Effective Action from M-theory on twisted connected sums

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☆ Motivation

❶ 11D supergravity: An unique set up

❷ G_2 manifolds: An unique compactification class

❸ Yet many questions

☆ Constructing G_2 Manifolds

❶ Structure theorems for SU_n and G_2 holonomy mflds

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☆ Transitions between G_2 manifolds

❶ Abelian Gauge symmetry, charged matter spectrum

❷ Non-abelian Gauge symmetry, charged matter spectrum

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☆ Motivation:

❶ M-theory: An unique set up

D.o.f. spinors $2^{\lfloor \frac{D+1}{2} \rfloor -1}$, d.o.f. of bosons $p_r(D)$. So there is a maximal dimension D for super symmetric representations.

Supersymmetry representation of physical theories singles out especially two cases W. Nahm 1978.

- Eleven dimensional supergravity $D = 11$ ✓
- Six dimensional superconformal field theory

The action of eleven dimensional supergravity was found by Cremmer, Julia, Scherk 1978

$$S_{11d} = \frac{1}{2\kappa_{11}^2}\int\left(*\hat{R}_S - \frac{1}{2}d\hat{C}\wedge *d\hat{C} - *i\,\bar{\hat{\Psi}}_M\hat{\Gamma}^{MNP}\hat{D}_N\hat{\Psi}_P\right)$$
$$-\frac{1}{192\kappa_{11}^2}\int *\bar{\hat{\Psi}}_M\hat{\Gamma}^{MNPQRS}\hat{\Psi}_N(d\hat{C})_{[PQRS]} - \frac{1}{2\kappa_{11}^2}\int d\hat{C}\wedge *_{11}\hat{F}$$
$$-\frac{1}{12\kappa_{11}^2}\int d\hat{C}\wedge d\hat{C}\wedge C + \dots\ ,$$

Here $\kappa_{11} = \hat{G}_N$, $\hat{\Psi}$ the gravitino, $\hat{C}$ anti-symmetric 3-form, $\hat{F}_{[MNPQ]} = 3\bar{\hat{\Psi}}_{[M}\hat{\Gamma}_{NP}\hat{\Psi}_{Q]}$, $+\dots$ 4-fermion int.

Simple unique beautiful starting point for K-K reduction

❷ G2 manifolds: To get $\mathcal{N} = 1$ sugra in 4d, look in Berger's list of special holonomy manifolds. Beyond the generic cases, $\exists$ two entries

- (vi) $d = 7$: $\mathrm{Hol}(g) = G_2$, G_2-*mfld, Ricci-flat* $R_{ij}(g) = 0$, $N = 1$ covariant constant spinor; φ associative 3-form, $*\varphi$ coassociative 4-form. ✓
- (vii) $d = 8$: Spin(7) : $N_+ = 1$, ψ Cayley 4-form

That is we get an unique K-K compactification, supposingly unifying non-perturbative String theories

❸ Yet many questions:

- 1.) How to construct compact G_2 manifolds?
- 2.) How to geometrical engineer the ones that yield interesting $\mathcal{N} = 1$ supergravities including the standard model ?
- 3.) How to calculate Kaluza-Klein and M-theory corrections to the effective $\mathcal{N} = 1$ supergravity action?
- 4.) How those relate to other $\mathcal{N} = 1$ vacua?

☆ Constructing G_2 Manifolds

❶ Structure theor. for SU_n and G_2 holonomy mflds: To address 1.) recall the more familiar $N = 2$ situation of CY manifolds X $(n = 3)$

- (iii) $d = 2n$, $n \geq 2$: $\mathrm{Hol}(g) = SU_n$, *CY-mfld, Ricci-flat, Kähler*, $N_{\pm} = 1$ for n odd, $N_+ = 2$ for n even, ω Kähler $(1,1)$-form and Ω hol.harm. $(n,0)$-form.

Existence Theorem of Yau [1]

$$K_X = -c_1(T_X) = 0 \;\rightarrow\; \exists \text{ unique}^1 \; g \text{ with } R_{i\bar{j}}(g) = 0$$

[1] Given complex– (Ω) and Kähler structure (ω)

Controlling K_X is trivial by multiplicative properties of Chern characters. Take a Fano variety such as $\mathbb{P}^n$ and X_{n-1} as zero locus of a degree d hypersurface. Then

$$\mathrm{ch}(T_X) = \frac{\mathrm{ch}(T_{\mathbb{P}^n})}{\mathrm{ch}(\mathcal{N}_X)} = \frac{(1+H)^{n+1}}{1+dH} = 1 + c_1(T_X) + \dots$$
$$= 1 + [(n+1) - d]H + \dots, \quad \text{i.e.}$$

$c_1(T_X) = 0 \Leftrightarrow d = (n+1)$. Strategy:

$$c_1(T_X) = 0 \to^1 R_{i\bar{\jmath}}(g) = 0, \ (R_{i\bar{\jmath}}(g) = 0 \,\&\, \pi_1(X)\ fin.) \to$$
$$\mathrm{Hol}(g) = SU(n) \to \mathrm{N} = 2\ \mathrm{4d} - \mathrm{susy}$$

Remarks:

- Slight generalisations $\mathbb{P}^n \to \mathbb{P}_{\Delta_n}$ with (Δ_n, Δ_n^*) a pair of reflexive polyhedra, yields 10^8 families of compact CY 3-folds and 4319 non-compact CY 3 folds.

- Similar as above one can define a non-compact CY as $X_n = \mathbb{P}^n \setminus \{P_{d=n+1}(\underline{x}) = 0\}$. For an easy example think of $n = 1$, then $X_1 = \{S^2 \setminus 2 \text{ points}\} \sim \text{ cylinder}$, which clearly allows a flat metric. In higher dimensions Tian & Yau established the existence of a no-where vanishing Ω — trivializing K_X — together with a boundary asymptotic, so that Yau's theorem still applies.

G_2 structure manifolds: G_2 is a 14d simply connected subgroup of $SO(7)$. Geometrically it arises as follows. $\exists$ $\varphi \in \Lambda^3_+(\mathbb{R}^7)^*$ a 3-form on $\mathbb{R}^7$ such that

$$B_\varphi(X,Y) = -\frac{1}{3!}(X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi \qquad (1)$$

is a positive define bilinear form with respect to an oriented volume form. $GL(7,\mathbb{R})$ acts on φ and G_2 is its fourteen dimensional stabilizer group R.L. Bryant 1987.

A G_2 structure on an oriented 7d manifold Y is a 3-form φ which is $\forall p \in Y$ oriented isomorphic to $\Lambda^3 T^*_p Y \simeq \Lambda^3_+(\mathbb{R}^7)^*$. Via (1) this defines a Riemannian metric on Y

$$g_\varphi(X_p, Y_p) \;=\; \frac{B_\varphi(X_p, Y_p)(\partial_1|_p, \ldots, \partial_7|_p)}{\mathrm{vol}_\varphi(\partial_1|_p, \ldots, \partial_7|_p)} \;,$$

Theorem Fernández & Gray 1983 Y has a holomomy $Hol(g_\varphi) \subset G_2$ iff

$$d\varphi = 0, \qquad d *_{g_\varphi} \varphi = 0 \;.$$

$Hol(g_\varphi) = G_2$ iff $\pi_1(Y)$ is finite.

- Note the non-linearity in harmonicity condition for φ.

- A harmonic G_2 structure φ is called torsion free in the sense that the Levi-Cevita connection has G_2 holonomy.

- "Strategy": (1) Show existence of torsion free φ on Y. (2) show that $\pi_1(Y)$ is finite.

- The first part has boring solutions: $Y_0 = X \times S^1$, with θ the angl. coord. on S^1 one has torsion free G_2 structure

$$\varphi_0 = \gamma \, \mathrm{d}\theta \wedge \omega + \mathrm{Re}(\Omega), \; *\varphi_0 = \frac{1}{2}\omega^2 - \gamma \, \mathrm{d}\theta \wedge \mathrm{Im}(\Omega)$$

But $\pi_1(Y) = \mathbb{Z}$ and therefore $\rightarrow SU_3$ holonomy and

$N = 2$ 4d supergravity. $(\gamma \in \mathbb{R})$

❷ The twisted gluing construction

Mathematics : Donaldson (?) $\to$ Kovalev (2003) $\to$ Corti, Haskins, Nordström, Pacini (2013), Crowley Nordström (2015), Haskin, Hein, Nordström (2015)

Physics: Halverson and Morrison (2014,2015) $\to$ Braun (2016) $\to$ Braun and del Zotto (2017)

Alternative approach: First compact examples constructed by *Dominic Joyce (1994)* as $Y = \widehat{T^7/G}$ resolutions. . . . Simons Collaboration

Basic idea of the twisted connected gluing:

- Construct two non-compact Calabi-Yau 3-fold as discussed by Tian and Yau, called $X_{L/R}$, where L/R stands for Left and Right
- Construct two product 7-folds $Y_{L/R} = X_{L/R} \times S^1_{L/R}$, with the "trivial" torsion free G_2-structures $\varphi_{0\ L/R}$.
- Each $X_{L/R}$ has a K3 called $S_{L/R}$ removed.
- There is another canonical $S^{*1}_{L/R}$ in $\mathcal{N}_{S_{L/R}} \subset X_{L/R}$ parametrizing in polar coordinates $|z| = e^t$ and θ^* a disk

$D_{L/R} \in X_{L/R}$.

- Glue Y_L to Y_R to obtain Y so that
  - a) $\varphi_{0\ L/R}$ extend to a torsion free G_2 structure on Y. This requires a hyperkähler rotation on the $K3$ boundaries
  - b) the infinite $\pi_1(Y_{L/R})$ becomes finite ($\pi_1(Y) = 0$). This this achieved as in the Hopf gluing of two solid tori here $D_{L/R} \times S^1_{L/R}$ to an S^3 with $\pi_1(S^3) = 0$.

By the structure theorem Y is then a manifold whose metric has the full G_2 holonomy.

Let us visualize this as good as we can:

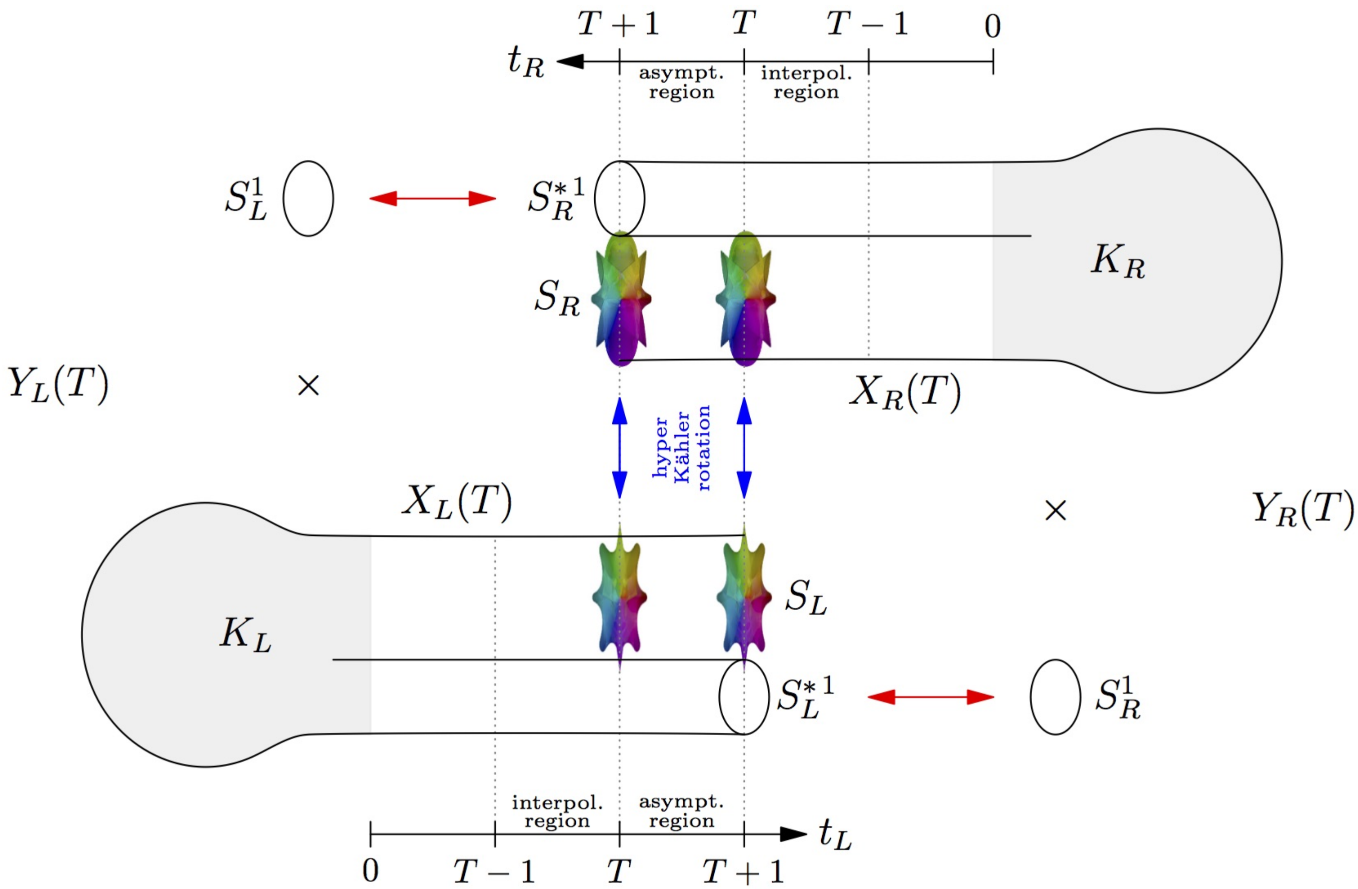

Figure 1: Kovalev's twisted connected sum construction.

Gluing the asymptotic regions: Before the $K3$ is cut out, one needs to remove its self-intersection $[S^2] = [C]$. That is achieved by blowing up along C. The asymptotic region near the $K3$, that is cut out, has hence a simple form, called asymptotically cylindrical Calabi-Yau 3-fold $X^\infty = S \times \Delta^{cyl}$, with $\Delta^{cyl} = \{z \in \mathbb{C} | |z| > 1\}$. In this region the Kähler- and the holomorphic 3-form are given by

$$\begin{aligned} \omega^\infty &= \gamma^{*2} \frac{i \mathrm{d}z \wedge \mathrm{d}\bar{z}}{2 z \bar{z}} + \omega_S = \gamma^{*2} \mathrm{d}t \wedge \mathrm{d}\theta^* + \omega_S \ , \\ \Omega^\infty &= -\gamma^* \frac{i \mathrm{d}z}{z} \wedge \Omega_S = \gamma^* (\mathrm{d}\theta^* - i \mathrm{d}t) \wedge \Omega_S \ . \end{aligned} \tag{2}$$

Here (ω_S, Ω_S) are Kähler- and holomorphic two-form of S, $z = e^{t+i\theta^*}$ and γ^* the length scale of Δ^{cyl}. Let K be the compact part of X then there is a diffeomorphism $\eta : X^\infty \to X \setminus K$ so that

$$\eta^*\omega - \omega^\infty = \mathrm{d}\mu \ \text{ with } |\nabla^k \mu| = O(e^{-\lambda\gamma^*\xi}) \ ,$$
$$\eta^*\Omega - \Omega^\infty = \mathrm{d}\nu \ \text{ with } |\nabla^k \nu| = O(e^{-\lambda\gamma^*\xi}) \ ,$$

with $\lambda = \min\left\{\frac{1}{\gamma^*}, \lambda_S\right\}$. Here λ_S is the smallest positive eigenvalue of ∇^2_S. This ensures that one can glue the asymptotic forms (2), which are fast enough approximated to yield a torsion free φ on Y as follows:

On the gluing region
$Y^\infty_{L/R} = X^\infty_{L/R} \times S^1_{L/R} = S_{L/R} \times \Delta^{\mathrm{cyl}}_{L/R} \times S^1_{L/R}$, we define

$$\omega^\infty_{S_{L/R}} = \omega^I_{L/R}\,, \qquad \Omega^\infty_{S_{L/R}} = \omega^J_{L/R} + i\,\omega^K_{L/R}\,.$$

We get then on $Y^\infty_{L/R}$ a torsion free $3-form$

$$\varphi^\infty_{0\,L/R} = \gamma_{L/R}\mathrm{d}\theta_{\mathrm{L/R}} \wedge \left(\gamma^{*\,2}_{\mathrm{L/R}}\mathrm{dt}_{\mathrm{L/R}} \wedge \mathrm{d}\theta^*_{\mathrm{L/R}} + \omega^\infty_{\mathrm{S}_{\mathrm{L/R}}}\right)$$
$$+ \gamma^*_{L/R}\mathrm{d}\theta^*_{\mathrm{L/R}} \wedge \mathrm{Re}(\Omega^\infty_{\mathrm{S}_{\mathrm{L/R}}}) + \gamma^*_{\mathrm{L/R}}\mathrm{dt}_{\mathrm{L/R}} \wedge \mathrm{Im}(\Omega^\infty_{\mathrm{S}_{\mathrm{L/R}}})\;.$$

The gluing diffeomorphism: First we need the $K3$ to be isometric with respect to a hyperkähler rotation

$r : S_L \to S_R$

$$r^* \omega_R^I = \omega_L^J \ , \quad r^* \omega_R^J = \omega_L^I \ , \quad r^* \omega_R^K = -\omega_L^K \ .$$

Then there is a family ($\Lambda \in \mathbb{R}$) of gluing diffeomorphisms defined as Kovalev (2003)

$$F_\Lambda : \ (\theta_L^*, t_L, u_L^\alpha, \theta_L) \mapsto (\theta_R^*, t_R, u_R^\alpha, \theta_R) = (\theta_L, \Lambda - t_R, r(u_L^\alpha), \theta_L^*)$$

$(\theta_{L/R}^*, t_{R/L})$ of $\Delta_{L/R}^{\rm cyl}$, $u_{L/R}^\alpha$ coords of $S_{L/R}$, and $\theta_{L/R}$ of $S_{L/R}^1$. With $\gamma := \gamma_L = \gamma_R = \gamma_L^* = \gamma_R^*$ it is easy to check that

$$\boxed{F_\Lambda^* \varphi_{0\,R} \ = \ \varphi_{0\,L} \ .}$$

With $X_{L/R}(T) = K_{L/R} \cup \eta_{L/R}(\mathbb{R}_{<T+1})$, $\quad Y_{L/R}(T) = X_{L/R}(T) \times S^1_{L/R}$

$$\boxed{Y = Y_L(T) \cup_{F_{2T+1}} Y_R(T) \ .}$$

Kovalev's checks that Y is a G_2 manifold in two steps: First he establishes analytically that

$$\widetilde{\omega}^T_{L/R} = \omega_{L/R} - d\left(\alpha(t-T)\mu_{L/R}\right) \ ,$$
$$\widetilde{\Omega}^T_{L/R} = \Omega_{L/R} - d\left(\alpha(t-T)\nu_{L/R}\right) \ ,$$

define an interpolating G_2 structure on $Y_{L/R}$ as

$$\widetilde{\varphi}_{L/R}(\gamma, T) \;=\; \gamma\,\mathrm{d}\theta \wedge \widetilde{\omega}^{\mathrm{T}}_{\mathrm{L/R}} + \mathrm{Re}(\widetilde{\Omega}^{\mathrm{T}}_{\mathrm{L/R}}) \;,$$

which extend to a torsion free G_2 structure on Y, because the latter can be approximated as

$$\varphi(\gamma, T) = \widetilde{\varphi}(\gamma, T) + \mathrm{d}\rho(\gamma, \mathrm{T}) \;\text{ with }\; \left|\nabla^{\mathrm{k}}\rho(\gamma, \mathrm{T})\right| \;=\; \mathrm{O}(\mathrm{e}^{-\gamma\lambda\mathrm{T}})$$

in terms of the norm $|\cdot|$ and the Levi–Civita connection ∇ of the metric induced from the asymptotic G_2-structure.

Secondly he shows topologically that

$$\boxed{\pi_1(Y) = \pi_1(X_L) \times \pi_1(X_R)}$$

which completes his proof that Y has full G_2 holonomy •

Fibration Structures: $Y_{L/R}$ is $K3$ fibrations over solid tori $T_{L/R} \equiv S^1_{L/R} \times D_{L/R}$

$$\begin{array}{ccc} S_{L/R} & \rightarrow & Y_{L/R} \\ & & \downarrow \pi \\ & & T_{L/R} \ . \end{array}$$

The gluing of two solid tori $T_{L/R}$ to an S^3 induces the fibration.

$$\begin{array}{ccc} S & \rightarrow & Y \\ & & \downarrow \pi \\ & & S^3 \ . \end{array}$$

❸ Examples As we already mentioned there are 4319 examples of non-compact CY-folds X constructed in weak Fano toric ambient spaces. To easily establish the isometric map $r : S_L \to S_R$ one needs the technical condition of semi ample canonical class (semi Fano), which leaves 899 $\mathbb{P}_\Delta$, for which Δ contains no codim 2-points lattice points. The latter have different Kähler cones, so roughly one gets $10^8 \times m_g$ examples, where m_g is the gluing multiplicity of order $O(10)$.

They have to be distinguished by their Betti numbers. Using the Mayer-Vietoris sequence of the gluing map one gets these topological data as

$$H^2(Y,\mathbb{Z}) \simeq (k_L \oplus k_R) \oplus (N_L \cap N_R) \ ,$$

$$H^3(Y,\mathbb{Z}) \simeq H^3(Z_L,\mathbb{Z}) \oplus H^3(Z_R,\mathbb{Z}) \oplus k_L \oplus k_R \oplus N_L \cap T_R$$
$$\oplus \ N_R \cap T_L \oplus \mathbb{Z}[S] \oplus L/\left(N_L + N_R\right) \ .$$

Here $[S]$ is the Poincaré dual three-form of a the K3 fibre S in $(Z_{L/R}, S_{L/R})$, $L \simeq H^2(S_L,\mathbb{Z}) \simeq H^2(S_R,\mathbb{Z})$. The inclusion $\rho_{L/R} : S_{L/R} \hookrightarrow X_{L/R}$ induce maps $\rho^*_{L/R} : H^2(X_{L/R},\mathbb{Z}) \to L$ defining kernels $k_{L/R} := \ker \rho^*_{L/R}$, images $N_{L/R} := \mathrm{Im}\, \rho^*_{L/R}$, and the transc. lattices $T_{L/R} = N^{\perp}_{L/R} = \left\{ l \in L \;\middle|\; \langle l, N_{L/R} \rangle = 0 \right\}$.

Orthonogal gluing examples:

$W = N_L + N_R = N_L \perp_R N_R$, $R = N_L \cap N_R$ and $N_L^\perp \subset N_R$. $N_R^\perp \subset N_L$. Building blocks

No.	rk N	$-K^3$	κ	e	e^2	$b_3(Z)$
MM27$_3$, K62 (Fano)	3	48	$\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$	-4	50
MM25$_3$, K68 (Fano)	3	44	$\begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & -2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$	-4	46
MM31$_3$, K105 (Fano)	3	52	$\begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & -2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$	-4	54

Table 1: Data of low rank toric terminal Fano threefolds. R generated by a vector of length square -4.

The formulas for the Betti numbers simplify for the orthogonal gluing

$$
\begin{aligned}
b_2(Y) &= \operatorname{rk} R + \dim k_L + \dim k_R, \\
b_3(Y) &= b_3(Z_L) + b_3(Z_R) + \dim k_L + \dim k_R - \operatorname{rk} R + 23
\end{aligned}
$$

$b_3(Y_{\dots}^{\dots})$	$\mathrm{MM27}_3$	$\mathrm{MM25}_3$	$\mathrm{MM31}_3$	K124	$\mathrm{MM12}_4$	$\mathrm{MM10}_4$	K221	K232	K233	K247	K257
$\mathrm{MM27}_3$	122	118	126	122	120	116	112	114	112	118	120
$\mathrm{MM25}_3$	118	114	122	118	116	112	108	110	108	114	116
$\mathrm{MM31}_3$	126	122	130	126	124	120	116	118	116	122	124
K124	122	118	126	122	120	116	112	114	112	118	120
$\mathrm{MM12}_4$	120	116	124	120	118	114	110	112	110	116	118
$\mathrm{MM10}_4$	116	112	120	116	114	110	106	108	106	112	114
K221	112	108	116	112	110	106	102	104	102	108	110
K232	114	110	118	114	112	108	104	106	104	110	112
K233	112	108	116	112	110	106	102	104	102	108	110
K247	118	114	122	118	116	112	108	110	108	114	116
K257	120	116	124	120	118	114	110	112	110	116	118

☆ The effective action

❶ Homology and Spectrum:

Massless four-dimensional modes arise from the coefficients in the decomposition of the eleven-dimensional anti-symmetric three-form tensor $\hat{C}$ as

$$\hat{C}(x,y) \;= \sum_I A^I(x) \wedge \omega_I^{(2)}(y) + \sum_i P^i(x) \wedge \rho_i^{(3)}(y) \;,$$

in terms of the harmonic two-forms $\omega_I^{(2)}$ and three-forms $\rho_i^{(3)}$ identified with non-trivial cohomology representatives of $H^2(Y)$ and $H^3(Y)$ of dimension $b_2(Y)$ and $b_3(Y)$, respectively. In the paper further details about the dimensional reduction of the fermions can be found. The spectrum is summarized as follows

Multiplicity	Massless 4d component fields		Massless 4d
	bosonic fields	fermionic fields	$\mathcal{N} = 1$ multiplets
1	metric $g_{\mu\nu}$	gravitino ψ_μ, ψ_μ^*	gravity multiplet
$i = 1, \ldots, b_3(Y)$	scalars (S^i, P^i)	spinors $\chi^i, \chi^{*\,i}$	chiral multiplets Φ^i
$I = 1, \ldots, b_2(Y)$	vectors A_μ^I	gauginos λ_α^I	vector multiplets V^I

Note that on a smooth G_2 manifold there are neither non-abelian Gauge groups nor charged matter.

❷ Kähler potential, gauge kinetic function, superpotential: The dimensional reduction of the Einstein–Hilbert term and the three-form tensor $\hat{C}$ yields the four-dimensional action Beasley Witten (2002db)

$$S_{4d}^{\text{bos}} = \frac{1}{2\kappa_4^2}\int \Big[*_4 R_S + \frac{\kappa_{IJk}}{2V_{Y_0}}\left(S^k F^I \wedge *_4 F^J - P^k F^I \wedge F^J\right) - \frac{7}{2V_{Y_0}}\int_Y \rho_i^{(3)} \wedge *_{g_\varphi}\rho_j^{(3)} \left(dP^i \wedge *_4 dP^j - dS^i \wedge *_4 dS^j\right)$$

in terms of the four-dimensional Hodge star $*_4$, the Ricci scalar R_S with respect to the metric $g_{\mu\nu}$, the reference volume V_{Y_0}, and the seven-dimensional Hodge star $*_7$.

From this we can derive the Kähler potential and the gauge kinetic coupling matrix

$$\begin{aligned} K(\phi,\bar{\phi}) &= -3\,\log\left(\tfrac{1}{7}\int_Y \varphi \wedge *_{g_\varphi}\,\varphi\right)\;, \\ f_{IJ}(\phi) &= 2V_{Y_0}\sum_k \phi^k \int_Y \omega_I^{(2)} \wedge \omega_J^{(2)} \wedge \rho_k^{(3)} \\ &= 2V_{Y_0}\sum_k \kappa_{IJk}\phi^k, \end{aligned}$$

as well as the superpotential

$$W(\phi^i) \;=\; \frac{1}{4V_{Y_0}}\int_Y G \wedge (C + i\varphi)\;.$$

❹ $\mathcal{N}=2$ and $\mathcal{N}=4$ sectors: Note first that the Kovalev construction implies that there is a scale $2T+1$, which seperates Z_L from Z_L. Taking this scale to ∞ one roughly expects the spectrum to seperate into sectors

local geometry (Kovalev limit)	multiplicity of $\mathcal{N}=1$ multiplets		$U(1)$ vector multiplets	
	$U(1)$ vectors	chirals	multiplicity	supersym.
$Y_L = S^1_L \times X_L$ $SU(3)$ holonomy	$\dim k_L$	$\dim k_L$	$\dim k_L$	$\mathcal{N}=2$
$Y_R = S^1_R \times X_R$ $SU(3)$ holonomy	$\dim k_R$	$\dim k_R$	$\dim k_R$	$\mathcal{N}=2$
$T^2 \times S \times (0,1)$ $SU(2)$ holonomy	$\dim N_L \cap N_R$	$3 \cdot \dim N_L \cap N_R$	$\dim N_L \cap N_R$	$\mathcal{N}=4$

❸ The Kovalon: More precisely one has to define of course complex $\mathcal{N}=1$ neutral chiral moduli. We identify two universal moduli: ν related to the overall volume and $\varkappa$ related to the gluing parameters so that

$$\mathrm{Re}(\nu) = v \; , \qquad \mathrm{Re}(\varkappa) = vb \; .$$

Here b ist the squashing parameter of the S^3. We refer to the chiral multiplet $\varkappa$ as the Kovalon, as it describes in the limit $\mathrm{Re}(\varkappa) \to +\infty$ — while keeping $\mathrm{Re}(\nu)$ constant — the Kovalev limit.

The remaining real moduli fields are not universal and relate to the non-universal neutral chiral multiplets as

$$\mathrm{Re}(\phi^{\hat{\imath}}) = v\tilde{S}^{\hat{\imath}} \; , \qquad \mathrm{Re}(\phi^{\tilde{\imath}}) = vb\tilde{S}^{\tilde{\imath}} \; .$$

They depend on the topological details of the building blocks $(Z_{L/R}, S_{L/R})$ and the choice of gluing diffeomorphism.

Analysing the gluing maps allows e.g. to determine the leading order dependence on the universal moduli

$$K = -\log\left[\left(V_S^{\tilde{g}}(\tilde{S})\right)^3 (\nu+\bar{\nu})^4(\varkappa+\bar{\varkappa})^3 + A(\tilde{S},\nu+\bar{\nu},\varkappa+\bar{\varkappa})e^{-\lambda\frac{\varkappa+\bar{\varkappa}}{(\nu+\bar{\nu})^{1/3}}}\right] \ ,$$

where the coefficient of the exponentially suppressed correction is expected to generically depends on both universal and non-universal geometric moduli fields.

☆ Transitions between G_2 manifolds:

Generally one expects in M-theory on G_2 manifolds gauge symmetry from codimension four singularities, charged matter from codimension six singularities and chiral spectra at codimension seven. We can realize the first two phenomena on the building blocks using essentially $\mathcal{N} = 2$ techniques. Our implications are that there are Higgs transitions these sectors which are compatible with the gluing diffemorphism and lead to genuine G_2 transitions.

❶ Abelian Gauge symmetry, charged matter spectrum:

For the semi-Fano threefold P we pick two global sections s_0 and s_1 of the anti-canonical divisor $-K_P$. However, instead of choosing a generic section s_0, we assume that the global section s_0 factors into a product

$$s_0 \;=\; s_{0,1} \cdots s_{0,n} \;,$$

such that $s_{0,i}$ are global sections of line bundles $\mathcal{L}_i$ with $-K_P = \mathcal{L}_1 \otimes \ldots \otimes \mathcal{L}_n$. As a consequence the curve $\mathcal{C}_{\text{sing}} = \{s_0 = 0\} \cap \{s_1 = 0\}$ becomes reducible and decomposes into

$$\mathcal{C}_{\mathsf{sing}} = \sum_{i=1}^{n} \mathcal{C}_i \ , \qquad \mathcal{C}_i = \{s_{0,i} = 0\} \cap \{s_1 = 0\} \ ,$$

where we assume that the individual curves $\mathcal{C}_i$ are smooth and reduced. Following Kovalev and Lee 2008, we construct the building block $(Z^\sharp, S)$ associated to P by the sequence of blow-ups $\pi_{\{\mathcal{C}_1,\ldots,\mathcal{C}_n\}} : Z^\sharp \to P$ along the individual curves $\mathcal{C}_i$ according to

$$Z^\sharp = \mathrm{Bl}_{\{\mathcal{C}_1,\ldots,\mathcal{C}_n\}} P = \mathrm{Bl}_{\mathcal{C}_n} \mathrm{Bl}_{\mathcal{C}_{n-1}} \cdots \mathrm{Bl}_{\mathcal{C}_1} P \ .$$

Since the curves $\mathcal{C}_i$ and the semi-Fano threefold P are smooth, the blow-up $Z^\sharp$ is smooth as well. As before, the K3 surface S arises as the proper transform of a smooth anti-canonical divisor $S^\sharp = \{\alpha_0 s_0 + \alpha_1 s_1 = 0\} \subset P$ for some $[\alpha_0 : \alpha_1] \in \mathbb{P}^1$. By blowing up a semi-Fano threefold P the resulting dimension of the kernel k of ρ

$$\dim k \;=\; n - 1 \;.$$

The three-form Betti number $b_3(Z^\sharp)$ of the blown-up threefold $Z^\sharp$ becomes

$$b_3(Z^\sharp) \;=\; b_3(P) + 2\sum_{i=1}^{n} g(\mathcal{C}_i) \;,$$

in terms of the three-form Betti number $b_3(P)$ of the semi-Fano threefold P and the genera $g(\mathcal{C}_i)$ of the smooth curve components $\mathcal{C}_i$. As all these curves $\mathcal{C}_i$ lie in the K3 fiber S, the genus $g(\mathcal{C}_i)$ is readily computed by the adjunction formula

$$g(\mathcal{C}_i) = \frac{1}{2}\mathcal{C}_i.\mathcal{C}_i + 1 ,$$

with the self-intersections $\mathcal{C}_i.\mathcal{C}_i$ in S.

Locally the singularity looks near $\pi^{-1}([1,0])$ with

$t = \alpha_1/\alpha_0$ like

$$s_{0,1} \cdots s_{0,n} + t s_1 = 0 \ .$$

In particular near the intersection locii $\mathcal{I}_{ij} = \{t = 0\} \cap \{s_1 = 0\} \cap \{s_{0,i} = 0\} \cap \{s_{0,j} = 0\}$ it looks like conifolds, which are away from the asymptotic $K3$ fibres at $\alpha_0, \alpha_1 \neq 0$ involved in the gluing. This yields the following abelian gauge group

$$U(1)^{n-1} \sim \frac{U(1)_1 \times \ldots \times U(1)_n}{U(1)_{diag}}$$

with the matter whose charges are determined by the

intersections $\mathcal{X}_{ij} = C_j \cdot C_j$ and as displayed in the table:

Multiplicity	$\mathcal{N}=2$ multiplets		$\mathcal{N}=1$ multiplets	
	$U(1)^{n-1}$ charges	multiplet	$U(1)^{n-1}$ charges	multiplet
$n-1$	$(0,0,\ldots,0)$	vector	$(0,\ldots,0)$	vector
			$(0,\ldots,0)$	chiral
χ_{ij}	$(0,\ldots,+1_i,\ldots,+1_j,\ldots,0)$	hyper	$(0,\ldots,+1_i,\ldots,+1_j,\ldots,0)$	chiral
$1 \leq i < j < n$			$(0,\ldots,-1_i,\ldots,-1_j,\ldots,0)$	chiral
χ_{in}	$(0,\ldots,+1_i,\ldots,0)$	hyper	$(0,\ldots,+1_i,\ldots,0)$	chiral
$1 \leq i < n$			$(0,\ldots,-1_i,\ldots,0)$	chiral

Table 2: The table shows the spectrum of the Abelian $\mathcal{N}=2$ gauge theory sector arising from the conifold singularities in the building block (Z_{sing}, S).

Higgs Transitions:

$$\begin{aligned} b_2(Y^a) &= b_2(Y^b) - n - 1 \\ b_3(Y^a) &= b_3(Y^b) + 2\left(\sum_{1\leq i<j\leq n} \mathcal{X}_{ij}\right) - 3(n-1) \end{aligned}$$

❷ Non-abelian Gauge symmetry, charged matter spectrum

Let us now turn to the enhancement to non-Abelian $\mathcal{N} = 2$ gauge theory sectors in the context of twisted connected G_2-manifolds. Let us assume that the anti-canonical line bundle $-K_P$ of the semi-Fano threefold P factors as

$$-K_P = \tilde{\mathcal{L}}_1^{\otimes k_1} \otimes \ldots \otimes \tilde{\mathcal{L}}_s^{\otimes k_s} \quad \text{with} \quad n = k_1 + \ldots + k_s \ ,$$

where $\tilde{\mathcal{L}}_i$ are line bundles with global sections $\tilde{s}_{0,i}$. Then the global section s_0 of $-K_P$ can further degenerate to $s_0 = \tilde{s}_{0,1}^{k_1} \cdots \tilde{s}_{0,s}^{k_s}$ and the singular building block is of the form

$$Z_{\mathsf{sing}} = \left\{ (x,z) \in P \times \mathbb{P}^1 \,\middle|\, \alpha_0 \tilde{s}_{0,1}^{k_1} \cdots \tilde{s}_{0,s}^{k_s} + \alpha_1 s_1 = 0 \right\} \ , \tag{3}$$

with the singular equation in the affine coordinate $t = \frac{z_1}{z_0}$

given by

$$\tilde{s}_{0,1}^{k_1} \cdots \tilde{s}_{0,s}^{k_s} + t s_1 = 0 \ .$$

As before we assume that all curves $\tilde{\mathcal{C}}_i = \{\tilde{s}_{0,i} = 0\} \cap \{s_1 = 0\}$ are smooth. In the vicinity of the singular fiber $\pi^{-1}([1,0]) \subset Z_{\mathsf{sing}}$ the singular building block (Z_{sing}, S) develops A_{k_i-1}-singularities along those curves $\tilde{\mathcal{C}}_i$ with $k_i > 1$.

Multiplicity	$\mathcal{N}=2$ multiplets		$\mathcal{N}=1$ multiplets	
	G reps.	multiplet	G reps.	multiplet
$s-1$	$\mathbf{1}$	$U(1)$ vector	$\mathbf{1}$	$U(1)$ vector
			$\mathbf{1}$	chiral
$i=1,\ldots,s$	$\mathbf{adj}_{SU(k_i)}$	$SU(k_i)$ vector	$\mathbf{adj}_{SU(k_i)}$	$SU(k_i)$ vector
			$\mathbf{adj}_{SU(k_i)}$	chiral
$g(\tilde{\mathcal{C}}_i)$	$\mathbf{adj}_{SU(k_i)}$	hyper	$\mathbf{adj}_{SU(k_i)}$	chiral
$1 \leq i \leq s$			$\mathbf{adj}_{SU(k_i)}$	chiral
$\tilde{\chi}_{ij}$	$(\mathbf{k_i},\mathbf{k_j})_{(+1_i,+1_j)}$	hyper	$(\mathbf{k_i},\mathbf{k_j})_{(+1_i,+1_j)}$	chiral
$1 \leq i < j < s$			$(\bar{\mathbf{k}}_\mathbf{i},\bar{\mathbf{k}}_\mathbf{j})_{(-1_i,-1_j)}$	chiral
$\tilde{\chi}_{in}$	$(\mathbf{k_i},\mathbf{k_n})_{(+1_i)}$	hyper	$(\mathbf{k_i},\mathbf{k_n})_{(+1_i)}$	chiral
$1 \leq i < n$			$(\bar{\mathbf{k}}_\mathbf{i},\bar{\mathbf{k}}_\mathbf{n})_{(-1_i)}$	chiral

Table 3: The table shows the spectrum of the $\mathcal{N}=2$ gauge theory sector with gauge group $G = SU(k_1)\times\ldots\times SU(k_s)\times U(1)^{s-1}$ as arising from the non-Abelian building blocks (Z_{sing}, S). It lists both the four-dimensional $\mathcal{N}=2$ and the four-dimensional $\mathcal{N}=1$ multiplet structure. The adjoint matter is determined by the genus $g(\tilde{\mathcal{C}}_i)$ of the curves $\tilde{\mathcal{C}}_i$, whereas the bi-fundamental matter from their intersection numbers $\tilde{\chi}_{ij}$ within the K3 surface S.

No.	ρ	Gauge Group	$\mathcal{N}=2$ Hypermultiplet spectrum	$h^\flat$	$c^\sharp$	$b_3^\flat$	$b_3^\sharp$	$k^\sharp$
K24, $\mathrm{MM}34_2$	2	$SU(3)\times SU(2) \times U(1)$	$2\times(\mathbf{adj},\mathbf{1});(\mathbf{1},\mathbf{adj});3\times(\mathbf{3},\mathbf{2})_{+1}$	50	14	79	43	4
K32	2	$SU(3)^2\times U(1)$	$(\mathbf{adj},\mathbf{1});(\mathbf{1},\mathbf{adj});3\times(\mathbf{3},\mathbf{3})_{+1}$	52	13	79	40	5
K35, $\mathrm{MM}36_2$	2	$SU(5)\times SU(2) \times U(1)$	$2\times(\mathbf{adj},\mathbf{1});(\mathbf{5},\mathbf{2})_{+1}$	60	22	87	49	6
K36, $\mathrm{MM}35_2$	2	$SU(4)\times SU(2) \times U(1)$	$2\times(\mathbf{adj},\mathbf{1});2\times(\mathbf{4},\mathbf{2})_{+1}$	54	17	81	44	5
K37, $\mathrm{MM}33_2$	2	$SU(4)\times SU(3) \times U(1)$	$(\mathbf{adj},\mathbf{1});3\times(\mathbf{4},\mathbf{3})_{+1}$	54	12	79	37	6

Table 4: The table exhibits the $\mathcal{N}=2$ gauge theory sectors for some smooth toric semi-Fano threefolds P_Σ of Picard rank two and higher. The columns display the number of the threefold P_Σ in the Mori–Mukai and/or Kasprzyk classification, its Picard rank ρ, the maximally enhanced gauge group of maximal rank by factorizing the anti-canonical bundle, the $\mathcal{N}=2$ matter hypermultiplets, the complex dimensions $h^\flat$ and $c^\sharp$ of the Higgs and Coulomb branches, the reduced three-form Betti numbers $b_3^\flat$ and $b_3^\sharp$, and the kernel $k^\sharp$ of the Coulomb branch.

☆ Conclusions

- 10^8 new G_2 manifolds
- universal moduli, splitting of spectrum in sectors
- abelian and non abelian gauge symmetries with non chiral matter
- G_2 transitions