Supergravity, topological gravity and localization

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Localization

- Localization is a long-known property of both supersymmetric (SQFT) and topological (TQFT) theories, by virtue of which semi-classical approximation becomes, in certain cases, exact.
- The traditional link between SQFT and TQFT is topological twist (Witten:1991).

Topological twist

- Topological twisting involves putting a supersymmetric theory on a curved background and identifying the spin connection with a background gauge field associated to a conserved R-symmetry.
- For example, in $\mathcal{N} = 2$, $d = 2$, which will be one of the examples of my discussion, one sets

$$\mathcal{A}_\mu \to \frac{1}{2}\,\omega_\mu$$

Localization and supergravity

- More recently (Pestun: 2007) a new localization paradigm emerged, which makes no explicit reference to TQFTs.
- In this viewpoint localization is the special property of SQFTs when defined on supersymmetry-preserving curved backgrounds.
- These external backgrounds are best thought of (Festuccia et al:2011) as off-shell, supersymmetric configurations of the supergravity (SUGRA) multiplet which the SQFT can couple to.

Generalized covariantly constant spinors

- Supersymmetric SUGRA configurations are determined by generalized covariantly constant spinors ζ (GCCS) that set the supersymmetry variations of fermionic fields of the SUGRA multiplet to zero.

GCCS for $\mathcal{N} = 2$ $d = 2$

- For example, for $\mathcal{N} = (2,2)$ $d = 2$, the GCCS equations write

$$S\,\psi_\mu = (D_\mu - i\mathcal{A}_\mu)\,\zeta - \frac{i}{2}\,F\,\Gamma_\mu\zeta - \frac{i}{2}\,G\,\Gamma_\mu\Gamma_3\zeta = 0$$

  where $\mathcal{A}_\mu$ is the $U(1)_R$ gauge field and F and G are graviphoton backgrounds.
- For generic space-time topologies, 2d GCCS equations admit the solution

$$\mathcal{A}_\mu = \frac{1}{2}\,\omega_\mu \qquad G = F$$

  which corresponds to the "old" topologically twisted A-model.

New supersymmetric solutions

- However, more solutions of the 2*d* GCCS eqs can be found for spheric world-sheet topology (Benini& Cremonesi:2012, Doroud et al: 2012, Closset&Cremonesi:2014, Closset, Cremonesi, Park:2015).
- GCCS equations have also been studied in higher dimensions and a host of new solutions have been found as well (Hama et al 2012, Klare et al. 2012...)

A classification?

- There is no general strategy to construct solutions of GCCS.
- More importantly, in the supergravity framework no “a priori” way to classify inequivalent solutions of GCCS equations has been developed.
- Several localizing solutions have been shown “a posteriori” to be equivalent by comparing explicit computations of partition functions.

A cohomological reformulation

- I want to describe an approach to this problem which recasts GCCS equations as cohomological eqs: this both
  - uncovers some universal structure of the localization equations and
  - introduces an “a priori” notion of equivalence for localization backgrounds which makes the study of their moduli space possible (and in 2d, even straightforward).

The topological approach

- In this approach background supergravity is replaced by a background topological gravity+topological gauge system.
- The topological symmetry of the topological system is inherited from the supergravity BRST symmetry.
- The cohomological reformulation of GCCS equations of supergravity are then obtained by setting to zero the BRST variations of the fermionic topological fields.

Fermionic ghost bilinears

- The basic ingredients of this construction are ghost number 2 bilinears of the commuting ghosts of the BRST formulation of supergravity.
- E.g., in $\mathcal{N} = 2$ $d = 2$, the bilinears are

$$\varphi = \bar{\zeta}\,\zeta \quad \gamma^{\mu} = \bar{\zeta}\Gamma^{\mu}\zeta \quad \widetilde{\varphi} = \bar{\zeta}\,\Gamma^{3}\,\zeta$$

- In $\mathcal{N} = 1$ $d = 4$

$$\gamma^{\mu} = \bar{\zeta}\Gamma^{\mu}\zeta \quad \varphi^{\mu\nu} = \bar{\zeta}\,\sigma^{\mu\nu}\,\zeta$$

The equations for ghost bilinears

- It is well known that the GCCS eqs for ζ translate into differential equations for the ghost bilinears.
- This fact has been widely used to study GCCS eqs. (P. Meessen & T. Ortin 2006, Gupta& Murthy 2012).

The equations for ghost bilinears for $\mathcal{N} = 2$ $d = 2$

- For example, the 3 bilinears of $\mathcal{N} = 2$ $d = 2$ obey the eqs:

$$D_\mu \varphi = G \sqrt{g}\, \epsilon_{\mu\nu} \gamma^\nu$$
$$D_\mu \widetilde{\varphi} = F \sqrt{g}\, \epsilon_{\mu\nu} \gamma^\nu$$
$$D_\mu \gamma_\nu = \sqrt{g}\, \epsilon_{\mu\nu} (G \varphi - F \widetilde{\varphi})$$

- The first two equations (in blue) admit an immediate cohomological reformulation (Bae, C.I. Rey, Rosa 2016)

Cohomological equations

- Introducing $F^{(2)} = \star F$ and $G^{(2)} = \star G$, one can write

$$(d - i_\gamma)\,(F^{(2)} + \widetilde{\varphi}) = 0$$
$$(d - i_\gamma)\,(G^{(2)} + \varphi) = 0$$

- $D \equiv d - i_\gamma$ is the Cartan equivariant differential associated to the S^1-action on the world-sheet defined by the vector field γ^μ:

$$D^2 = \mathcal{L}_\gamma$$

- $F^{(2)} + \widetilde{\varphi}$ and $G^{(2)} + \varphi$ are S^1-equivariant closed forms.

Topologically equivalent backgrounds

- The cohomological reformulation provides an a priori natural definition of topologically equivalent backgrounds:

$$
\begin{aligned}
& F^{(2)} \to F^{(2)} + \mathrm{d}\,\omega^{(1)} \\
& \tilde{\varphi} \to \tilde{\varphi} + i_\gamma(\omega^{(1)}) \\
& \mathcal{L}_\gamma\,\omega^{(1)} = 0
\end{aligned}
$$

The minimal subset of bilinear equations

- Moreover it turns out that a solution of the GCCS equation is completely specified by a solution of a subset of the equations for the bilinears

$$d\,\widetilde{\varphi} + i_\gamma(F) = 0$$
$$D^\mu\,\gamma^\nu + D^\nu\,\gamma^\mu = 0$$

- This is due, basically, to the fact that the bilinears are not independent, but obey Fierz identities.
- This “minimal’ subset of the bilinear equations is naturally obtained from an underlying topological system.

The topological gauge-gravity system in 2d

- To understand why, let us review the basic topological multiplets.
- Topological gravity field content is

$$\begin{array}{cccc} & g_{\mu\nu} & \psi_{\mu\nu} & \gamma^{\mu} \\ \text{ghost\#} & 0 & 1 & 2 \end{array}$$

- It can be coupled to an abelian topological gauge multiplet:

$$\begin{array}{cccc} & F^{(2)} & \psi^{(1)} & \widetilde{\varphi} \\ \text{ghost\#} & 0 & 1 & 2 \end{array}$$

The topological BRST transformations

The BRST rules for topological gravity coupled to topological abelian gauge multiplet are

$$
\begin{aligned}
S\, g_{\mu\nu} &= \psi_{\mu\nu} \\
S\, \psi_{\mu\nu} &= \mathcal{L}_\gamma g_{\mu\nu} \\
S\, \gamma^\mu &= 0 \\
S\, F^{(2)} &= -d\, \psi^{(1)} \\
S\, \psi^{(1)} &= -d\, \tilde{\varphi} + i_\gamma F^{(2)} \\
S\, \tilde{\varphi} &= i_\gamma \psi^{(1)}
\end{aligned}
$$

where

$$
s \equiv S + \mathcal{L}_\xi \qquad s^2 = 0 \Leftrightarrow S^2 = \mathcal{L}_\gamma
$$

The topological equations in 2d

- By setting to zero the BRST variations of the fermionic backgrounds $S\,\psi_{\mu\nu} = S\,\psi^{(1)} = 0$ one obtains indeed the minimal set of equations for the bilinears, if one identifies that superghost γ^μ of topological gravity with the vector fermionic bilinear $\bar{\zeta}\Gamma^\mu\,\zeta$, and one of the graviphoton field strengths with the field strength of the topological abelian gauge multiplet.
- For a fixed isometry γ^μ, the topologically inequivalent backgrounds are therefore classified by the S^1-equivariant classes of degree 2.

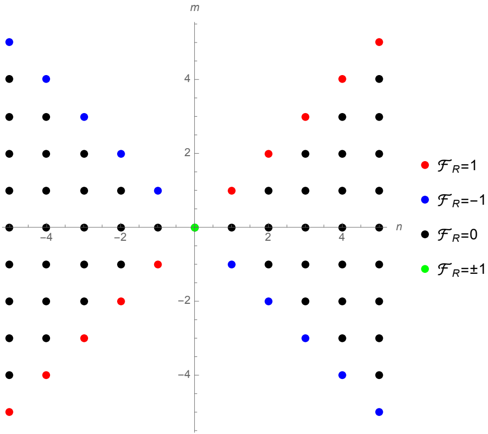
Classifying topologically inequivalent backgrounds on $\mathbb{S}^2$

- The topologically inequivalent backgrounds are determined by two parameters which can be identified with the two graviphoton fluxes

$$n \equiv \int_\Sigma F^{(2)} \qquad m \equiv \int_\Sigma G^{(2)}$$

- These are quantized if F and G are field strength of abelian gauge fields associated to compact gauge group.

Fluxes of localizing backgrounds on the 2-sphere: $n = \int_\Sigma F$, $m = \int_\Sigma G$.



Topological eqs vs GCCS

- But are two backgrounds which are equivalent in the topological sense equivalent also in the supergravity sense?
- The answer is yes. Indeed, we will see in the following, that the action of the topological BRST operator on the "composite" topological fields coincides with the action of supergravity BRST operator on the "microscopic" supegravity fields.

Supergravity BRST symmetry

- The commuting supergravity ghost ζ transforms under the sugra BRST symmetry as

$$
\begin{aligned}
&s\,\zeta = i_\gamma(\psi) + \text{diffeos} + \text{gauge transfs}\\
&s^2 = 0\\
&\gamma^\mu \equiv \bar{\zeta}\,\Gamma^\mu\,\zeta\\
&\psi = \psi_\mu\, dx^\mu \equiv \text{the gravitino(s)}
\end{aligned}
$$

Reparametrization ghosts and superghosts

- The anti-commuting reparametrization ghosts ξ^μ transform as

$$
\begin{aligned}
s\,\xi^\mu &= -\frac{1}{2}\,\mathcal{L}_\xi\,\xi + \gamma^\mu \qquad\qquad \gamma^\mu \equiv \bar{\zeta}\,\Gamma^\mu\,\zeta \\
s\,\gamma^\mu &= -\mathcal{L}_\xi\,\gamma^\mu
\end{aligned}
$$

- Therefore the vector bilinear $\bar{\zeta}\,\Gamma^\mu\,\zeta$ transforms precisely under supergravity BRST as the superghost γ^μ of topological gravity. (Baulieu& Bellon: 1986).

Gauge symmetries ghosts

- The anti-commuting ghosts c associated to bosonic symmetries (Lorentz and R-symmetries) transform as

$$s\,c = -\mathcal{L}_\xi\, c - \frac{1}{2}\delta_c\, c + i_\gamma(A) + \phi$$

- A is the collection of bosonic 1-form gauge fields associated to local Lorentz (spin-connection) and local R-symmetries.
- ϕ is a bilinear of the supergravity ghost ζ, with values in the bosonic gauge Lie algebra, whose specific form depends on the particular supergravity theory.

The structure of the supergravity BRST operator

- It is convenient to subtract from the nilpotent BRST operator s both reparametrization and R-symmetry gauge transformations

$$S \equiv s + \mathcal{L}_\xi + \delta_c$$

- This defines the “topological gaugino”

$$S\,A \equiv \lambda$$

which is linear in the supergravity ghost ζ.

Supergravity BRST algebra

- It turns out that nilpotency of the supergravity BRST s is equivalent to the universal algebra, valid on all fields with the exclusion of the bosonic ghosts:

$$S^2 = \mathcal{L}_\gamma + \delta_{i_\gamma(A)+\phi}$$

The topological gauge multiplet

- The consequence of this algebra is that the A, λ and ϕ transform under sugra BRST in the same way as a (possibly non-abelian) topological gauge multiplet

$$
\begin{aligned}
S\,A &= \lambda \\
S\,\lambda &= D\,\phi + i_\gamma(F) \\
S\,\phi &= i_\gamma(\lambda)
\end{aligned}
$$

where $F \equiv dA + A^2$.

The universal localization equations for the curvatures

- The conclusion is that, on the supergravity localization locus, the following cohomological equations hold:

$$\begin{aligned} D\,\phi_{Lorentz} + i_\gamma(R) &= 0 \\ D\,\phi_R + i_\gamma(\mathcal{F}_R) &= 0 \end{aligned}$$

  where we projected ϕ and F along the Lorentz and R-symmetry part of the gauge algebra.

The BRST descent eqs for the ghost bilinears

- One obtains one more cohomological equation, by considering a Lorentz and gauge scalar ghost bilinear

$$\varphi = \bar{\zeta}\,\zeta$$

- This is BRST invariant modulo i_γ

$$S\,\varphi = i_\gamma(\bar{\zeta}\psi + \bar{\psi}\,\zeta) \equiv i_\gamma(\chi^{(1)})$$

The BRST descent eqs for the ghost bilinears

- The BRST algebra ensures that there exists a 2-form of ghost number zero 2-form $T^{(2)}$ satisfying the descent equations:

$$
\begin{aligned}
S\,\varphi &= i_\gamma(\chi^{(1)}) \\
S\,\chi^{(1)} &= d\,\varphi + i_\gamma(T^{(2)}) \\
S\,(T^{(2)}) &= d\,\chi^{(1)}
\end{aligned}
$$

The localization equations for the scalar ghost bilinear

- Again this impies that, on the supergravity localization locus, the topological equations

$$d\,\varphi + i_\gamma(T^{(2)}) = 0$$

hold universally.

Summary of cohomological equations

- Summarizing, on the supergravity localization locus, we have recast the equations for the bilinears in the cohomological form:

$$
\begin{aligned}
d\,\varphi + i_\gamma(T^{(2)}) &= 0 \\
D\,\phi_{Lorentz} + i_\gamma(R) &= 0 \\
D\,\phi_R + i_\gamma(\mathcal{F}_R) &= 0
\end{aligned}
$$

- φ, $\phi_{Lorentz}$ and ϕ_R are ghost bilinears of ghost number 2
- R is the Lorentz curvature, $\mathcal{F}_R$ is the curvature of the R-symmetry gauge fields, and $T^{(2)}$ is the ghost number zero 2-form which solves the BRST descent equations associated to the scalar bilinear φ.

The topological structure of $N = 2$ $d = 2$ supergravity

- In this case the bosonic local gauge symmetries are $U(1)_{Lor} \times U(1)_R$:

$$
\begin{aligned}
T^{(2)} &= F^{(2)} + \bar{\psi}\,\Gamma^3\,\psi \\
\phi_{Lor} &= \tilde{\varphi}\,F - \varphi G \\
\phi_R &= \varphi\,F - \tilde{\varphi}\,G
\end{aligned}
$$

The $N = 2$ $d = 2$ cohomological localization equations

- The cohomological localization equations

$$d\,\phi_{Lorentz} + i_\gamma(R) = 0$$
$$d\,\phi_R + i_\gamma(\mathcal{F}_R) = 0$$

turn out to be precisely equivalent to the equations for the vector bilinears which did not have a “manifest” cohomological form

$$D_\mu\,\gamma_\nu = \sqrt{g}\,\epsilon_{\mu\nu}\,(G\,\varphi - F\,\widetilde{\varphi})$$

The topological structure of $N = 1$ $d = 4$ supergravity

- In this case the bosonic gauge symmetry includes, beyond Lorentz, also a $U(1)_R$ axial gauge group.
- The corresponding superghost ϕ_{Lor} and ϕ_R turn out to be

$$
\begin{aligned}
\phi_R &= \frac{3}{2}\, i_\gamma(H) \\
(\phi_{Lorentz})^{ab} &= \frac{1}{4}\, i_\gamma(\epsilon^{abcd}\, H_c\, e_d) \\
e_a &\equiv e_{a\mu}\, dx^\mu
\end{aligned}
$$

where $H = H_\mu\, dx^\mu$ is the auxiliary 1-form whose divergence vanishes (up to fermion bilinears).

The $N = 1$ $d = 4$ topological localization equations

- In this case both ϕ_R and $\phi_{Lorentz}$ are i_γ-trivial:

$$\phi = i_\gamma(\Delta)$$

- This means that the localization equations can be recast in the form

$$i_\gamma(F^-) = 0$$

  where F^- is the total gauge curvature built with a connection $A^- = A + \Delta$

- The existence of the special A^- connection for gauged $\mathcal{N} = 1$ $d = 4$ sugra was noted long ago by Ferrara and Sabharwal (1989).

Conclusions

- We saw how “composite” topological gravity coupled to “composite” topological gauge multiplets emerge from the BRST formulation of — gauged, off-shell — supergravity.
- We obtained in this way a cohomological rewriting of GCCS eqs and thus the “a priori” notion of equivalent localizing backgrounds we were seeking.

Conclusions

- In 2d this allowed for a complete classification of inequivalent localizing backgrounds.
- $d = 4$, $\mathcal{N} = 1$ cohomological localization equations have not been analyzed yet.
- Extending these results to $d = 4$ $\mathcal{N} = 2$ sugra requires the corresponding off-shell gauged formulation.