q-Virasoro constraints for $3d$ partition functions

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Content

- Virasoro constraints in hermitian matrix model
- Motivation from physics
- Free fields representation
- q-Virasoro constraints
  - q-Virasoro algebra
  - Free field representation
  - $3d$ generating functions
- Conclusions and Outlook

Ward identities

Hermitian matrix model with most general polynomial potential

Dijkgraaf, Verlinde, Verlinde '91

Morozov, Mironov '91

$$Z_N(\{t\}) = \int \prod_{i=1}^{N} d\,x_i \prod_{i<j}(x_i - x_j)^2\, e^{\sum\limits_{k\geq 0} t_k \sum\limits_i x_i^k}$$

Invariant under

$$x_i \to x_i + \epsilon_n\, x_i^{n+1}$$

Ward Identities: $\langle \sum_{i,j} \sum_{k=0}^{n} x_i^k x_j^{n-k} + \sum_{k\geq 0} k t_k \sum_i x_i^{k+n} \rangle = 0\,, \quad n > 0$

Virasoro constraints: $\hat{L}_n Z_N(\{t\}) = 0$ for $n > 0$

$$\hat{L}_n = \sum_{k=0}^{n} \frac{\partial^2}{\partial t_k \partial t_{n-k}} + \sum_{k=0}^{\infty} k t_k \frac{\partial}{\partial t_{k+n}}$$

$$\left[\hat{L}_n,\, \hat{L}_m\right] = (n-m)\,\hat{L}_{n+m}$$

Clue to the integrability of the matrix model! *Morozov '93, '95*

Motivation

Supersymmetric localization Pestun, '07

Path integral of SUSY gauge theory

$$Z = \int \mathcal{D}\phi \, e^{-S(\phi)}$$

Matrix integral

$$Z = \int d^N x \, Z_{1-loop}(\{x\}) \, e^{-\frac{1}{g^2}\sum_i x_i^2}$$

Rich variety of "new" matrix models coming from various gauge theories (different multiplet content, space-time dimensions, manifolds etc.)

Can we generalize Virasoro constraints construction to the localization inspired matrix models?

Generating functions

- Typical localization matrix model

$$Z = \int d^N x \, Z_{1-loop}(\{x\}) \, e^{-\frac{1}{g^2}\sum_i x_i^2}$$

Yang-Mills coupling or CS level. Gaussian potential. We need general potential $\sum_k \sum_i t_k x_i^k$!

1-loop determinant $\sim \prod_{i\neq j} f(x_i - x_j)$

- Consider *Wilson loop:*

$$\langle W_R \rangle = \int d^N x \, s_R\left(\{e^{2\pi x}\}\right) Z_{1-loop}(\{x\}) \, e^{-\frac{1}{g^2}\sum_i x_i^2},$$

$s_R(x)$ - Schur pol.

- Construct WL *generating function:* $Z(\{t\}) \equiv \sum_R s_R(\{\hat{t}\})\langle W_R \rangle$

$$Z(\{t\}) = \int d^N x \, Z_{1-loop}(\{x\}) \, e^{-\frac{1}{g^2}\sum_i x_i^2 + \sum_{k>0} t_k \sum_i e^{2\pi k x_i}}$$

$$t_n = \sum_k \frac{\hat{t}_k^n}{n}$$

For $g^2 \to \infty$ ($k_{CS} = 0$ in $3d$) it is exactly what we need!

Constraints from Ward Identities

How to generalize Virasoro constraints construction?

First way: Choose different basis of transformations $x_i \to f_i(\{x\})$

Example: *A.N., M.Zabzine, '15*

- Matrix model

$$Z_N(\{t\}) = \oint \prod_{i=1}^{N} \frac{dz_i}{z_i} \prod_{i<j} \theta\left(\frac{z_i}{z_j};q\right)\theta\left(\frac{z_j}{z_i};q\right) e^{\sum_{s=0}^{\infty}\frac{t_s}{s!}\sum_i z_i^s}$$

vector multiplet on $S^3 \times S^1$

Wilson loop on S^1 fiber

- Transformations generated by the operator $D_q\left(x^{n+1}\dots\right)$
- Constraints:

$$T_n^q Z_N(\{t\}) = 0 \qquad T_n^{q^2} Z_N(\{t\}) = 0$$

$$\left[T_n^q, T_m^q\right] = q^{-n-m}\left([n]_q - [m]_q\right)\left([2]_q T_{n+m}^{q^2} - T_{n+m}^q\right)$$

Proper choice of basis: WI=Differential Constraints

Hard (a lot of guessing and playing with expressions)

Constraints from free bosons

- Hermitian matrix model → State in free bosons theory

$$Z_N(\{t\}) = \int \prod_{i=1}^N d\,x_i \prod_{i<j}(x_i - x_j)^2\, e^{\sum\limits_{k\geq 0} t_k \sum\limits_i x_i^k} \simeq \int d^N x \prod_i S(x_i)|0\rangle$$

Heisenberg algebra representation

$$a_{-n} \simeq n\,t_n\,,\quad a_n \simeq 2\frac{\partial}{\partial t_n}\,,\quad |0\rangle \simeq 1$$

Screening current

$$S(x) \equiv :e^{-\sum_{n\neq 0}\frac{x^{-n}}{n}a_n}:\,e^Q\,x^P$$

- Crucial property of screening currents:

$$\left[\hat{L}_n,\, S(w)\right] = \frac{d}{dw}O(w) \quad \Rightarrow \quad \left[\hat{L}_n,\, \int dw\, S(w)\right] = 0$$

Virasoro operators: $\hat{L}_n = \frac{1}{4}\sum\limits_{k\in\mathbb{Z}} :a_{n-k}\,a_k: \simeq \sum\limits_{k=0}^{n}\frac{\partial^2}{\partial t_k \partial t_{n-k}} + \sum\limits_{k=0}^{\infty} k t_k \frac{\partial}{\partial t_{k+n}}$

- Hence we obtain *Virasoro constraints!*

$$\hat{L}_n Z_N(\{t\}) = 0 \qquad n > 0$$

q-Virasoro algebra

First natural generalization: *q-deformed Virasoro*

Shiraishi, Kubo, Awata, Odake '95

$$[T_n, T_m] = -\sum_\ell f_\ell (T_{n-\ell} T_{m+\ell} - T_{m-\ell} T_{n+\ell}) - \frac{(1-q)(1-t^{-1})}{(1-p)} (p^n - p^{-n}) \delta_{n+m,0}$$

- Deformation is parametrized by $q, t \in \mathbb{C}$, $p = q\, t^{-1}$. Another parametrization $t = q^\beta$.
- *Structure constants* f_l fixed by associativity

$$\sum_{\ell>0} f_\ell z^\ell = \exp\left(\sum_{n>0} \frac{1}{n} \frac{(1-q^n)(1-t^{-n})}{(1+p^n)} z^n \right)$$

- *Virasoro limit:* $q = e^\hbar$, $\hbar \to 0$

$$T_n \to 2\delta_{n,0} + \beta\hbar^2 \left(\hat{L}_n + \frac{1}{4} Q_\beta^2\, \delta_{n,0} \right) + O\left(\hbar^4\right)$$

$\hat{L}_n$ - Virasoro operator with central charge $c = 1 - 6Q_\beta^2$

$$Q_\beta = \sqrt{\beta} - \frac{1}{\sqrt{\beta}}$$

q-Virasoro: free field representation

- *Heisenberg algebra:*

$$[a_n,\, a_m] = \frac{1}{n}(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})(p^{\frac{n}{2}} + p^{-\frac{n}{2}})\delta_{n+m,0}\,, \quad [P, Q] = 2$$

- *Screening current:*

$$S_q(w) \equiv :e^{-\sum_{n\neq 0} \frac{w^{-n}}{q^{n/2}-q^{-n/2}} a_n} : e^{\sqrt{\beta}Q}\, w^{\sqrt{\beta}P}$$

- *q-Virasoro operator:*

$$\sum_{n\in\mathbb{Z}} T_n z^{-n} = \sum_{\sigma=\pm 1} \Lambda_\sigma(z)$$

$$\Lambda_\sigma(z) =: e^{\sigma \sum_{n\neq 0} \frac{z^{-n}}{(1+p^{-\sigma n})} a_n} : q^{\sigma \frac{\sqrt{\beta}}{2} P} p^{\frac{\sigma}{2}}$$

- *q-Virasoro matrix model:*

$$Z(\{t\}) \simeq \oint d^N w \prod_i S_q(w_i)|0\rangle$$

$$a_{-n} \simeq (q^{\frac{n}{2}} - q^{-\frac{n}{2}})t_n\,,\quad a_n \simeq \frac{1}{n}(t^{\frac{n}{2}} - t^{-\frac{n}{2}})(p^{\frac{n}{2}} + p^{-\frac{n}{2}})\frac{\partial}{\partial t_n}$$

q-Virasoro matrix model

Nieri, A.N., Zabzine '16

- *q-Virasoro matrix model generating function:*

$$Z_q(\{t\}) = \mathcal{N}_0 \oint \frac{d^N w}{w} \prod_{k\neq j} \frac{\left(w_k w_j^{-1}; q\right)_\infty}{\left(t\, w_k w_j^{-1}; q\right)_\infty} \prod_i w_i^{\kappa_1} e^{\sum_{k>0} t_k \sum_j w_j^k}$$

Wilson loop along S^1

1-loop for vec. + adj. multiplets of mass t on $D^2_\epsilon \times S^1$ $\left(q = e^{2\pi i \epsilon}\right)$
Aganagic, Haouzi, Kozcaz, Shakirov '13

FI term: $\kappa_1 = \beta N - \sqrt{\beta} Q_\beta$

- By construction $T_n Z_q(\{t\}) = 0 \quad n > 0$

Bell polynomials

$$A_n^{(\sigma)} = \sigma \frac{a_n \, |n|!}{1+p^{-\sigma n}}$$

$$T_n = \sum_{\sigma=\pm1} q^{\sigma \frac{\sqrt{\beta}}{2}} p^{\frac{\sigma}{2}} \sum_{k\leq 0} \frac{B_k\left(\{A_{-k}^{(\sigma)}\}\right) B_{n+k}\left(\{A_{n+k}^{(\sigma)}\}\right)}{(n+k)!\, k!}$$

S_b^3 partition function: modular double

- Gluing two copies of $\mathcal{N}=2$ on $D^2_\epsilon \times S^1 \Rightarrow \mathcal{N}=2$ generating function on the *squashed sphere* S_b^3: $\omega_1^2|z_1|^2+\omega_2^2|z_2|^2=1$

Copy 1	Copy 2
$q_1 = e^{2\pi i \frac{\omega_2}{\omega_1}}$	$q_2 = e^{2\pi i \frac{\omega_1}{\omega_2}}$
$t_1 = e^{2\pi i \beta \frac{\omega_2}{\omega_1}}$	$t_2 = e^{2\pi i \beta \frac{\omega_1}{\omega_2}}$
$\beta_1 = \beta$	$\beta_2 = \beta$

$SL(2,\mathbb{Z})$ gluing: $\epsilon \to g\cdot\epsilon = \frac{\epsilon}{1-\epsilon}$

$q_1 = e^{2\pi i \epsilon}$; $q_2 = e^{2\pi i g\cdot\epsilon}$; $\epsilon = \frac{\omega_2}{\omega_1}$;

Two Wilson loops: length $L_{1(2)} = \frac{2\pi}{\omega_{1(2)}}$ around $z_{2(1)}=0$

- *Partition function*

$$Z_{S_b^3}(\{t\}) = \mathcal{N}_0 \oint d^N x \prod_{k\neq j} \frac{S_2(x_i - x_j|\underline{\omega})}{S_2(x_i-x_j+M|\underline{\omega})} \prod_i e^{\frac{2\pi i \kappa_1}{\omega_1\omega_2} x_i} \prod_{j=1,2} e^{\sum_{k>0} t_{k,j} \sum_i e^{2\pi i k \frac{\omega}{\omega_j} x_i}}$$

1-loop for vec.+adj. of mass $M = \beta(\omega_1+\omega_2)$

FI term

- *q-Virasoro constraints*

$$T_{n,1} Z_{S_b^3}(\{t\}) = 0$$
$$T_{n,2} Z_{S_b^3}(\{t\}) = 0$$

Conclusions and Outlook

- Construction of Virasoro constraints is generalized to constraints closing q-Virasoro algebra
- Resulting matrix model is generation function of the Wilson loops in $\mathcal{N} = 2$ theory on $D^2 \times S^1$
- S_b^3 Wilson loop generating function is annihilated by two sets of commuting q-Virasoro operators.
- Generalization not mentioned in the talk
  - Other spaces: $S_b^3/\mathbb{Z}_r$, $S^2 \times S^1$, twisted $S^2 \times S^1$
  - Non-trivial CS level: inclusion of vertex operators
  - Adding fundamental hypermultiplets: inclusion of vertex operators
  - Quiver theories: quiver $W^{(q,t)}$-algebras
- Going to higher dimensions: $4d$ (*Lodin, Nieri, Zabzine '17*) and $5d$ theories?
- Integrability of the constraints?
- Correspondence to q-deformed integrable systems (Toda, Macdonald, etc.)

Thank you!