Soft and collinear limits of scattering amplitudes beyond the leading order

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based on work with Johannes Brödel, Marius de Leeuw, Dhritiman Nandan, Matteo Rosso, Oliver Schlotterer and Wadim Wormsbecher
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Overview

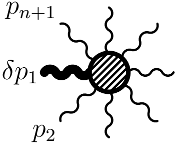
Introduction:

- Soft and collinear limits of scattering amplitudes display universality.
- Renewed interest in soft limits & discovery of subleading soft theorems for gluon and graviton amplitudes. Relation to hidden BMS symmetry?
- New formalisms for expressing scattering amplitudes $\Rightarrow$ BCFW, Grassmannian, CHY.

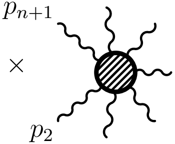
Outline:

1. Soft theorems and how they are constrained
2. Einstein-Yang-Mills (EYM) amplitudes from YM via CHY formalism
3. Collinear limits of gluons, gravitons and scalars and subleading structure
4. Summary

Soft Theorems

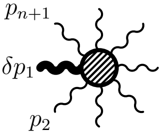


$$\xrightarrow{\delta \to 0} \quad S^{[j]}(\delta\, p_1, \{p_a\}) \times$$



Theorems of Low (1958) and Weinberg (1964)

Scattering amplitudes display universal factorization when a single photon, gluon or graviton becomes soft: Parametrize soft momentum as $\delta \cdot q^\mu$ and take $\delta \to 0$



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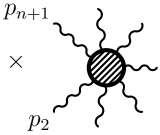

$$\boxed{\mathcal{A}_{n+1}(\delta\, q, p_1, \ldots, p_n) \underset{\delta\to 0}{=} S^{[0]}(\delta\, q, \{p_a\}) \cdot \mathcal{A}_n(p_1, \ldots, p_n) + \mathcal{O}(\delta^0)}$$

At tree-level with soft leg q polarization $E_{\mu(\nu)}$:

$$S^{[0]}(\delta\, q, \{p_a\}) = \begin{cases} \sum_{a=1}^{n} \frac{1}{\delta} \frac{E_\mu\, p_a^\mu}{p_a \cdot q} & : \text{ photon} \quad \to \text{gluon (color ordered)} \\ \sum_{a=1}^{n} \frac{1}{\delta} \frac{E_{\mu\nu}\, p_a^\mu\, p_a^\nu}{p_a \cdot q} & : \text{ graviton} \end{cases}$$

Proof is elementary. Tree-level exact for gravity. IR divergent loop corrections in YM.

Subleading soft theorems

Universal factorization extends to subleading order [Cachazo, Strominger][Low,Burnett,Kroll;Casali]

$$\boxed{\mathcal{A}_{n+1}(\delta\, q, p_1, \ldots, p_n) \underset{\delta\to 0}{=} S^{[j]}(\delta\, q, \{p_a\}) \cdot \mathcal{A}_n(p_1, \ldots, p_n) + \mathcal{O}(\delta^j)}$$

with soft operators

$$S^{[j]}(\delta\, q, \{p_a\}) = \begin{cases} \frac{1}{\delta} S^{(0)}_{\mathsf{YM}} + S^{(1)}_{\mathsf{YM}} & : \text{ Yang-Mills } (j=1) \\ \frac{1}{\delta} S^{(0)}_{\mathsf{G}} + S^{(1)}_{\mathsf{G}} + \delta\, S^{(2)}_{\mathsf{G}} & : \text{ Gravity } (j=2) \end{cases}$$

Explicit constructions (using BCFW, CHY) @ tree-level yield

$$S^{(1)\text{tree}}_{\mathsf{YM}} = \frac{E_\mu\, q_\nu\, J_1^{\mu\nu}}{p_1 \cdot q} - \frac{E_\mu\, q_\nu\, J_n^{\mu\nu}}{p_n \cdot q}$$

$$S^{(1)\text{tree}}_{\mathsf{G}} = \sum_{a=1}^{n} \frac{(E\cdot p_a)\, E_\mu\, q_\nu\, J_a^{\mu\nu}}{p_a \cdot q}$$

$$S^{(2)\text{tree}}_{\mathsf{G}} = \sum_{a=1}^{n} \frac{(E_\mu\, q_\nu\, J_a^{\mu\nu})^2}{p_a \cdot q}$$

$$J_a^{\mu\nu} := p_a^\mu \partial_{p_a^\nu} + E_a^\mu \partial_{E_a^\nu} - \mu \leftrightarrow \nu$$

writing polarization $E_{\mu\nu} = E_\mu E_\nu$

$\longrightarrow$ hidden BMS symmetry? [Strominger et al]

Constraining soft theorems

$$\delta^{(D)}(\delta q + \sum_{i=1}^{n} p_i) \quad \text{vs.} \quad \delta^{(D)}(\sum_{i=1}^{n} p_i)$$

A subtle momentum conservation issue

- Amplitudes are distributions: $$\mathcal{A}_n(\{p_a\}) = \delta^{(D)}(\sum_{a=1}^{n} p_a)\, A_n(\{p_a\})$$
- Soft theorem should be stated with δ-functions:

$$\boxed{\delta^{(D)}(\delta\, q + P)\, A_{n+1}(\delta\, q, \{p_a\}) \underset{\delta\to 0}{=} S^{[j]}(\delta\, q, \{p_a\})\, \delta^{(D)}(P)\, A_n(\{p_a\}) + \mathcal{O}(\delta^j)}$$

with $P = \sum_{a=1}^{n} p_a$ and $S^{[j]} = \frac{1}{\delta} S^{(0)} + S^{(1)} + \ldots$

- Mismatch in arguments of delta-functions! If we want to state the theorem on the level of stripped amplitudes, i.e.

$$A_{n+1}(\delta\, q, \{p_a\}) = \tilde{S}^{[j]}(\delta\, q, \{p_a\})\, A_n(\{p_a\})$$

must have a non-trivial commutator:

$$S^{[j]}(\delta\, q)\, \delta^{(D)}(P) = \delta^{(D)}(P + \delta\, q)\, \tilde{S}^{[j]}(\delta\, q)$$

Consistency condition [Broedel, de Leeuw, JP, Rosso]

Relation at leading orders: $P = \sum_{a=1}^{n} p_a$

$$\left(\frac{1}{\delta}S^{(0)} + S^{(1)}\right)\delta^{(D)}(P) = \left(\delta^{(D)}(P) + \delta\, q \cdot \partial_P \delta^{(D)}(P)\right)\left(\frac{1}{\delta}\tilde{S}^{(0)} + \tilde{S}^{(1)}\right) + \mathcal{O}(\delta)$$

- No issue at leading order in δ:

$$S^{(0)} = \tilde{S}^{(0)} \qquad \& \qquad [S^{(0)}, \delta^{(D)}(P)] = 0$$

- But non-trivial commutator at NLO:

$$S^{(1)} = \tilde{S}^{(1)} + \chi \qquad \& \qquad [S^{(1)}, \delta^{(D)}(P)] = S^{(0)}\, q \cdot \partial_P \delta^{(D)}(P) + \delta^{(D)}(P)\,\chi$$

$\Rightarrow$ implies that $S^{(1)}(\delta\, q, \{p_a\})$ must contain differential operator ∂_{p_a}.

- Similar story at NNLO (relevant for gravity)
- Message: ∂_{p_a} terms in $S^{(j)}$ are constrained by lower order $S^{(j'<j)}$ ops.
- Moreover, it turns out that $\chi = \chi' = 0$

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Constraining subleading soft theorems [Broedel, de Leeuw, JP, Rosso]

Collect all known constraints on soft operators:

$$\mathcal{A}_{n+1}(\delta\, q, E, \{E_a, p_a\}) \underset{\delta\to 0}{=} S^{[j]}(\delta\, q, E, \{E_a, p_a, \partial_{E_a}, \partial_{p_a}\}) \cdot \mathcal{A}_n(\{E_a, p_a\}) + \mathcal{O}(\delta^j)$$

- Gauge invariances on soft and hard legs
- Distributional constraint: (as discussed)

$$S^{[j]}(\delta\, q)\, \delta^{(D)}(\sum_a p_a) = \delta^{(D)}(\delta\, q + \sum_a p_a)\, \tilde{S}^{[j]}(\delta\, q)$$

- "Locality": $S^{(l)} = \sum_{a=1}^{n} S^{(l)}(q, E; E_a, p_a; \partial_{E_a}, \partial_{p_a})$

  as it would arise from a Ward identity. Is an assumption beyond tree-level
- Dimensional analysis: $[g_{\mathsf{YM}}] = 0 \quad [\kappa] = -1 \quad [S^{[j]}_{\mathsf{YM}}] = -1 \quad [S^{[j]}_{\mathsf{G}}] = 0$

Enforcing all constraints entirely fixes the subleading soft functions in 4d!

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4D: Gauge theory

- Use spinor helicity: $q^\mu \to q^\alpha \tilde{q}^{\dot\alpha}$ & consider $(+)$ helicity soft gluon:
  $E_\mu \to E^{(+)}_{\alpha\dot\alpha} = \frac{\mu_\alpha\, \tilde{q}_{\dot\alpha}}{\langle \mu q\rangle}$
- Ansatz: Local, linear in $E^{(+)}$, first order in ∂_a and $\partial_{\dot\alpha}$

$$S^{(1)}_{\text{YM}} = \sum_{a=1,n} E^{(+)}_{\alpha\dot\alpha}\left[\Omega_a^{\alpha\dot\alpha\beta}\frac{\partial}{\partial\lambda_a^\beta} + \bar{\Omega}_a^{\alpha\dot\alpha\dot\beta}\frac{\partial}{\partial\tilde{\lambda}_a^{\dot\beta}}\right]$$

  Coefficients $\Omega_a(\lambda_q, \tilde{\lambda}_q, \lambda_a, \tilde{\lambda}_a)$ constrained by little-group scaling
- Gauge invariance $\mu_\alpha \to \mu_\alpha + \eta\, q_\alpha$ and distributional constraint yield the unique result for subleading soft operator

$$\boxed{S^{(0)}_{\text{YM}} = \frac{\langle n\,1\rangle}{\langle n\,q\rangle\,\langle q\,1\rangle}} \quad \overset{\text{locality \& consistency}}{\Longrightarrow} \quad \boxed{S^{(1)}_{\text{YM}} = \frac{[\tilde{q}\tilde{\partial}_1]}{\langle q1\rangle} - \frac{[\tilde{q}\tilde{\partial}_n]}{\langle qn\rangle}}$$

- N.B: Does not prove the existence of subleading soft thm, but says that if a sub-leading universal soft factorization holds, it must be of this form.

4D: Gravity

Positive helicity soft graviton: $\boxed{S_{\mathsf{G}}^{(0)} = \sum_{a=1}^{n} \frac{\langle xa\rangle \langle ya\rangle [qa]}{\langle xq\rangle \langle yq\rangle \langle aq\rangle}}$ x & y reference spinors

- Analogous arguments: Local, first order ansatz

$$S_{\mathsf{G}}^{(1)} = \sum_{a=1}^{n} E^{(+)}_{\alpha\dot{\alpha}\beta\dot{\beta}} \left[\Omega_a^{\alpha\dot{\alpha}\beta\dot{\beta}\gamma} \frac{\partial}{\partial \lambda_a^{\gamma}} + \bar{\Omega}_a^{\alpha\dot{\alpha}\beta\dot{\beta}\dot{\gamma}} \frac{\partial}{\partial \tilde{\lambda}_a^{\dot{\gamma}}}\right]$$

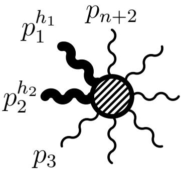
  Ω_a & $\bar{\Omega}_a$ contain 4 local constants

- Again constraints (gauge invariance & distributional constraint) nail down subleading operator completely:

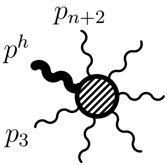
$$\Rightarrow \quad \boxed{S_{\mathsf{G}}^{(1)} = \frac{1}{2} \sum_{a=1}^{n} \frac{[a\,q]}{\langle a\,q\rangle} \left(\frac{\langle a\,x\rangle}{\langle q\,x\rangle} + \frac{\langle a\,y\rangle}{\langle q\,y\rangle} \right) [\tilde{q}\,\tilde{\partial}_a]}$$

- Same reasoning also fixes sub-subleading soft operator $S_{\mathsf{G}}^{(2)}$ in 4d.

Collinear Limits

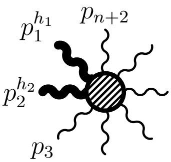


$$\xrightarrow{1\|2} \sum_{h=\pm} \mathrm{Split}_{-h}(c; 1^{h_1}, 2^{h_2}) \times$$



Collinear limit

- Collinear factorization is central property for gluon amplitudes



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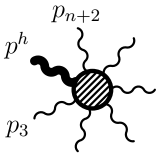

- Parametrize the collinear limit $\epsilon \to 0$: $(c = \cos\phi\,, \quad s = \sin\phi)$ [Stieberger,Taylor]

$$|1\rangle = c\,|p\rangle - \epsilon\, s\,|r\rangle \qquad |1] = c\,|p] - \epsilon\, s\,|r] \qquad |1\rangle^{\alpha} = \lambda_1^{\alpha}\,, \quad |1]^{\dot\alpha} = \tilde\lambda_1^{\dot\alpha}$$

$$|2\rangle = s\,|p\rangle + \epsilon\, c\,|r\rangle \qquad |2] = s\,|p] + \epsilon\, c\,|r] \qquad \Rightarrow \quad \langle 12\rangle = \epsilon_{\alpha\beta}\lambda_1^{\alpha}\,\lambda_2^{\beta} = \epsilon\,\langle pr\rangle$$

with reference momentum r^{μ}. This translates to 4-momenta

$$p_1^{\mu} = c^2\, p^{\mu} - \epsilon\, cs\, q^{\mu} + \epsilon^2\, s^2\, r^{\mu}$$

$$p_2^{\mu} = s^2\, p^{\mu} + \epsilon\, cs\, q^{\mu} + \epsilon^2\, c^2\, r^{\mu} \qquad q := |p\rangle[r| + |r\rangle[p|$$

- Momentum conservation up to order ϵ^2: $\quad p_1 + p_2 = p + \epsilon^2\, r$

Splitting functions for gluons

- Collinear factorization

$$A_{n+2}(1^{h_1}, 2^{h_2}, \ldots) \xrightarrow{1\|2} \sum_{h=\pm} \mathsf{Split}_{-h}(c; 1^{h_1}, 2^{h_2})\, A_{n+1}(p^h, \ldots) + \mathcal{O}(\epsilon^0)$$

- One has to leading order in ϵ:

$$\mathsf{Split}_+(c; 1^+, 2^+) = 0 \qquad \mathsf{Split}_-(c; 1^+, 2^+) = \frac{1}{\epsilon}\frac{1}{cs\,\langle pr\rangle}$$

$$\mathsf{Split}_+(c; 1^+, 2^-) = -\frac{1}{\epsilon}\frac{s^3}{c\,\langle pr\rangle} \qquad \mathsf{Split}_-(c; 1^+, 2^-) = \frac{1}{\epsilon}\frac{c^3}{s\,[pr]}$$

- Question:

Is there universal factorization at subleading order in ϵ?

Natural question to ask in view of subleading soft theorems. However, no potential hidden symmetry at the horizon.

Intriguing relation to Einstein-Yang-Mills amplitudes [Stieberger, Taylor '15]

- Consider linear combinations of collinear amplitudes s.t. $1/\epsilon$ pole cancels
  - N=5:

$$s_{5p}\, A(1^+,2^+,3,4,5) - s_{4p}\, A(1^+,2^+,3,5,4) \xrightarrow{1\|2} \frac{g^2}{\kappa\, c^2}\, A_{\mathsf{EYM}}(p^{++},3,4,5) + \mathcal{O}(\epsilon)$$

  - N=6:

$$s_{6p}\, A(1^+,2^+,3,4,5,6) - s_{5p}\, [A(1^+,2^+,3,4,6,5) + A(1^+,2^+,3,6,4,5)]$$
$$+ s_{4p}\, A(1^+,2^+,3,6,5,4) \xrightarrow{1\|2} \frac{g^2}{\kappa\, c^2}\, A_{\mathsf{EYM}}(p^{++},3,4,5,6) + \mathcal{O}(\epsilon)$$

- In general: $(N-3)!/2$ constraints for the independent $(N-3)!$ gluon amplitudes in $1 \| 2$ limit.
- $\Rightarrow$ Suggests possible existence of subleading splitting into EYM-amplitudes

$$A_{\mathsf{YM}}(1^+,2^+,3,\ldots,N)\Big|_{1\|2}^{\text{subleading}} \overset{????}{=} S_-^{[1]}(c;1^+,2^+;3,N)\, A_{\mathsf{EYM}}(p^{++},3,\ldots,N)$$

Calls for unified description of gluon & graviton amplitudes

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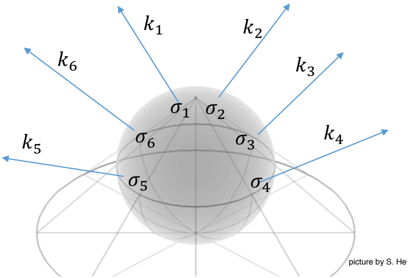
Calls for unified description of gluon & graviton amplitudes

The Cachazo-He-Yuan formalism

$$\sum_{b=1, b\neq a}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0$$



Best formalism for unified description of amplitudes: CHY

- Scattering equations: $$f_a = \sum_{\substack{b=1\\ b\neq a}}^{n} \frac{p_a\cdot p_b}{\sigma_a-\sigma_b} = 0 \qquad \sigma_{ab} := \sigma_a - \sigma_b \in \mathbb{C}$$

  For N-particle kinematics these have $(N-3)!$ solutions

- $$\mathcal{A}_n = \int \frac{d^n\sigma_a}{\text{vol SL}(2,\mathbb{C})} \left(\prod_{a=1}^{n}{}' \, \delta(f_a) \right) \mathcal{I}_n(\{p,E,\sigma\})$$ [Cachazo,He,Yuan]

  Integrand $\mathcal{I}_n$ built from 2 building blocks: 'Parke-Taylor factor' and CHY-Matrix

$$C(1,\ldots,n) = \frac{1}{\sigma_{12}\ldots\sigma_{n1}} \qquad \Psi_n = \begin{pmatrix} A & -C^{\mathrm{T}} \\ C & B \end{pmatrix}$$

$$A_{ab} = \begin{cases} \frac{p_a\cdot p_b}{\sigma_a-\sigma_b} \\ 0 \end{cases} \qquad B_{ab} = \begin{cases} \frac{E_a\cdot E_b}{\sigma_a-\sigma_b} \\ 0 \end{cases} \qquad C_{ab} = \begin{cases} \frac{E_a\cdot p_b}{\sigma_a-\sigma_b} & \text{for } a\neq b \\ -\sum_{c\neq a} \frac{E_a\cdot p_c}{\sigma_a-\sigma_c} & \text{for } a=b \end{cases}$$

- Integrand $\mathcal{I}_n$ defines the theory in question.

CHY: Unifying picture of gluon & graviton amplitdues

- CHY formula

$$\mathcal{A}_n = \int \frac{d^n\sigma_a}{\text{vol SL}(2,\mathbb{C})} \left(\prod_{a=1}^{n}{}' \delta(f_a)\right) \mathcal{I}_n(\{p, E, \sigma\})$$

- Integrands $\mathcal{I}_n$:

Gravitons: $\mathcal{I}_n^{\text{Einstein}}(1, 2, \ldots, n) = \text{Pf}'\,\Psi_n^2$

Color ordered gluons: $\mathcal{I}_n^{\text{YM}}(1, 2, \ldots, n) = C(1, 2, \ldots, n)\,\text{Pf}'\,\Psi_n$

Graviton-gluon: $\mathcal{I}_{n+1}^{\text{EYM}}(1, 2, \ldots, n; p) = \mathcal{C}(1, 2, \ldots, n)\, C_{pp}\,\text{Pf}'\,\Psi_{n+1}$

$$\mathcal{I}_{n+r}^{\text{EYM}}(1, 2, \ldots, n; p_1, \ldots, p_r) = \mathcal{C}(1, 2, \ldots, n)\,\text{Pf}\,\Psi_r(\{p, E_p, \sigma\})\text{Pf}'\,\Psi_{n+r}$$

with $C_{pp} = -\sum_{b=1}^{n} \frac{E_p \cdot p_b}{\sigma_p - \sigma_b} = \text{Pf}\Psi_1(p, E, \sigma)$

- Unified description of gluon-graviton trees $\Rightarrow$ formalism we were seeking.

Short proof of EYM-YM amplitude relation

Derived from string amplitudes: $h\, g^n$-amplitude from pure YM [Stieberger,Taylor]

$$A_{\mathsf{EYM}}(1,2,\ldots,n;p^{\pm\pm}) = \frac{\kappa}{g}\sum_{l=1}^{n-1} E_p^{\pm}\cdot x_l\, A_{\mathsf{YM}}(1,2,\ldots,l,p^{\pm},l+1,\ldots,n)$$

with region momentum $x_l = \sum_{i=1}^{l} p_i$.

- Follows straightforwardly from CHY representation:

$$\mathcal{I}_n^{\mathsf{YM}}(1,2,\ldots,n) = \frac{1}{\sigma_{12}\ldots\sigma_{n1}}\,\mathrm{Pf}'\,\Psi_n$$

$$\mathcal{I}_{n+1}^{\mathsf{EYM}}(1,2,\ldots,n;p^{\pm\pm}) = C_{pp}\,\frac{1}{\sigma_{12}\ldots\sigma_{n1}}\,\mathrm{Pf}'\Psi_{n+1}$$

- Using the simple identity: [Nandan,JP,Schlotterer,Wen]

$$C_{pp} = \sum_{i=1}^{n}\frac{E_p\cdot p_i}{\sigma_{i,\rho}} = \sum_{i=1}^{n-1} E_p\cdot x_i\,\frac{\sigma_{i,i+1}}{\sigma_{i,\rho}\,\sigma_{\rho,i+1}}$$

Trivially shown using $\frac{\sigma_{i,i+1}}{\sigma_{i,p}\,\sigma_{p,i+1}} = \frac{1}{\sigma_{i,p}} - \frac{1}{\sigma_{i+1,p}}$ and telescoping sum.

Higher level EYM to YM relations [Nandan,JP,Schlotterer,Wen]

- May be generalized to more gravitons. One seeks identities of the type

$$\boxed{\mathrm{Pf}\,\Psi_r(\{p,E_p,\sigma\}) = \sum_{\{i,j,a\}} F_a(\{p,E_p\})\,\frac{\sigma_{i\,j}}{\sigma_{ia}\sigma_{aj}}}$$

- Two gravitons with momenta p & q:

$$\mathrm{Pf}\,\Psi_{r=2} = C_{pp}\,C_{qq} - \frac{s_{pq}\,(\epsilon_p\cdot\epsilon_q)}{\sigma_{p,q}^2} + \frac{(\epsilon_p\cdot q)\,(\epsilon_q\cdot p)}{\sigma_{p,q}^2}\,, \qquad s_{pq}\equiv p\cdot q\,.$$

- Toolbox of identities:

Schouten's $\quad \sigma_{i,i+1}\,\sigma_{p,q} = -\sigma_{i,p}\,\sigma_{q,i+1} + \sigma_{i,q}\,\sigma_{p,i+1}$

Kleiss-Kuijf $\quad \mathcal{C}(1,A,n,B) = (-)^{|B|}\sum_{\sigma\in A\sqcup\!\sqcup B^t}\mathcal{C}(1,\sigma,n)$

with shuffles $\quad A\sqcup\!\sqcup B \equiv\{\alpha_1(\alpha_2\ldots\alpha_{|A|}\sqcup\!\sqcup B)\} + \{\beta_1(\beta_2\ldots\beta_{|B|}\sqcup\!\sqcup A)\}$

Cross-ratio identity: $\quad \frac{s_{pq}}{\sigma_{p,q}^2} = \sum_{i\neq a,p,q} s_{pi}\,\frac{\sigma_{i,a}}{\sigma_{i,p}\,\sigma_{p,q}\,\sigma_{q,a}}$ [Cardona,Feng,Gomez,Huang]

New EYM to YM relations [Nandan,JP,Schlotterer,Wen]

- Two gravitons

$$\mathcal{A}_{\text{EYM}}(1,2,\ldots,n;p,q) =$$

$$\frac{\kappa^2}{g^2}\Big[\sum_{1=i\leq j}^{n-1}(\epsilon_p\cdot x_i)\,(\epsilon_q\cdot x_j)\,\mathcal{A}(1,\ldots,i,p,i{+}1,\ldots,j,q,j{+}1,\ldots,n)$$

$$-\,(\epsilon_q\cdot p)\sum_{j=1}^{n-1}(\epsilon_p\cdot x_j)\sum_{i=1}^{j+1}\mathcal{A}(1,2,\ldots,i{-}1,q,i,\ldots,j,p,j{+}1,\ldots,n)$$

$$-\,\frac{(\epsilon_p\cdot\epsilon_q)}{2}\sum_{l=1}^{n-1}(p\cdot k_l)\sum_{1=i\leq j}^{l}\mathcal{A}(1,2,\ldots,i{-}1,q,i,\ldots,j{-}1,p,j,\ldots,n)$$

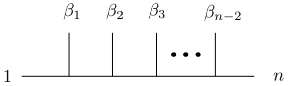
$$+\,(p\leftrightarrow q)\Big]$$

- Similar yet more complicated expressions have been worked out for 3 and 4 gravitons as well as double-trace EYM amplitudes with zero or one graviton.
- All multipicity problem solved recently [Fu,Du,Huang,Feng] [Chiodaroli,Gunaydin,Johansson,Roiban] [Teng,Feng]

Soft constraints on kinematical numerators [JP,Wormsbecher; ongoing]

- EYM as YM amplitude relations are special cases of doubly copy relation between graviton and gluon amplitudes: [Bern,Carrasco,Johansson]

$$M_n(1,\ldots,n) = \sum_{\beta\in S_{n-2}} n(1,\{\beta\},n)\, A_n(1,\{\beta\},n)$$



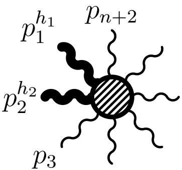
$n(1,\beta,n)$: Kinematical numerators in DDM form. [Del Ducca,Dixon,Maltoni]. Are polynomials in ϵ_i and p_i. Recent combinatorial construction established [Du,Teng]

- Question: What constraints do soft theorems impose on $n(1,\beta,n)$?
- Preliminary result: Emerging recursive structure for $n(1,\beta,n)$ e.g.

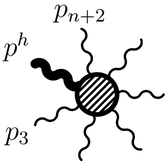
$$n(1,\{q,\beta\},n) = \Big(-\epsilon_q\cdot p_1 - \epsilon_q\cdot J_1\cdot q\Big)\, n(1,\{\beta\},n) + (q\cdot p_1)[\ldots] + \mathcal{O}(q^2)$$

- Combine with gauge invariance $\Longrightarrow$ Completely determined? [Arkani-Hamed,Rodina,Trnka]

Back to (subleading) Collinear Limits



$$\xrightarrow{1\|2} \quad \sum_{h=\pm} \mathrm{Split}_{-h}(c; 1^{h_1}, 2^{h_2}) \times$$



The collinear scattering equations

- We take $1 \parallel 2$ with $\begin{pmatrix} |1\rangle \\ |2\rangle \end{pmatrix} = \begin{pmatrix} c & -\epsilon\, s \\ s & +\epsilon\, c \end{pmatrix} \begin{pmatrix} |p\rangle \\ |r\rangle \end{pmatrix}$
- Change of variables:

$$\sigma_1 = \rho - \frac{\xi}{2} \qquad \sigma_2 = \rho + \frac{\xi}{2}$$

- In fact solutions with $\xi \to 0$ imply collinearity of $1 \parallel 2$: [Dolan,Goddard]
- But is the opposite also true?

- Numerical studies (for $N \leq 8$) of the $(N-3)!$ solutions reveal:
  - Always found $2 \cdot (N-4)!$ degenerate $(\xi \to 0)$ solutions
  - Remaining $(N-5)\,(N-4)!$ solutions are non-degenerate $(\xi = \text{finite})$
- Degenerate solutions are numerically seen to be dominant in the CHY integral at leading $\mathcal{O}(\frac{1}{\epsilon})$ and sub-leading $\mathcal{O}(1)$ order in the collinear limit.

$\Rightarrow$ May focus on degenerate solutions!

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Finding the degenerate solutions

$$\delta(f_1)\,\delta(f_2) = 2\,\delta(f_1+f_2)\,\delta(f_1-f_2) = 2\delta(f_1+f_2)\,\delta(f_-)$$

- Degenerate solution ansatz $\xi = \epsilon\xi_1 + \epsilon^2\xi_2 + \mathcal{O}(\epsilon^3)$:

$$f_- := (f_1-f_2) - (c^2-s^2)\,(f_1+f_2) = \epsilon\left[2c^2s^2\,\xi_1\mathcal{P}_2 - 2cs\,\mathcal{Q}_1 - \frac{2(p\cdot r)}{\xi_1}\right] + \mathcal{O}(\epsilon^2)$$

with shorthands $\mathcal{Q}_i = \sum_{b=3}^{n} \frac{p_b\cdot q}{(\rho-\sigma_b)^i} \qquad \mathcal{P}_i = \sum_{b=3}^{n} \frac{p_b\cdot p}{(\sigma_b-\rho)^i} \quad i\geq 2$

- Yields two solutions for $\xi_1 = \xi_\pm$.

$$\xi_{1,\pm} = \frac{\mathcal{Q}_1}{2cs\mathcal{P}_2} \pm \sqrt{\frac{\mathcal{Q}_1^2 + 4(p\cdot r)\mathcal{P}_2}{4(c^2s^2)\mathcal{P}_2^2}}$$

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Expanding out: 4 contributions to the near collinear expansion

$$A(1,2,3,\ldots,n+2) \overset{1\|2}{=} \sum_{\xi_\pm} \int \prod_{a=3}^{n+2} \left[d'\sigma_a \, \delta(f_a) \right] d\rho \, \delta\Big(\underbrace{f_p + \epsilon \, (c^2 - s^2) \frac{\xi_1}{2} \mathcal{P}_2}_{f_1+f_2} \Big)$$

$$\times \left[\underbrace{\frac{1}{\left| \frac{\partial f_-}{\partial \xi} \right|}}_{\text{Jacobian}} \underbrace{\frac{1}{\sigma_{1,2} \ldots \sigma_{n+2,1}}}_{\mathcal{C}_{n+2}} \underbrace{\text{Pf}' \, \Psi_{n+2}}_{\text{CHY matrix}} \right] \Bigg|_{\xi = \epsilon \xi_1 + \epsilon^2 \xi_2 + \ldots}$$

- Jacobian:

$$\mathcal{J} = \underbrace{\frac{1}{2} \frac{\xi_1^2}{(p\cdot r) + c^2 s^2 \mathcal{P}_2 \, \xi_1^2}}_{\mathcal{J}_0} + \epsilon \, \mathcal{J}_0^2 \left(4(p\cdot r) \frac{\xi_2}{\xi_1^3} - cs(c^2 - s^2) \mathcal{Q}_2 \right) + \mathcal{O}(\epsilon^2)$$

- Parke-Taylor factor:

$$\mathcal{C}_{n+2} = -\frac{1}{\epsilon} \frac{\mathcal{C}_{n+1}}{\xi_1} + \mathcal{C}_{n+1} \left(\frac{\xi_2}{\xi_1^2} + \frac{1}{2} S_{n+2,\rho,3} \right) + \mathcal{O}(\epsilon)$$

- CHY-matrix: Needs more involved computation, but $\text{Pf}' \, \Psi_{n+2} \sim \epsilon^0$

Same helicity $1^{\pm}2^{\pm}$: Leading order

- CHY matrix:

$$\mathrm{Pf}'(\Psi_{n+2}) = \left(C_{pp} - \frac{2}{s\,c\,\xi_1} \begin{Bmatrix} -[pr] \\ +\langle pr\rangle \end{Bmatrix} \right) \mathrm{Pf}'(\Psi_{n+1}) \qquad \text{for helicities} \quad \begin{Bmatrix} 1^+2^+ \\ 1^-2^- \end{Bmatrix}$$

- Putting everything together we recover the splitting functions:

$$\mathcal{A}_{n+2} \stackrel{1\|2}{=} \sum_{\xi_1} \int d\mu_n \, d\rho \, \delta(f_p) \, \frac{-\mathcal{J}_0}{\epsilon\,\xi_1} C_{n+1} \left(C_{pp} - \frac{2}{s\,c\,\xi_1} \begin{Bmatrix} -[pr] \\ +\langle pr\rangle \end{Bmatrix} \right) \mathrm{Pf}'(\Psi_{n+1})$$

$$= \mathrm{Split}^{\mathrm{tree}}_{\mp}(c; 1^{\pm}, 2^{\pm}) \, \mathcal{A}_{n+1}(p^{\pm}, 3, \ldots, n+2)$$

using the sum identities:

$$\sum_{\{\xi_1\}} \frac{\mathcal{J}_0}{\xi_1} = 0 \qquad \sum_{\{\xi_1\}} \frac{\mathcal{J}_0}{\xi_1^2} = \frac{1}{2\,p\cdot r}$$

- Leadign order opposite helicity case $1^{\pm}2^{\mp}$ works similarly.

Subleading order: Same helicity $1^{\pm}2^{\pm}$

Final result (dropping a total derivative):

$$\mathcal{A}(1,2,3,\ldots,n+2)\Big|_{1\|2}^{\text{subleading}} = \int d\mu_{n+1} \left(\frac{C_{pp}}{\mathcal{P}_2}\left(\frac{1}{c^2}\,\frac{1}{\sigma_{n+2,\rho}} + \frac{1}{s^2}\,\frac{1}{\sigma_{\rho,3}}\right) + \frac{c^2-s^2}{c^2\,s^2\,\mathcal{P}_2}\left(C_{pp}^{(2)} - \frac{C_{pp}\mathcal{P}_3}{\mathcal{P}_2}\right)\right) C_{n+1}\ \mathrm{Pf}'(\Psi_{n+1})$$

- Reproduces the Stieberger-Taylor relations.
- Curious identity: Consider the differential operator $\boxed{p\cdot\partial_{E_p}}$ (gauge transf. in effective collinear leg)

$$p\cdot\partial_{E_p}\,\mathcal{A}^{\text{YM}}(1,2,3,\ldots,n+2)\Big|_{1\|2}^{\text{subleading}} = \frac{s^2-c^2}{c^2 s^2}\,\mathcal{A}^{\text{YM}}(p,3,\ldots,n+2)$$

recall: $p_1 = c^2\,p - \epsilon\, cs\, q + \mathcal{O}(\epsilon^2)\,,\qquad p_2 = s^2\,p + \epsilon\, cs\, q + \mathcal{O}(\epsilon^2)$

Factorization at subleading collinear order?

- **Nicest result** in democratic collinear limit $c = s$

$$\mathcal{A}(1^{\pm}, 2^{\pm}, 3, \ldots, n+2)\Big|_{1\|2, c=s}^{\text{subleading}} = \int d\mu_{n+1} \frac{1}{\mathcal{P}_2} \frac{\sigma_{n+2,3}}{\sigma_{n+2,\rho}\, \sigma_{\rho,3}}\, C_{pp}\, C_{n+1}\, \text{Pf}'(\Psi_{n+1})$$

where $\mathcal{P}_2 = \sum_{b=3}^{n+2} \frac{p_b \cdot p}{(\sigma_b - \rho)^2} = \frac{\partial}{\partial \rho} f_p$, derivative of scattering equation.

- Still, have not (yet) been able to write this in factorized form!!

$$\mathcal{A}(1^{\pm}, 2^{\pm}, 3, \ldots, n+2)\Big|_{1\|2, c=s}^{\mathcal{O}(\epsilon^0)} \neq \text{Split}^{(1)}(p_a, E_a, \partial_{p_a}, \ldots)\, \mathcal{A}(p^{\pm\pm}, 3, \ldots, n+2)$$

- $\Rightarrow$ Absence of a subleading collinear theorem for gluons.

Collinear gravitons

With the collinear expansion of CHY building blocks in place, can deduce collinear limits of scalar and gravitons:

- Gravitons: $\mathcal{A}_n = \int d\mu_n \, \mathrm{Pf}'(\Psi_n) \, \mathrm{Pf}'(\Psi_n)$
- In the collinear expansion this yields the leading behavior

$$\mathcal{A}_n^{1^{++},2^{++}} \overset{1\|2}{=} 2 \sum_{\xi_1} \int d\mu_{n-1} \, \mathcal{J}_0 \left(C_{pp} + \frac{2\,[pr]}{c\,s\,\xi_1} \right)^2 \mathrm{Pf}'(\Psi_{n-1}) \, \mathrm{Pf}'(\Psi_{n-1})$$

with the result [Bern,Dixon,Perelstein,Rozowsky]

$$\boxed{\mathcal{A}_n \overset{1\|2}{=} \frac{[p\,r]}{c^2\,s^2\,\langle r\,p\rangle} \mathcal{A}_{n-1} + \frac{1}{c^2\,s^2} \int d\mu_{n-1} \frac{C_{pp}^2}{\mathcal{P}_2} \, \mathrm{Pf}'(\Psi_{n-1}) \, \mathrm{Pf}'(\Psi_{n-1})}$$

- This result is universal. Identical behaviour for scattering of m gravitons and k gluons, $\mathcal{A}_n = \int d\mu_n \, C_k \, \mathrm{Pf}(\Psi_m) \, \mathrm{Pf}'(\Psi_n)$. Collinear graviton limit:

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Collinear scalars

- Pure scalar amplitudes

$$\mathcal{A}_{n+2} = \int d\mu_{n+2}\, \mathcal{C}_{n+2}^2$$

- Working out the leading and subleading collinear limit $1 \parallel 2$ one finds

$$\mathcal{A}_{n+2}(1,2,3,\ldots,n+2) \stackrel{1\|2}{=} \frac{1}{\epsilon^2\, 2p\cdot r}\, \mathcal{A}_{n+1}(p,3,\ldots,n+2)$$
$$-\frac{1}{\epsilon}\int d\mu_{n+1}\Big(\underbrace{\#}_{\text{Jacobian}} - \underbrace{\#}_{\text{Parke-Taylor}} + \underbrace{0}_{\delta'(f_+)}\Big)\, \mathcal{C}_{n+1}^2 + \mathcal{O}(1)$$

Scalars have vanishing subleading collinear behaviour!

Summary:

Subleading soft limit

- Form of subleading soft operators strongly constrained by symmetries and commutator with δ-fct
- Open problem: Derive BMS symmetry algebra from scattering amplitudes $\Rightarrow$ Double soft limits

Soft constraints on color-kinematic numerators

Understanding the subleading collinear limit

- Intriguing relations:
  Linear combinations of subleading collinear gluon amplitudes = Einstein-Yang Mills amplitudes [Stieberger, Taylor]
- Reproduced tree-level splitting function from collinear limit of CHY.
- Gluons: We do not see factorization in the subleading collinear limit for pure glue. Stieberger-Taylor relations proven. Curious identity between gauge transformation of subleading collinear limit and gluon amplitude.

Thank you!