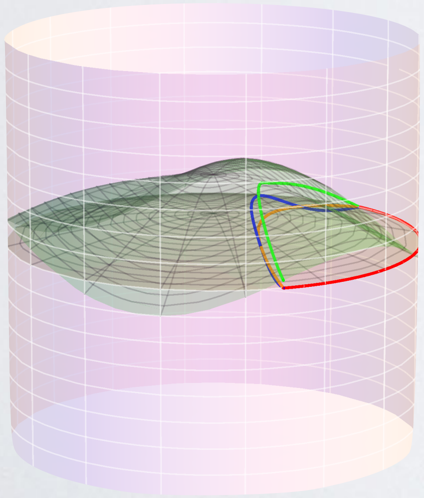
- $\mathfrak{E}$ = Extremal surface
  - (relatively) easy to find
  - minimal set of ingredients required in specification
  - need to include homology constraint as extra requirement



- Φ = Surface with zero null expansions
  - (cf. light sheet construction & covariant entropy bound [Bousso, '99]:

    Bulk entropy through light sheet of surface $\sigma \leq \text{Area}(\sigma)/4$

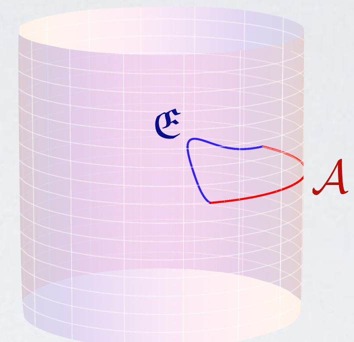
    Φ = surface admitting a light sheet closest to bdy

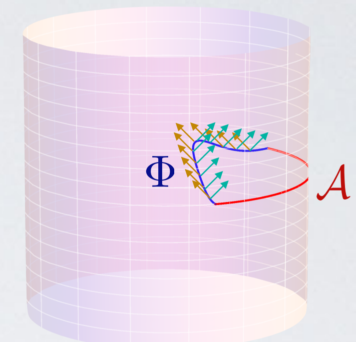


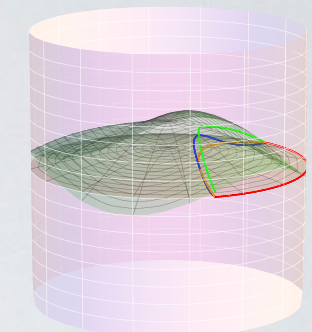
- Maximin surface [Wall, '12]
  - maximize over minimal-area surface on a spacelike slice
  - requires the entire collection of slices & surfaces
  - implements homology constraint automatically
  - useful for proofs (e.g. SSA)

Covariant re-formulations of HEE

All of these are the same geometrical construct.







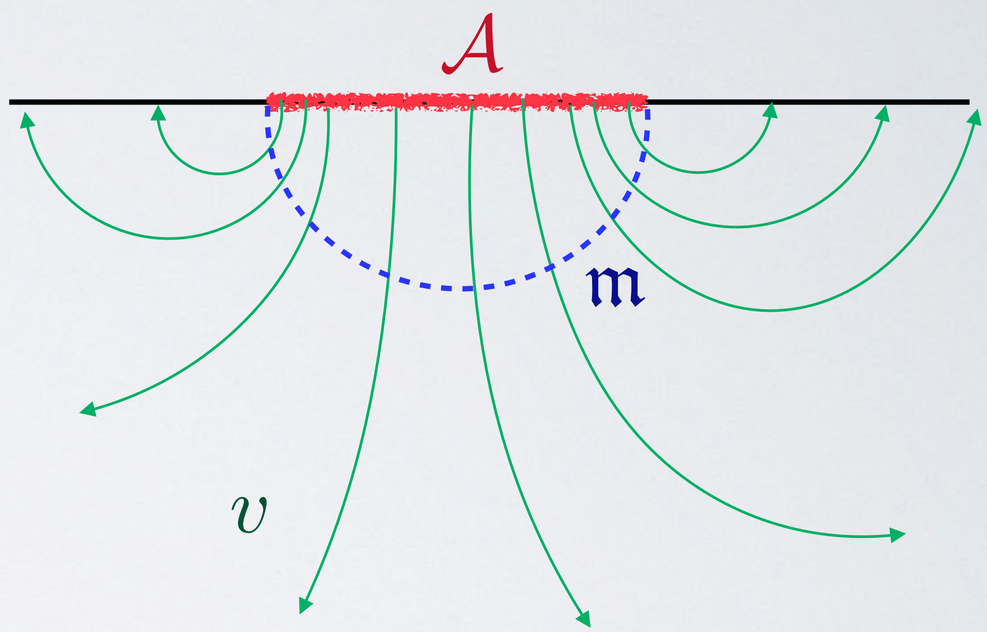
BUT it does not elucidate the relation to quantum information:

- Where does the information live?
- Mutual information I(A:B) = S(A) + S(B) - S(AB) is given by surfaces located in different spacetime regions.
- Geometric proof of SSA (S(AB) + S(BC) ≥ S(B) + S(ABC)) obscures its meaning as monotonicity under inclusion of correlations

Bit thread picture of (static) EE

- Reformulate EE in terms of flux of flow lines [Freedman & Headrick, '16]
  - let v be a vector field satisfying $\nabla \cdot v = 0$ and $|v| \leq 1$. Then EE is given by

$$S_{\mathcal{A}} = \max_{v} \int_{\mathcal{A}} v$$



  - By Max Flow - Min Cut theorem, equivalent to RT: (bottleneck for flow = minimal surface)

- Useful reformulation of holographic EE
  - flow continuous under varying region (while bottlenecks can jump discontinuously)
  - automatically implements homology constraint and global minimization of RT
  - maximal flow defined even without a regulator (when flux has UV divergence)
  - can be computed more efficiently (via linear programming methods)
  - implements QI meaning of EE and its inequalities more naturally
  - provides more intuition: think of each bit thread as connecting an EPR pair

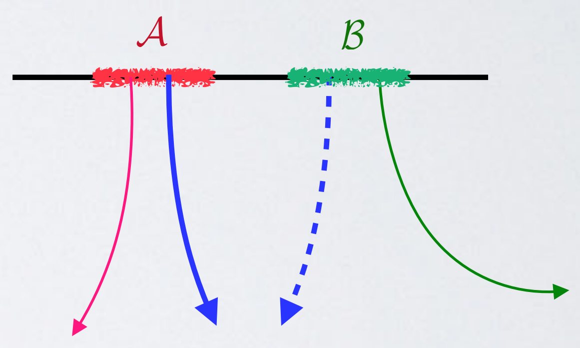
Bit threads - interpretation

Nesting: ∃ common maximizer flow for nested regions

Suppose we maximize on AB.
— Then we can additionally maximize on either A or B, but not both.

- Conditional entropy H(A:B) = S(AB) - S(B)
  ~ bits in A which are uncorrelated with B
  = # of threads left on A when we measure B
- Mutual information I(A:B) = S(A) + S(B) - S(AB)
  ~ correlations (redundancy) between A and B
  = # of threads which can flop between A and B
- Entangled qubits are threads between A and B
  which switch direction.

S(A)=S(B)=2, S(AB)=3:



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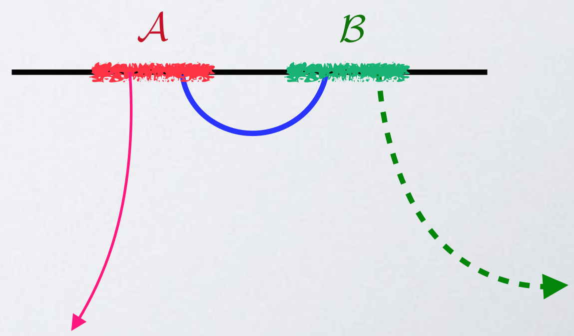

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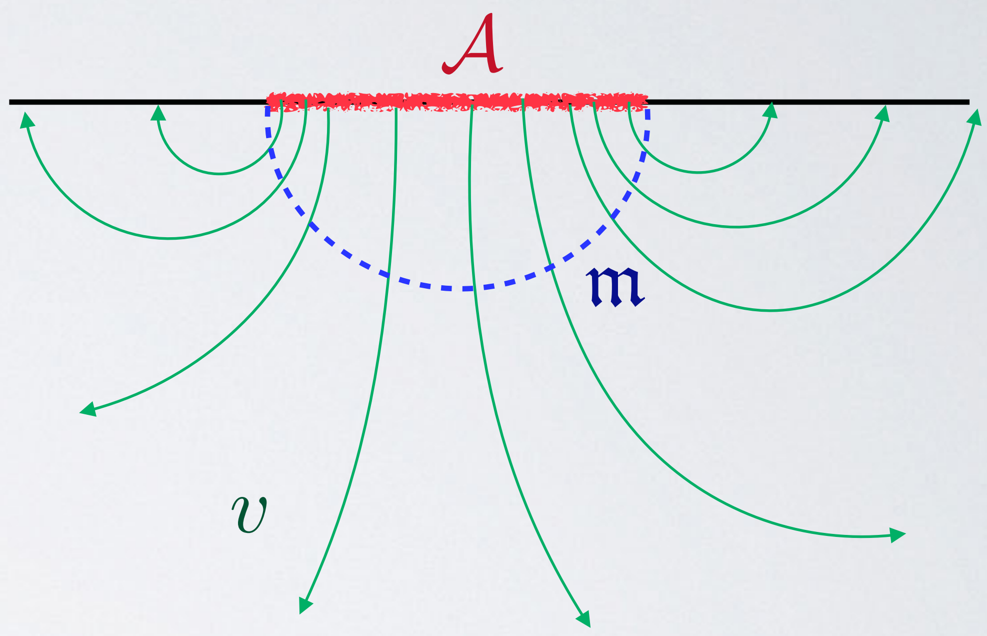
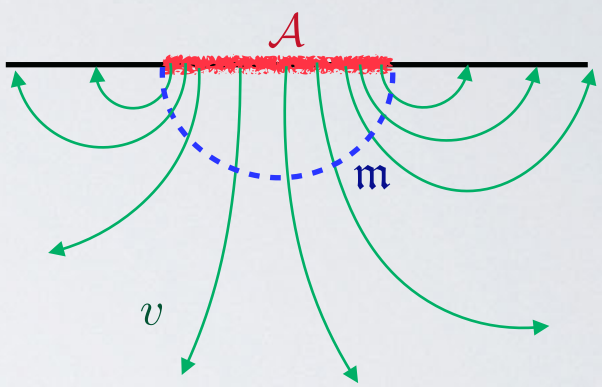


    - By Max Flow - Min Cut theorem, equivalent to RT: (bottleneck for flow = minimal surface)
- Useful reformulation of holographic EE
    - behaves more naturally
    - is more computationally efficient
    - ties better to QI quantities
    - provides more intuition
- How does this extend to time-dependent settings?

Covariantizing bit threads

1. Identify the correct geometrical quantities of interest

   Analogous to flow lines (vector field v) in



2. Identify the constraints they must satisfy

   Analogous to $\nabla \cdot v = 0$ and $|v| \leq 1$

3. Identify the expression for EE obtained from these

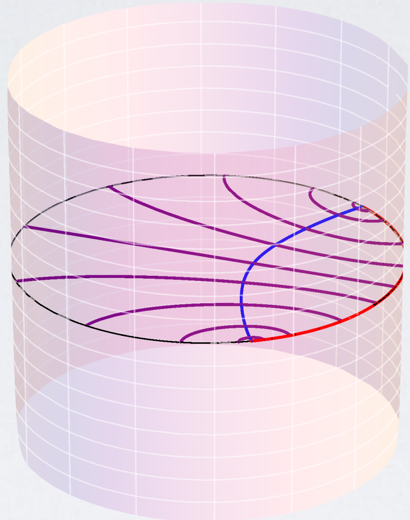
   Analogous to $$S_{\mathcal{A}} = \max_v \int_{\mathcal{A}} v$$

4. Test that it fulfills all requisite requirements

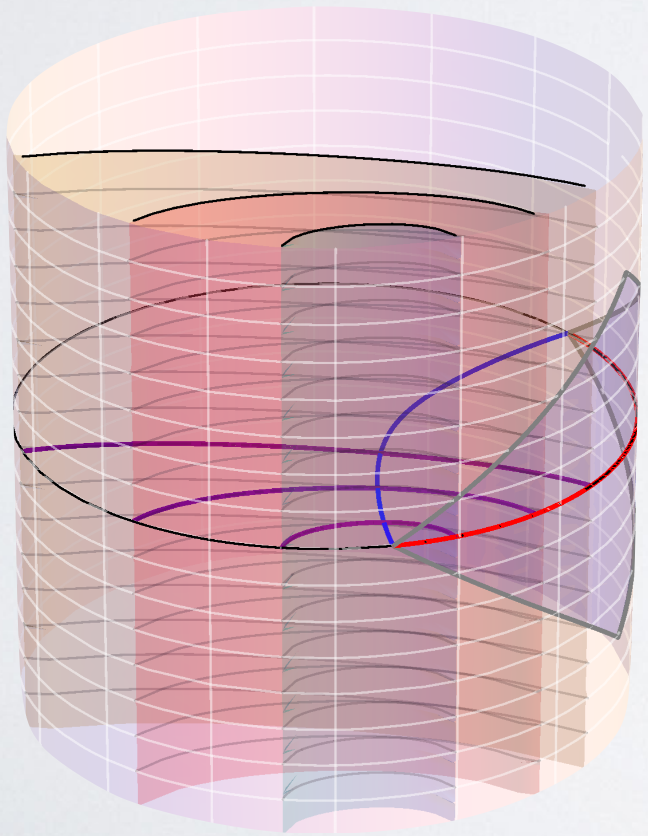
5. Extract lessons / implications

Two natural possibilities

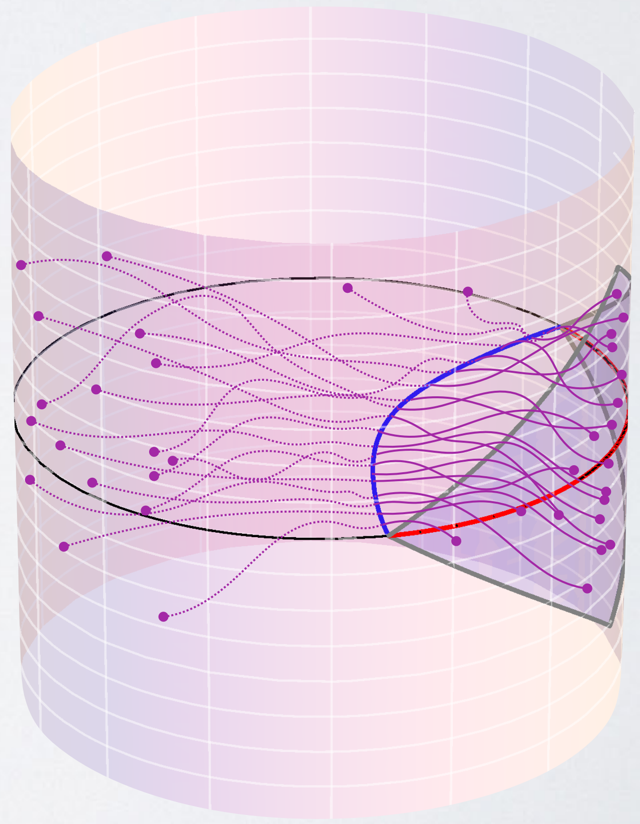
Step 1:



extend threads in time
flow sheets



keep 1-d threads
flow lines

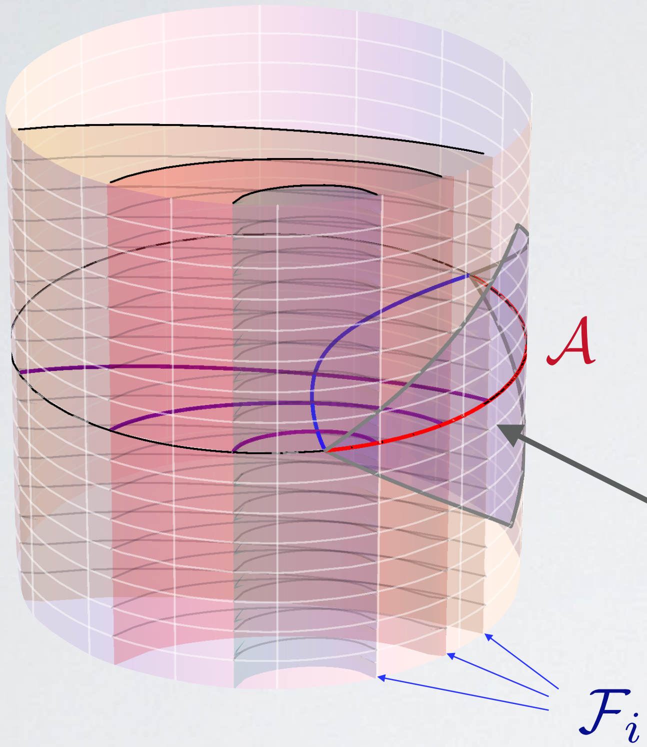


Requirements on constructs

- Imperative:
  - Reduces to bit threads in static case
  - Equivalent to HRT (when null energy condition (NEC) is obeyed)
  - Depends only on $D[\mathcal{A}]$ (i.e. $\partial\mathcal{A}$ + orientation), not on $\mathcal{A}$ itself

- Useful:
  - Manifests CFT causality (directly rather than via equivalence to HRT)
  - Manifests area law, positivity, subadditivity, SSA, etc.
  - Elucidates role of NEC
  - Elucidates role of homology constraint w/ time-dependence

Flow sheets



Require:

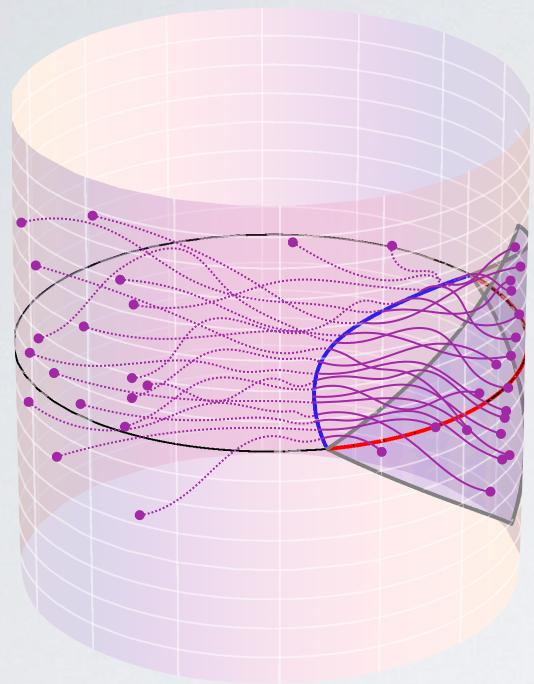
- flow sheets are timelike everywhere
- flow sheets cannot end in the bulk
- density is bounded

$$\rho = \frac{\sum_i \text{Area}[\mathcal{F}_i \cap \mathcal{R}]}{\text{Vol}[\mathcal{R}]} \leq 1$$

EE = sheets through $D[\mathcal{A}]$

- Most "obvious" generalization of bit threads
  - entanglement lasts in time & cannot be changed a-causally ✔
- Danger:
  - Potentially too global (e.g. future singularity may prevent sheets in past)
  - Too many sheets through $D[\mathcal{A}]$ by local boost

Flow lines



Require:

- flow lines start from $D[\mathcal{A}]$
- flow lines don't end: i.e. keep v s.t. $\nabla \cdot v = 0$
- but use integrated norm bound:

For a unit normal vector w on any worldline γ, $\int_\gamma w \cdot v \leq 1$

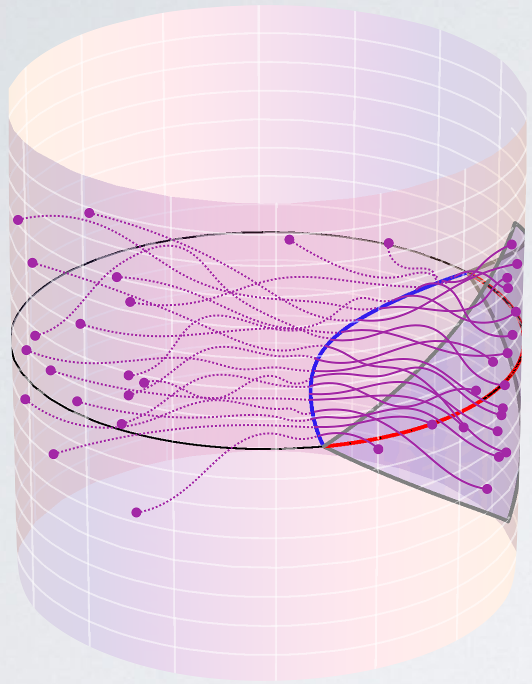
$\therefore$ Over full lifetime, any observer sees at most 1 thread / 4 Planck areas

EE counts bit threads in $D[\mathcal{A}]$: $S_{\mathcal{A}} = \max_v \int_{D[\mathcal{A}]} v$

- More promising:
  - reduces to bit threads at const time in static case
  - threads must all pass through extremal surface (for max flow)
  - endpoints are floppy and can lie anywhere within $D[\mathcal{A}]$
  - Bonus: naturally picks out the entanglement wedge
  - does not depend on spacetime in the far future

But: needs further specification for large regions

Flow lines



But what is the QI interpretation ?

- Entanglement entropy counted by events ?
  - e.g. # of indep. measurements that can be performed within $D[\mathcal{A}]$
  - novel interpretation…
- Why are 1-d structures natural?
  - why is a specific measurement connected to another instantaneous event somewhere in $\mathcal{A}^c$?

Convex optimization as a tool

- Max-flow/min-cut is an example of Lagrangian duality in theory of convex optimization

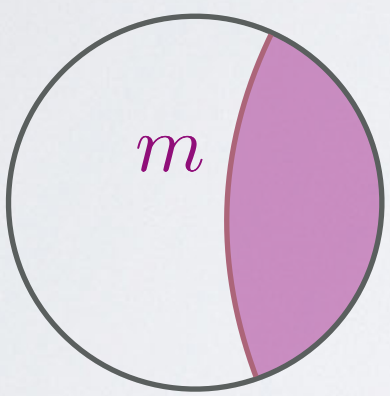
- Setup:
  - start with:
    - a vector space V
    - a non-empty convex subset $\mathcal{D}$ of V called the domain;
    - a convex function $f_0 : \mathcal{D} \to \mathbf{R}$ called the objective function;
    - a set of convex functions $f_i : \mathcal{D} \to \mathbf{R}$ called the *inequality* constraint functions;
    - and a set of affine functions $h_j : \mathcal{D} \to \mathbf{R}$ called the *equality* constraint functions.
  - Convex program P: minimize $f_0(y)$ over $y \in \mathcal{D}$ such that $\forall i,\ f_i(y) \le 0,\ \forall j,\ h_j(y) = 0$
  - use Lagrange multipliers $L(y, \lambda, \nu) \equiv f_0(y) + \sum_i \lambda_i f_i(y) + \sum_j \nu_j h_j(y)$
  - solution via convex optimization: $p^* = \inf_y \sup_{\lambda,\nu} L(y, \lambda, \nu)$

- Lagrangian duality: swap order
  - new extremization problem, in new variables

Convex optimization as a tool

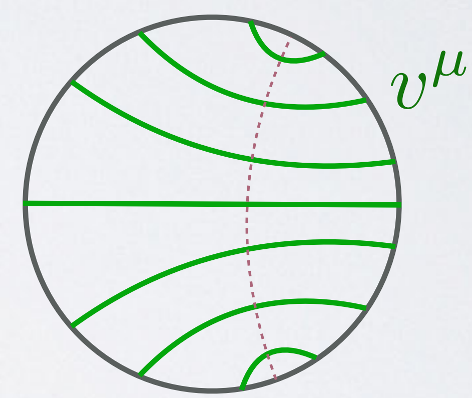
- Strategy:
  - Formulate the (Lorentzian) min cut side as convex relaxation
  - Interpret the dual geometrically
- Eg. for static case:

Min cut (RT): = Max flow (Bit threads):



$$S = \min[\mathrm{Area}(m)]$$

$$= \int_{\mathcal{M}} |\partial_\mu \psi| + \cdots \quad = \max \int_{\mathcal{A}} v^\mu$$



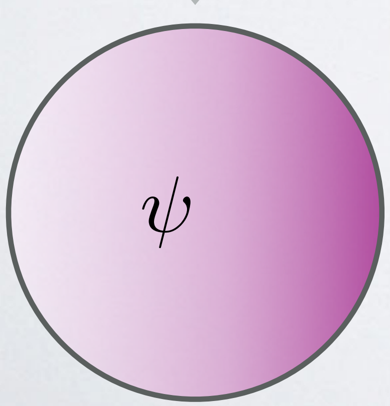
convex relaxation

Lagrangian duality

constraints:

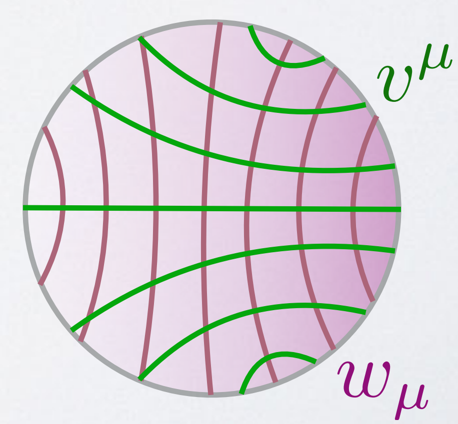
$$\nabla_\mu v^\mu = 0$$

$$|v^\mu| \leq 1$$



Recast by introducing a Lagrange multiplier

$$v^\mu \left(w_\mu - \partial_\mu \psi\right)$$



Summary & Outlook

- Holography conveniently geometrizes entanglement
  - Finding bulk geometrical constructs is (relatively) easy!
  - Useful in proving important properties!
  - Why is EE related to geometry so simply?
  - Duals of other measures of entanglement?
- General covariance is a powerful guiding principle
  - Motivated entanglement wedge, causal wedge, …
  - Covariantize bit threads to elucidate essence of holographic EE
  - Significance of instantaneous nature: (Why) 1-d threads?
- Convex relaxation and Lagrangian duality is a powerful tool
  - Motivates new geometric constructs, new elegant proofs, connections…
  - Other applications?
- Relation between spacetime (gravity) and entanglement?


Thank you