UCDAVIS
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THERMAL EQUIVARIANCE AND ITS APPLICATIONS

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- Basic philosophy: [1510.02494]*
- Dissipative hydrodynamic actions: [1511.07809]
- Origins in Schwinger-Keldysh: [1610.01940]
- **Thermal Equivariance: [1610.01941]**

- * Classification of solutions to hydro axioms: [1412.1090] [1502.00636]

Prelude: On that which came before....

CONTEXTUALIZATION

Schwinger 1961
Keldysh 1964
Feynman, Vernon 1963

Kubo 1957
Martin, Schwinger 1959

Review: Chou, Su, Hao, Yu 1985

Schwinger-Keldysh & KMS

Martin, Siggia, Rose 1973
Parisi, Sourlas 1982

Review: Zinn-Justin 2002

cf., Janssen 1976 &
de Dominicis, Peliti 1978

Topological symmetry & Dissipation

Witten 1988
Vafa, Witten 1994
Dijkgraaf, Moore 1996

Review: Cordes, Moore, Ramgoolam 1995
Guillemin, Sternberg 2013

cf., Blau, Thompson 1997
Zucchini 1998
Gozzi, Reuter 1990

Equivariance &
Topological Sigma Models

CONTEXTUALIZATION

Banerjee, Bhattacharya, Bhattacharyya, Jain, Minwalla, Sharma 2012
Jensen, Kaminski, Kovtun, Myer, Ritz, Yarom 2012

Sayantani Bhattacharyya 2012, 2014

Equilibrium, Entropy, and all that…

Jarzynski 1996, 1997
Crooks 1999

Mallick, Moshe, Orland 2010
Gaspard 2012

Nickel, Son 2010

Dubovsky, Hui, Nicolis 2011

Kovtun, Moore, Romatschke 2014

Haehl, Loganayagam, MR 2014-15

Non-equilibrium 2nd law

Effective actions for hydro

and that which is...

RELATED WORK

Crossley, Glorioso, Liu [1511.03646]
Glorioso, Liu [1612.07705]

Crossley, Glorioso, Liu [1701.07817]
Gao, Liu [1701.07445]

Jensen, Pinzani-Fokeeva, Yarom [1701.07436]

Commentary & comparison of 1st two papers (CGL/GL) in 1701.07896 (HLR)

Act I

in which we meet

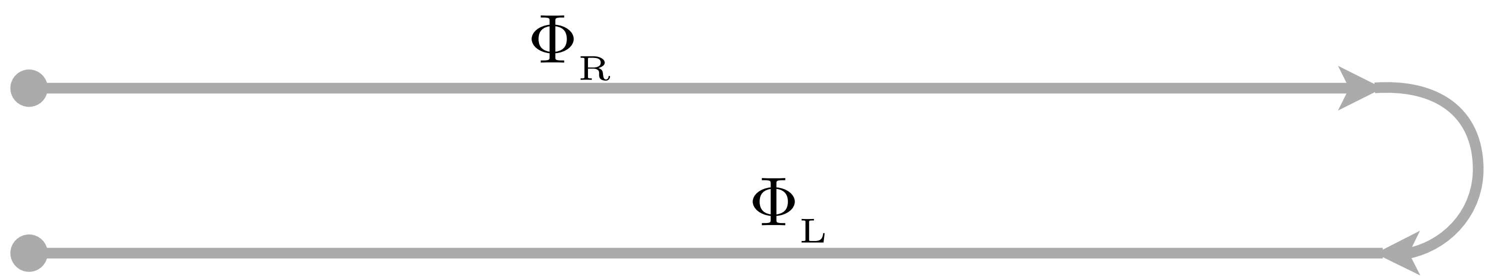
Schwinger-Keldysh and Kubo-Martin-Schwinger

and find a useful way to represent them...

SCHWINGER-KELDYSH FORMALISM

- The Schwinger-Keldysh formalism computes singly out-of-time ordered correlation functions in a generic (mixed) state.

density matrix



turning point

identity operator

$$S_{SK} = S[\Phi_{\rm R}] - S[\Phi_{\rm L}]$$

$$\delta S_{SK} = \int d^dx \sqrt{-g}\, (\mathcal{J}_{\rm R}\, \mathbb{O}_{\rm R} - \mathcal{J}_{\rm L}\, \mathbb{O}_{\rm L})$$

Generating functional

$$\mathcal{Z}_{SK}[J_{\rm R}, J_{\rm L}] \equiv {\rm Tr}\Big\{U[J_{\rm R}]\ \hat{\rho}_{\rm initial}\ (U[J_{\rm L}])^\dagger\Big\}$$

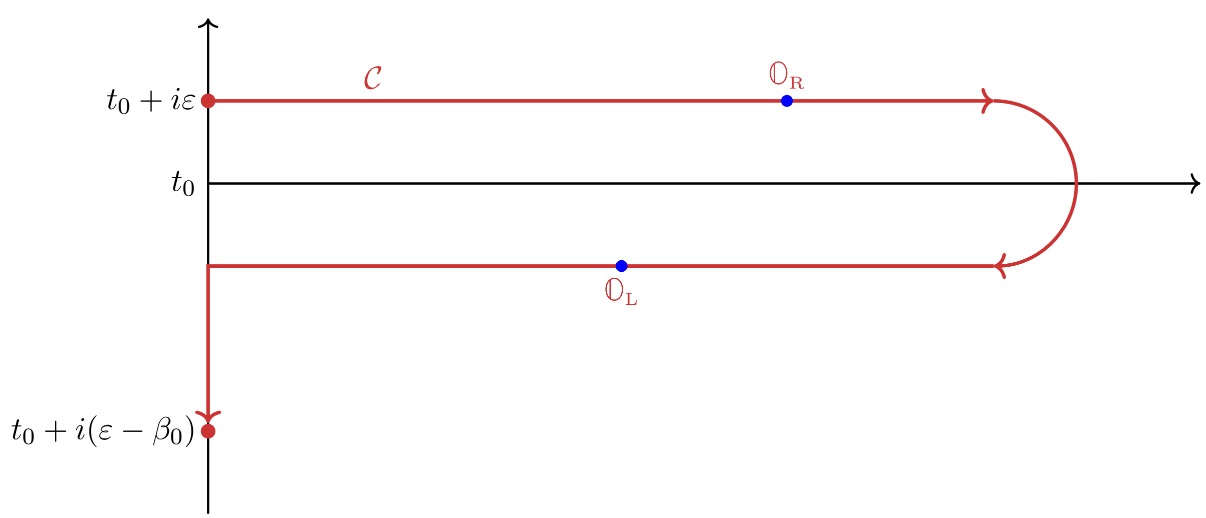
Time ordered correlations

$${\rm Tr}\Big(\ \hat{\rho}_{\rm initial}\ \bar{\mathcal{T}}\left(U^\dagger \mathbb{O}_{\rm L} U^\dagger \mathbb{O}_{\rm L} \ldots\right)\ \mathcal{T}\,(U\mathbb{O}_{\rm R} U\mathbb{O}_{\rm R} \ldots)\ \Big)$$

THERMAL DENSITY MATRICES & KMS CONDITION

- Thermal density matrices $\hat{\rho}_T = e^{-\beta\left(\widehat{\mathbb{H}}-\mu_I\,\widehat{\mathbb{Q}}^I\right)}$ define stationary equilibrium configurations.

- Correlation functions have analyticity properties which allows for a Euclidean (Matsubara) formulation, cf., $\mathcal{Z}_T(\beta,\mu_I) = \mathrm{Tr}\left(\hat{\rho}_T\right)$



$$\mathcal{Z}_T[\mathcal{J}_\mathrm{R},\mathcal{J}_\mathrm{L}] = \mathrm{Tr}\left(U[\mathcal{J}_\mathrm{R}]\,\hat{\rho}_T\,(U[\mathcal{J}_\mathrm{L}])^\dagger\right)$$

- KMS condition asserts that the correlation functions are analytic in the time strip $0 < \Im(t) < \beta$.

- This can be rephrased as a thermal Ward identity for correlation functions which involve operators shifted by a imaginary thermal period.

TWO SUM RULES

- *Unitarity* of Schwinger-Keldysh path integral implies vanishing difference operator correlators:

$$\langle \mathcal{T}_{SK} \prod_k \left(\mathbb{O}_{\mathrm{R}}^{(k)} - \mathbb{O}_{\mathrm{L}}^{(k)} \right) \rangle \equiv \langle \mathcal{T}_{SK} \prod_k \ \mathbb{O}_{dif}^{(k)} \rangle = 0$$

Weldon '05

- The KMS condition translates into a second sum rule for thermal differences:

$$\langle \mathcal{T}_{SK} \prod_{k=1}^{n} \left(\mathbb{O}_{\mathrm{R}}^{(k)} - \tilde{\mathbb{O}}_{\mathrm{L}}^{(k)} \right) \rangle = \langle \mathcal{T}_{SK} \prod_{i=1}^{n} \mathbb{O}_{ret} \rangle = 0$$

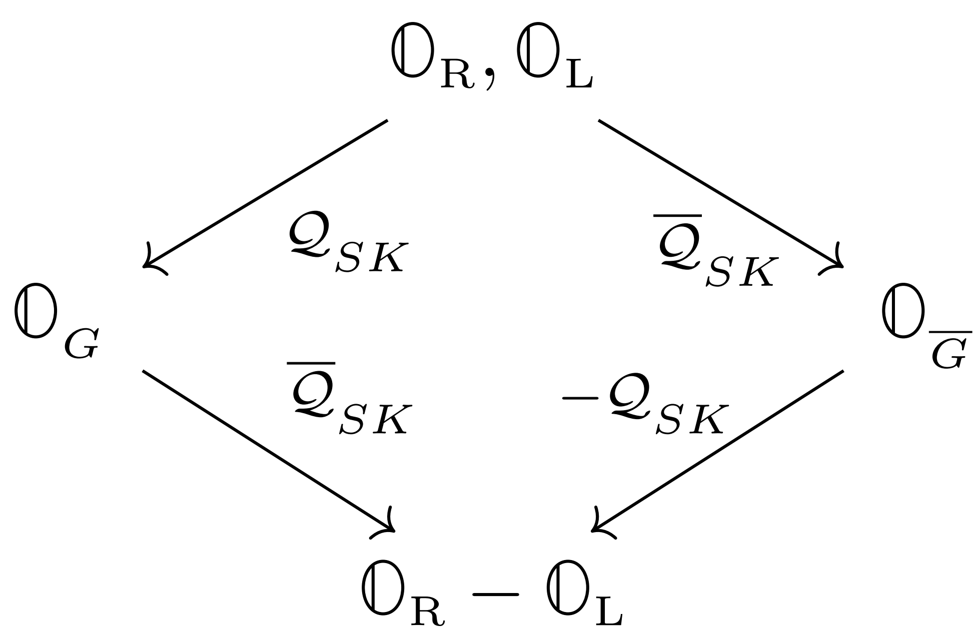
- Keldysh (light-cone) basis $\qquad \mathbb{O}_{dif} \equiv \mathbb{O}_{\mathrm{R}} - \mathbb{O}_{\mathrm{L}} \ , \qquad \mathbb{O}_{av} \equiv \frac{1}{2} \left(\mathbb{O}_{\mathrm{R}} + \mathbb{O}_{\mathrm{L}} \right)$

- Adv-Ret basis $\qquad \mathbb{O}_{adv} \equiv \mathbb{O}_{\mathrm{R}} - \mathbb{O}_{\mathrm{L}} \ , \qquad i\Delta_{\boldsymbol{\beta}} \mathbb{O}_{ret} = \mathbb{O}_{\mathrm{R}} - e^{-i\delta_{\boldsymbol{\beta}}} \, \mathbb{O}_{\mathrm{L}}$

- Furthermore, *a largest time* and *thermal smallest time* equations hold.

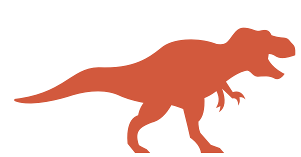
THE SCHWINGER-KELDYSH SUPER-QUARTET

- Difference operator correlation functions vanish because they are trivial elements of a BRST cohomology.
- There exists a pair of Grassmann odd charges which act on the doubled operator algebra.
- The SK theory is covariantly expressed in terms of a quartet of fields, which usual doubled formalism being a gauge fixed version (ghosts =0).



$$\mathcal{Q}_{SK}^2 = \overline{\mathcal{Q}}_{SK}^2 = \left[\mathcal{Q}_{SK}, \overline{\mathcal{Q}}_{SK}\right]_\pm = 0$$

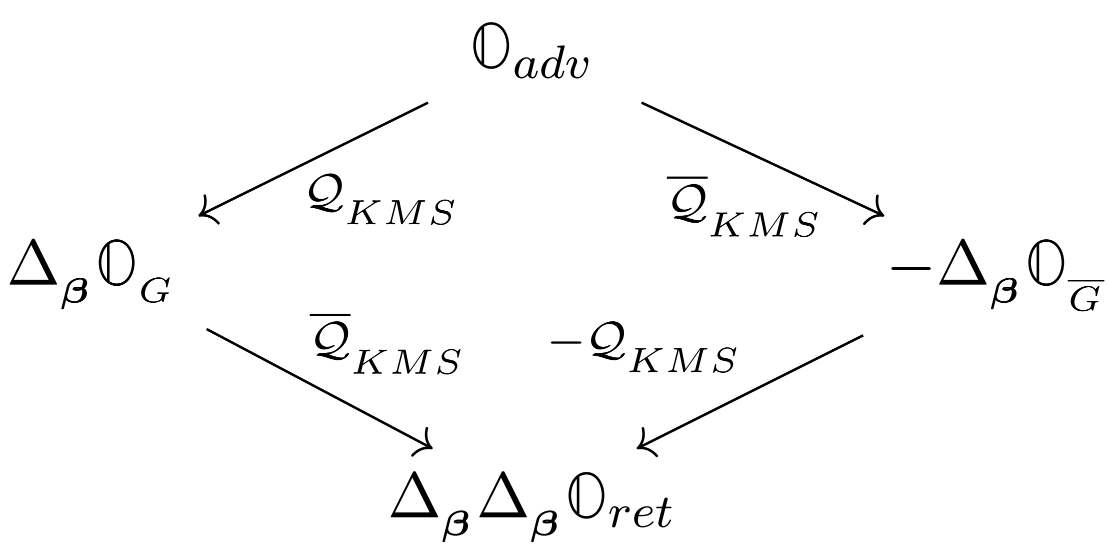
$$\left[\mathcal{Q}_{SK}, \mathbb{O}_{dif}\right]_\pm = 0 \;, \qquad \left[\overline{\mathcal{Q}}_{SK}, \mathbb{O}_{dif}\right]_\pm = 0$$



CGL argue that this should only be interpreted as a single supercharge δ, but the pair above are CPT conjugates (cf., anti-BRST).

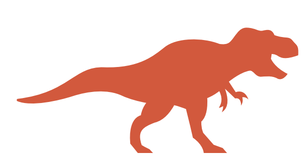
THE KMS SUPERCHARGES: I

- The second sum rule suggests an analogous structure should pertain in the thermal sector, with new supercharges aligned to the thermal translations.



$$\mathcal{Q}_{KMS}^2 = \overline{\mathcal{Q}}_{KMS}^2 = \left[\mathcal{Q}_{KMS}, \overline{\mathcal{Q}}_{KMS}\right]_\pm = 0$$

$$\left[\mathcal{Q}_{KMS}, \mathbb{O}_{ret}\right]_\pm = \left[\overline{\mathcal{Q}}_{KMS}, \mathbb{O}_{ret}\right]_\pm = 0$$



CGL posit that the KMS condition should be viewed as an involution leading to a second supercharge $\overline{\delta}$.

THE SK-KMS ALGEBRA

- The SK and KMS operations (Grassmann odd) form a closed super algebra with further two Grassmann even operations
- One the even operations is a thermal translation: Lie drag along the Euclidean thermal circle.

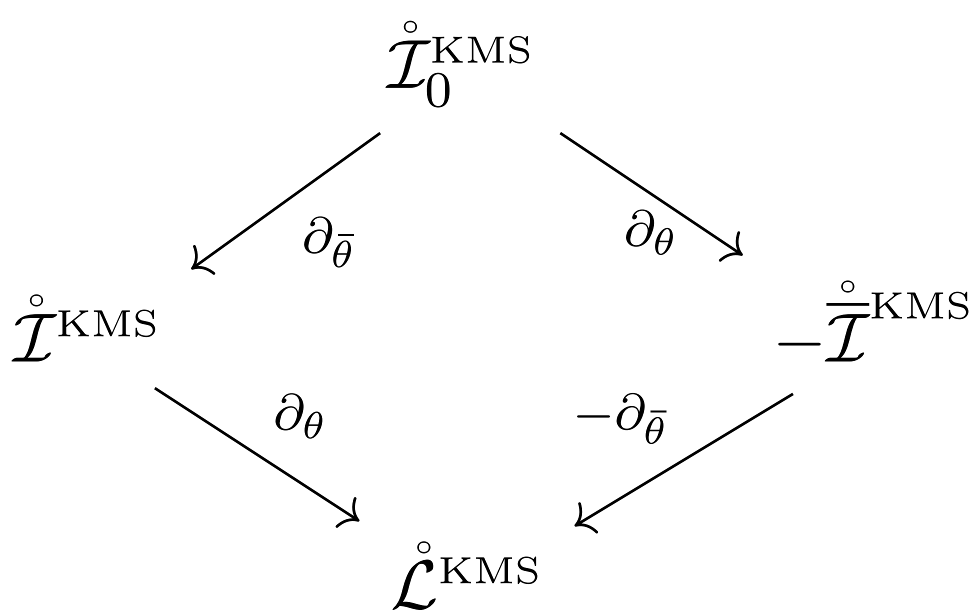
$$
\begin{gathered}
\mathcal{Q}_{SK}^2 = \overline{\mathcal{Q}}_{SK}^2 = \mathcal{Q}_{KMS}^2 = \overline{\mathcal{Q}}_{KMS}^2 = 0 \ , \\
[\mathcal{Q}_{SK}, \mathcal{Q}_{KMS}]_{\pm} = \left[\overline{\mathcal{Q}}_{SK}, \overline{\mathcal{Q}}_{KMS}\right]_{\pm} = \left[\overline{\mathcal{Q}}_{SK}, \mathcal{Q}_{SK}\right]_{\pm} = \left[\mathcal{Q}_{KMS}, \overline{\mathcal{Q}}_{KMS}\right]_{\pm} = 0 \ , \\
\left[\mathcal{Q}_{SK}, \overline{\mathcal{Q}}_{KMS}\right]_{\pm} = \left[\overline{\mathcal{Q}}_{SK}, \mathcal{Q}_{KMS}\right]_{\pm} = \mathcal{L}_{KMS} \ , \\
\left[\mathcal{Q}_{KMS}, \mathcal{Q}^0_{KMS}\right]_{\pm} = \left[\overline{\mathcal{Q}}_{KMS}, \mathcal{Q}^0_{KMS}\right]_{\pm} = 0 \, , \\
\left[\mathcal{Q}_{SK}, \mathcal{Q}^0_{KMS}\right]_{\pm} = \mathcal{Q}_{KMS} \ , \qquad \left[\overline{\mathcal{Q}}_{SK}, \mathcal{Q}^0_{KMS}\right]_{\pm} = -\overline{\mathcal{Q}}_{KMS} \, .
\end{gathered}
$$

SUPERSPACE CHARGES

- The structure is easily understood by passing onto a superspace construction, where the SK charges act as superderivations.

$$\mathring{\mathbb{O}} \equiv \mathbb{O}_{ret} + \theta\,\mathbb{O}_{\overline{G}} + \bar{\theta}\,\mathbb{O}_{G} + \bar{\theta}\theta\,\mathbb{O}_{adv}$$

$$\mathcal{Q}_{SK} \longrightarrow \partial_{\bar{\theta}}\,, \qquad \overline{\mathcal{Q}}_{SK} \longrightarrow \partial_{\theta}$$



$$\begin{aligned}
\mathring{\mathcal{I}}_0^{\rm KMS} &= \mathcal{Q}^0_{KMS} + \bar{\theta}\,\mathcal{Q}_{KMS} - \theta\,\overline{\mathcal{Q}}_{KMS} + \bar{\theta}\theta\,\mathcal{L}_{KMS}\,,\\
\mathring{\mathcal{I}}^{\rm KMS} &= \mathcal{Q}_{KMS} + \theta\,\mathcal{L}_{KMS}\,,\\
\mathring{\overline{\mathcal{I}}}^{\rm KMS} &= \overline{\mathcal{Q}}_{KMS} + \bar{\theta}\,\mathcal{L}_{KMS}\,,\\
\mathring{\mathcal{L}}^{\rm KMS} &= \mathcal{L}_{KMS}\,.
\end{aligned}$$

Act II

in which we meet Weil and Cartan and learn of equivariance….

EQUIVARIANCE

- Equivariance = cohomology with gauge symmetry
- To understand cohomology on an orbifold $\mathcal{M}/\mathcal{G}$ we use the Borel construction to work with the contractible universal $\mathcal{G}$ bundle $\mathcal{E}_\mathcal{G}$.
- Classifying space $\mathcal{B}_\mathcal{G} = \mathcal{E}_\mathcal{G}/\mathcal{G}$ is smooth as group action is free on the universal bundle.
- eg., $\mathbf{S}^1 = \mathbb{R}/\mathbb{Z}$
- cohomology of $\mathcal{M}/\mathcal{G}$ = cohomology of $(\mathcal{E}_\mathcal{G} \times \mathcal{M})/\mathcal{G}$
- Physically think of $\mathcal{E}_\mathcal{G}$ as the space of the universal $\mathcal{G}$ gauge connections and $\mathcal{B}_\mathcal{G}$ as the space of gauge orbits. This picture is efficient to write superspace Lagrangians.
- Cohomology of interest will be the space of invariant horizontal forms.

Matthai, Quillen '86
Kalkman '93

THE WEIL MODEL

- Gauge structure can be captured by a Grassmann odd gauge potential (fermions = differential forms) and its field strength

$$\mathbb{d}_{\mathrm{W}}^{\mathrm{E}} G^i + \frac{1}{2} f^i_{jk} G^j G^k = \phi^i \,,$$
$$\mathbb{d}_{\mathrm{W}}^{\mathrm{E}} \phi^i + f^i_{jk} G^j \phi^k = 0 \,.$$

Cartan equations for gauge structure

$$\delta^i_j + \overline{\mathcal{I}}_j^{\,\mathrm{E}} G^i = 0 \,, \qquad \overline{\mathcal{I}}_j^{\,\mathrm{E}} \phi^i = 0 \,.$$

interior contractions pick out gauge directions

$$\mathcal{L}_j^{\mathrm{E}} \equiv \left[\mathbb{d}_{\mathrm{W}}^{\mathrm{E}}, \overline{\mathcal{I}}_j^{\,\mathrm{E}}\right]_{\pm}$$

Lie derivations follow from above

Weil superalgebra

$$\left[\overline{\mathcal{I}}_i^{\,\mathrm{E}}, \overline{\mathcal{I}}_j^{\,\mathrm{E}}\right]_{\pm} = 0 \,, \qquad \left[\mathbb{d}_{\mathrm{W}}^{\mathrm{E}}, \overline{\mathcal{I}}_j^{\,\mathrm{E}}\right]_{\pm} = \mathcal{L}_j^{\mathrm{E}} \,,$$
$$\left[\mathcal{L}_i^{\mathrm{E}}, \overline{\mathcal{I}}_j^{\,\mathrm{E}}\right]_{\pm} = -f^k_{ij} \, \overline{\mathcal{I}}_k^{\,\mathrm{E}} \,, \qquad \left[\mathbb{d}_{\mathrm{W}}^{\mathrm{E}}, \mathcal{L}_j^{\mathrm{E}}\right]_{\pm} = 0 \,,$$
$$\left[\mathcal{L}_i^{\mathrm{E}}, \mathcal{L}_j^{\mathrm{E}}\right]_{\pm} = -f^k_{ij} \, \mathcal{L}_k^{\mathrm{E}} \,, \qquad \left[\mathbb{d}_{\mathrm{W}}^{\mathrm{E}}, \mathbb{d}_{\mathrm{W}}^{\mathrm{E}}\right]_{\pm} = 0 \,.$$

- *invariant horizontal forms are polynomial functions of field strengths.*

CARTAN MODEL

- One can similarly account for the group action on the manifold by working with coordinate vectors and Grassmann fields (for 1-forms).
- Since cohomology is in gauge invariant data, helpful to pass to gauge covariant language (Kalkman automorphism)

$$\mathbb{d}_{\mathrm{C}} X^\mu \equiv \mathbb{d}_{\mathrm{W}} X^\mu + G^k \xi_k^\mu = \psi_{\mathrm{C}}^\mu ,$$
$$\mathbb{d}_{\mathrm{C}} \psi_{\mathrm{C}}^\mu \equiv \mathbb{d}_{\mathrm{W}} \psi_{\mathrm{C}}^\mu + G^k (\partial_\nu \xi_k^\mu) \psi_{\mathrm{C}}^\nu = \phi^k \xi_k^\mu$$
$$\mathbb{d}_{\mathrm{C}} \phi^k \equiv \mathbb{d}_{\mathrm{W}} \phi^k + f_{ij}^k G^i \phi^j = 0 \ .$$

action of Cartan charges on target space and field strengths

The two charges act isomorphically on horizontal, invariant forms.

$$\mathbb{d}_{\mathrm{C}} = \mathbb{d}_{\mathrm{W}} + G^i \mathcal{L}_i + \left(\frac{1}{2} f_{ij}^k G^i G^j + \phi^k \right) \overline{\mathcal{I}}_k$$

The Cartan charge however squares to a gauge transformation

$$\mathbb{d}_{\mathrm{C}}^2 = \phi^k \mathcal{L}_k - [G, \phi]^k \overline{\mathcal{I}}_k$$

EXTENDED EQUIVARIANCE I

- One can extend the algebraic constructions to situations with more than one differential. We will focus on the case with 2 generators of the cohomology and swiftly pass to superspace: $\mathbb{d}_{\mathrm{W}} = \partial_{\bar\theta}(\ldots)| \,, \qquad \bar{\mathbb{d}}_{\mathrm{W}} = \partial_{\theta}(\ldots)| \,.$

- The Weil model closes on 6 generators: 2 derivations, 3 interior contraction, and one Lie derivation

Vafa, Witten '94
Dijkgraaf, Moore '96

$$\begin{gathered}
\mathbb{d}_{\mathrm{W}}^2 = \bar{\mathbb{d}}_{\mathrm{W}}^2 = \left[\mathbb{d}_{\mathrm{W}}, \bar{\mathbb{d}}_{\mathrm{W}}\right]_{\pm} = 0 \\
\left[\mathbb{d}_{\mathrm{W}}, \overline{\mathcal{I}}_j\right]_{\pm} = \left[\bar{\mathbb{d}}_{\mathrm{W}}, \mathcal{I}_j\right]_{\pm} = \mathcal{L}_j \,, \qquad \left[\mathbb{d}_{\mathrm{W}}, \mathcal{I}_j\right]_{\pm} = \left[\bar{\mathbb{d}}_{\mathrm{W}}, \overline{\mathcal{I}}_j\right]_{\pm} = 0 \\
\left[\mathbb{d}_{\mathrm{W}}, \mathcal{I}^0_j\right]_{\pm} = \mathcal{I}_j \,, \qquad \left[\bar{\mathbb{d}}_{\mathrm{W}}, \mathcal{I}^0_j\right]_{\pm} = -\overline{\mathcal{I}}_j \\
\left[\mathbb{d}_{\mathrm{W}}, \mathcal{L}_j\right]_{\pm} = \left[\bar{\mathbb{d}}_{\mathrm{W}}, \mathcal{L}_j\right]_{\pm} = 0 \\
\left[\overline{\mathcal{I}}_i, \mathcal{I}_j\right]_{\pm} = f_{ij}^k\, \mathcal{I}^0_k \\
\left[\mathcal{L}_i, \overline{\mathcal{I}}_j\right]_{\pm} = -f_{ij}^k\, \overline{\mathcal{I}}_k \,, \qquad \left[\mathcal{L}_i, \mathcal{I}_j\right]_{\pm} = -f_{ij}^k\, \mathcal{I}_k \,, \qquad \left[\mathcal{L}_i, \mathcal{I}^0_j\right]_{\pm} = -f_{ij}^k\, \mathcal{I}^0_k \,.
\end{gathered}$$

- *this should be reminiscent of structures in Act 1….*

EXTENDED EQUIVARIANCE II

- Package the universal data into a set of gauge superfield 1-form which we assume lives on a worldvolume with coordinates σ^a.

$$\mathring{\mathcal{A}} = \mathring{\mathcal{A}}_I \, dz^I = \mathring{\mathcal{A}}_a \, d\sigma^a + \mathring{\mathcal{A}}_\theta \, d\theta + \mathring{\mathcal{A}}_{\bar\theta} \, d\bar\theta \, .$$

$$\mathring{\mathcal{D}}_I = \partial_I + [\mathring{\mathcal{A}}_I, \ \cdot \ ] \, , \qquad \mathring{\mathfrak{F}}_{IJ} \equiv (1 - \frac{1}{2} \delta_{IJ}) \Big(\partial_I \mathring{\mathcal{A}}_J - (-)^{IJ} \partial_J \mathring{\mathcal{A}}_I + [\mathring{\mathcal{A}}_I, \mathring{\mathcal{A}}_J] \Big)$$

ghost charge	Faddeev-Popov ghost triplet	Vafa-Witten ghost of ghost quintet	Vector quartet
2		ϕ	
1	G	η	λ_a
0	B	ϕ^0	$\mathcal{A}_a$ $\mathcal{F}_a$
-1	$\bar G$	$\bar\eta$	$\bar\lambda_a$
-2		$\bar\phi$	

- Cartan charges are gauge-covariant super derivations and obey:

$$\mathring{\mathcal{D}}^2_{\bar\theta} = \mathring{\mathcal{L}}_{\mathring{\mathfrak{F}}_{\bar\theta\bar\theta}} \, , \qquad \mathring{\mathcal{D}}^2_{\theta} = \mathring{\mathcal{L}}_{\mathring{\mathfrak{F}}_{\theta\theta}} \, , \qquad \Big[\mathring{\mathcal{D}}_{\bar\theta}, \mathring{\mathcal{D}}_\theta \Big]_\pm = \mathring{\mathcal{L}}_{\mathring{\mathfrak{F}}_{\theta\bar\theta}} \, .$$

Act III

where we attempt a synthesis of SK-KMS with a touch of equivariance…

SK-KMS THERMAL EQUIVARIANCE

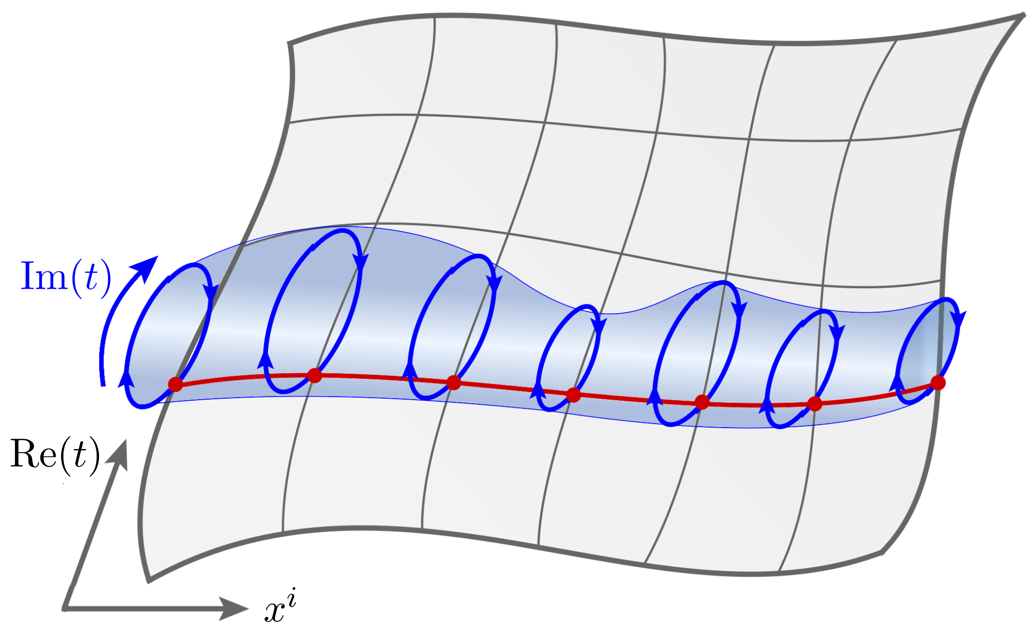
- SK charges are akin to Weil differentials, while the KMS charges fill out the interior contractions.
- The Lie derivation takes operators around the thermal circle.

$$
\begin{array}{rcl}
\underline{\mathcal{N}_{\mathsf{T}} = 2 \text{ algebra}} & | & \underline{\text{SK-KMS symmetries}} \\
\{\mathbb{d}_{\mathrm{W}}, \overline{\mathbb{d}}_{\mathrm{W}}\} & \leftrightarrow & \{\mathcal{Q}_{_{SK}}, \overline{\mathcal{Q}}_{_{SK}}\}\,, \\
\{\mathcal{I}_k, \overline{\mathcal{I}}_k\} & \leftrightarrow & \{\mathcal{Q}_{_{KMS}}, \overline{\mathcal{Q}}_{_{KMS}}\}\,, \\
\{\mathcal{L}_k, \mathcal{I}_k^0\} & \leftrightarrow & \{\mathcal{L}_{_{KMS}}, \mathcal{Q}^0_{_{KMS}}\}\,.
\end{array}
$$

- The algebraic structure for arbitrary temperature is complicated by non-locality of thermal translations.
- Some form of deformation of the group of circle diffeomorphisms…

SK-KMS THERMAL EQUIVARIANCE

- ✦ Life is simpler at high temperatures when thermal circle is small.



- ✦ Literally implement thermal translations as diffeomorphisms along the thermal circle and demand equivariance with this symmetry. $\mathring{\mathcal{L}}^{\rm KMS}\,\mathring{\mathbb{O}} = \Delta_{\boldsymbol{\beta}}\,\mathring{\mathbb{O}}$

$$\underline{\mathcal{N}_{\sf T} = 2 \text{ algebra}} \qquad \Big| \qquad \underline{\text{SK-KMS symmetries}}$$

$$\left[\mathring{\mathfrak{F}}_1, \mathring{\mathfrak{F}}_2\right]^k = f^k_{ij}\,\mathring{\mathfrak{F}}_1\mathring{\mathfrak{F}}_2 \quad \leftrightarrow \quad (\mathring{\mathfrak{F}}_1, \mathring{\mathfrak{F}}_2)_{\boldsymbol{\beta}} = \mathring{\mathfrak{F}}_1\,\Delta_{\boldsymbol{\beta}}\,\mathring{\mathfrak{F}}_2 - \mathring{\mathfrak{F}}_2\,\Delta_{\boldsymbol{\beta}}\,\mathring{\mathfrak{F}}_1$$

- ✦ This leads to the $U(1)_{\sf T}$ KMS symmetry discovered in during our attempt to classify hydro transport.

THERMAL CARTAN AND WEIL MODELS

- ✦ The gauge covariant Cartan charges (supercovariant derivations) can be mapped to the basic building blocks as follows:

$$\begin{aligned}\mathcal{Q} &\equiv \mathcal{Q}_{SK} + \phi_{\mathsf{T}}\, \overline{\mathcal{Q}}_{KMS} + \phi^0_{\mathsf{T}}\, \overline{\mathcal{Q}}_{KMS} + \eta_{\mathsf{T}}\, \mathcal{Q}^0_{KMS}\,, \\ \overline{\mathcal{Q}} &\equiv \overline{\mathcal{Q}}_{SK} + \overline{\phi}_{\mathsf{T}}\, \mathcal{Q}_{KMS}\,.\end{aligned}$$

- ✦ The superalgebra structure can then be captured by the anti-commutation relation among the Cartan charges as

$$\mathcal{Q}^2 = (\mathring{\mathfrak{F}}_{\bar\theta\bar\theta}|_{\bar\theta=\theta=0})\, \mathcal{L}_{KMS}\,, \qquad \overline{\mathcal{Q}}^2 = (\mathring{\mathfrak{F}}_{\theta\theta}|_{\bar\theta=\theta=0})\, \mathcal{L}_{KMS}\,, \qquad \left[\mathcal{Q}, \overline{\mathcal{Q}}\right]_\pm = (\mathring{\mathfrak{F}}_{\theta\bar\theta}|_{\bar\theta=\theta=0})\, \mathcal{L}_{KMS}$$

- ➤ Assume: dynamically consistent in dissipative systems to set all but the zero ghost number element of the Vafa-Witten quintet to zero: $\langle \mathring{\mathcal{F}}_{\theta\bar\theta}| \rangle = -i$

$$\mathcal{Q}^2 = 0\,, \qquad \overline{\mathcal{Q}}^2 = 0\,, \qquad \left[\mathcal{Q}, \overline{\mathcal{Q}}\right]_\pm = -i\, \mathcal{L}_{KMS} \mapsto i\pounds_{\boldsymbol{\beta}}$$

The final algebra is also the one CGL/GL work with in the high temperature limit and appears to be well known in the stat mech literature (Mallick, Moshe, Orland 2010).

$$\delta^2 = \bar\delta^2 = 0, \qquad \{\delta, \bar\delta\} = 2\, \tanh\left(\frac{i}{2}\,\beta\partial_t\right) \approx i\,\beta\partial_t$$

Act IV

in which the Brownian particle is thermally equivariantized…

TOY MODEL: LANGEVIN DYNAMICS

- Point particle in external potential subject to external forcing and noise

$$m\frac{d^2x}{dt^2} + \frac{\partial U}{\partial x} + \nu\, \Delta_{\boldsymbol{\beta}} x = \mathbb{N}$$

- One can write down a SK effective action for this dissipative dynamics

$$x = -i\,\Delta_{\boldsymbol{\beta}}^{-1}\left(x_{\rm R} - e^{-i\,\delta_{\boldsymbol{\beta}}}\, x_{\rm L}\right)\,, \qquad \tilde{x} = x_{\rm R} - x_{\rm L}$$

Martin, Siggia, Rose 1973

$$\mathcal{L}_{SK} = \left[\tilde{x}\,\frac{\partial U}{\partial x} + \bar{\psi}\frac{\partial^2 U}{\partial x^2}\psi\right] - m\left[\tilde{x}\,\frac{d^2x}{dt^2} + \bar{\psi}\,\frac{d^2\psi}{dt^2}\right] - \nu\left[\tilde{x}\,\Delta_{\boldsymbol{\beta}} x - \bar{\psi}\,\Delta_{\boldsymbol{\beta}}\psi\right] + i\,\nu\,\tilde{x}^2\,.$$

- The dissipative part of the action is controlled by ghosts and is related to the fluctuation terms difference fields - fluctuation/dissipation relation.
- Convergence of the path integral fixes the sign of dissipative terms.
- Simplest realization of the extended equivariant cohomology algebra.

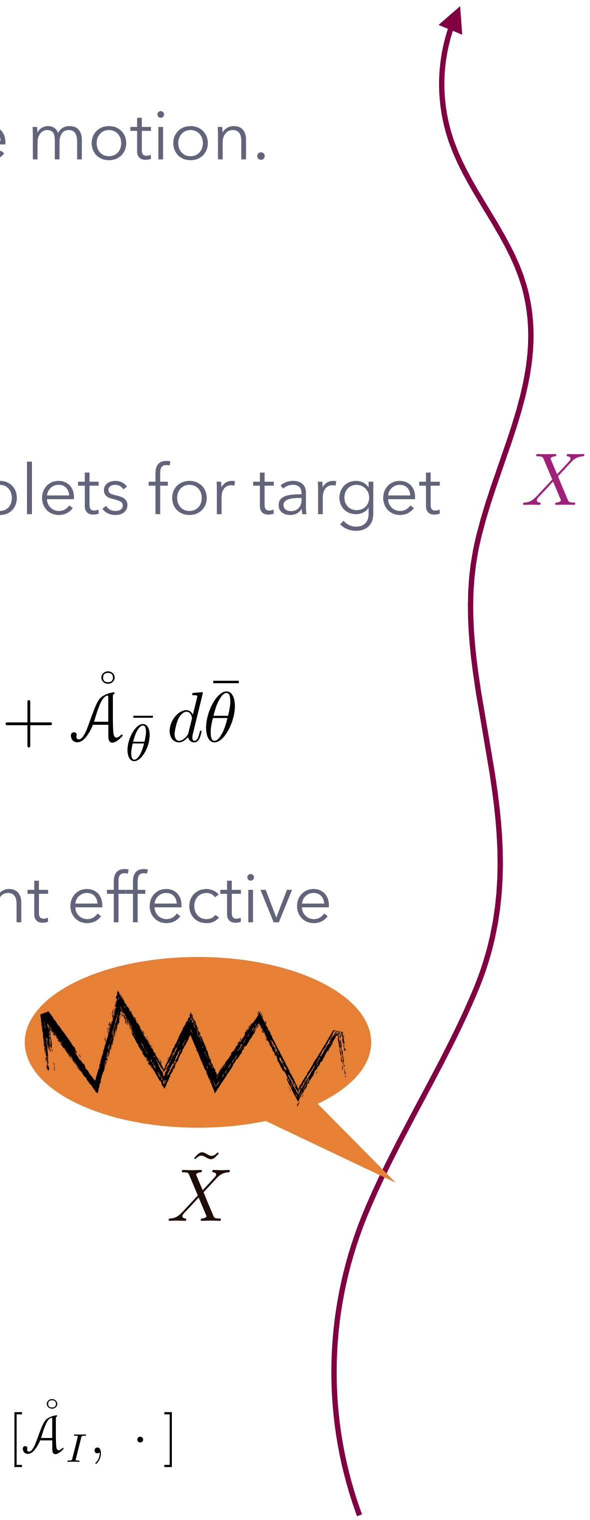
BROWNIAN BRANES

- Brownian particle immersed in a fluid undergoes dissipative motion.
- Langevin effective action: worldvolume B0-brane theory.
- Data for the worldvolume theory: thermal equivariant multiplets for target space coordinate map and thermal gauge field data.

$$\mathring{X} = \{X, X_\psi, X_{\bar{\psi}}, \tilde{X}\} \qquad\qquad \mathring{\mathcal{A}} \equiv \mathring{\mathcal{A}}_t\, dt + \mathring{\mathcal{A}}_\theta\, d\theta + \mathring{\mathcal{A}}_{\bar{\theta}}\, d\bar{\theta}$$

- MSR action follows as the basic thermal *U(1)$_T$* gauge invariant effective action of the worldline theory

$$S_{\text{B0}} = \int dt\, d\theta\, d\bar{\theta} \left\{ \frac{m}{2} \left(\mathring{\mathcal{D}}_t \mathring{X}\right)^2 - U(\mathring{X}) - i\,\nu\, \mathring{\mathcal{D}}_\theta \mathring{X} \mathring{\mathcal{D}}_{\bar{\theta}} \mathring{X} \right\}$$



$$(\mathring{\Lambda}, \mathring{X})_{\boldsymbol{\beta}} = \mathring{\Lambda}\, \pounds_{\boldsymbol{\beta}} \mathring{X} = \mathring{\Lambda}\, \Delta_{\boldsymbol{\beta}} \mathring{X} = \mathring{\Lambda}\, \boldsymbol{\beta} \frac{d}{dt} \mathring{X} \qquad\qquad \mathring{\mathcal{D}}_I = \partial_I + [\mathring{\mathcal{A}}_I,\ \cdot\ ]$$

FLUCTUATION DISSIPATION AS CPT BREAKING

- Stochasticity and dissipation arises because of spontaneous CPT symmetry breaking.

- BRST supersymmetry + spontaneous CPT breaking leads to Jarzynski relation which is a generalized fluctuation dissipation relation

Jarzynski 1997
Crooks 1999

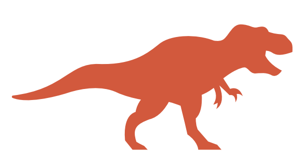
$$S_{\mathsf{B0}} \mapsto S_{\mathsf{B0}} - i \, \langle \mathring{\mathcal{F}}_{\theta\bar{\theta}}| \rangle \, \boldsymbol{\beta} \, (\Delta G + W) \quad \Longrightarrow \quad \langle e^{-\beta W} \rangle = e^{-\beta \, \Delta G}$$

Mallick, Moshe, Orland 2010

- The CPT symmetry in our construction is implemented as R-parity in superspace and its breaking encoded in the vev for the ghost number zero field strength: $\langle \mathring{\mathcal{F}}_{\theta\bar{\theta}}| \rangle = -i$

Gaspard 2012

- Useful moral: *dissipation = ghost condensation.*



The combined CPT + *$U(1)_T$ transformation ends up being the transformation used by GL to prove entropy positivity.*

Act V

in which thermal equivariance allows one to write down dissipative fluid dynamics in terms of an effective action, a topological sigma model….

FLUID DYNAMICS AS A SIGMA MODEL

- Hydrodynamics: low energy dynamics of conserved currents in near equilibrium situations.
- The hydrodynamic modes are Goldstone modes for spontaneously broken difference diffeomorphisms and difference gauge transformation.

Nickel, Son 2010

- The order parameter for broken difference diffeomorphisms is a vector field, which we identify with the hydrodynamic velocity rescaled by the local temperature (the pions of hydrodynamics):

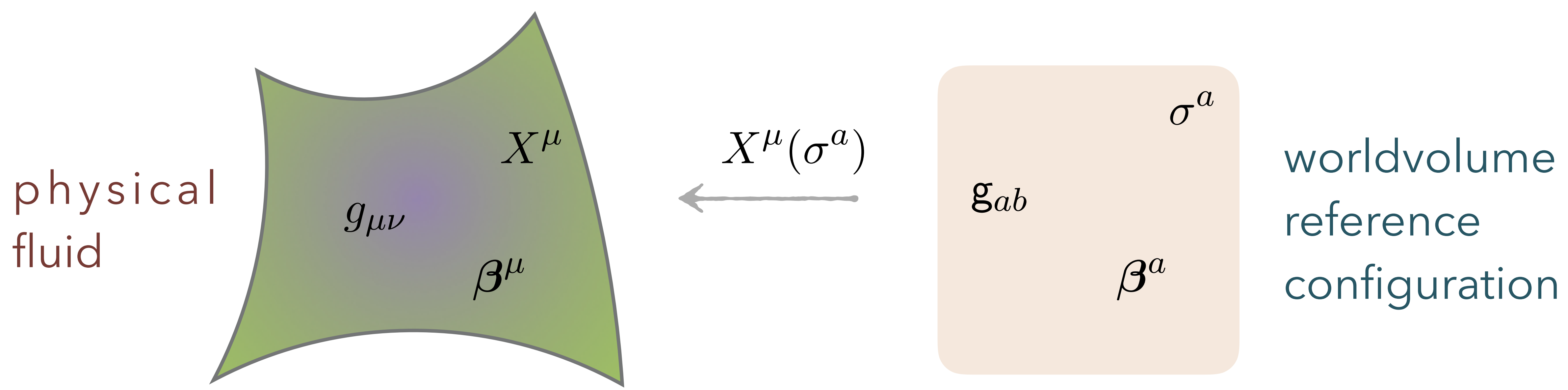
$$\boldsymbol{\beta}^\mu = \frac{u^\mu}{T}, \quad \Lambda_{\boldsymbol{\beta}} = \frac{\mu}{T} - \boldsymbol{\beta}^\alpha A_\alpha$$

- A Landau-Ginzburg theory of this vector field captures a part of hydrodynamic transport (Class L), but getting all of hydrodynamic transport requires more ingredients (cf., eightfold classification).

Haehl, Loganayagam, MR 2015

LANDAU-GINZBURG SIGMA MODELS

- *Class L:* effective action is just a sigma model parameterized by a scalar functional (free energy density) $\mathcal{L}[\boldsymbol{\beta}^a, g_{ab}(X)]$.
- *Adiabatic fluids:* Invariance under diffeomorphisms and flavour transformations forces non-dissipative dynamics.
- *Dynamics:* conservation follows from variational of the pullback maps with reference thermal vector being fixed.



THE EIGHTFOLD LAGRANGIAN

- More generally the full set of adiabatic transport derives from an Lagrangian density

$$\mathcal{L}_{\text{wv}} = \frac{1}{2}\,\mathbf{T}^{ab}\,\tilde{\mathrm{g}}_{ab} - \mathbf{N}^a_{\mathcal{L}}\,\tilde{\mathcal{A}}_a$$

- New variables $\tilde{\mathrm{g}}_{ab}$, $\tilde{\mathcal{A}}_a$: former is the SK partner of the worldvolume metric.

- The one-form is an abelian gauge field which couples to the entropy current.

Haehl, Loganayagam, MR 2015

- The linear couplings to the partners is highly suggestive of structures encountered in analysis of linear dissipative systems and topological sigma models.

Martin, Siggia, Rose 1973
Kovtun, Moore, Romatschke 2014

- Take the symmetry seriously and attempt to work out a full theory including dissipation.

TOPOLOGICAL SIGMA MODELS FOR HYDRODYNAMICS

- Hydrodynamic modes are equivariant maps from the worldvolume to the target space (physical manifold).

- The symmetry being gauged is thermal translations.

- Variables: superfields with top and bottom components being SK difference and average fields respectively

$$\mathcal{Y} \to \mathring{\mathcal{Y}} = \mathcal{Y} + \theta\, \mathcal{Y}_{\bar{\psi}} + \bar{\theta}\, \mathcal{Y}_{\psi} + \bar{\theta}\, \theta\, \tilde{\mathcal{Y}} \equiv \frac{\mathcal{Y}_{\text{L}} + \mathcal{Y}_{\text{R}}}{2} + \theta\, \mathcal{Y}_{\bar{\psi}} + \bar{\theta}\, \mathcal{Y}_{\psi} + \bar{\theta}\, \theta\, (\mathcal{Y}_{\text{R}} - \mathcal{Y}_{\text{L}})$$

- Thermal translations act via Lie drag along reference thermal vector $\mathring{\boldsymbol{\beta}}^I(z)$.

$$(\mathring{\Lambda}, \mathring{\mathcal{Y}})_{\boldsymbol{\beta}} = \mathring{\Lambda}\, \pounds_{\boldsymbol{\beta}} \mathring{\mathcal{Y}}$$

- KMS gauge superfield implements thermal equivariance.

TOPOLOGICAL SIGMA MODELS FOR HYDRODYNAMICS

- Symmetries we impose are:
  - Superdiffeomorphisms in target space and world volume
  - CPT symmetry of SK path integrals ($\mathcal{Z}^*_{SK}[\mathcal{J}_{\rm L}, \mathcal{J}_{\rm R}] = \mathcal{Z}_{SK}[\mathcal{J}_{\rm R}, \mathcal{J}_{\rm L}]$)
  - worldvolume ghost number conservation
  - KMS gauge invariance

- Dynamical fields are the pull-back maps which induce a worldvolume super-metric $\mathring{\mathrm{g}}_{IJ}(z) = g_{\mu\nu}(\mathring{X}(z))\, \mathring{\mathcal{D}}_I \mathring{X}^\mu\, \mathring{\mathcal{D}}_J \mathring{X}^\nu$

- Its top component is the SK difference metric which couples to the physical stress tensor.

- Physical fluid dynamics obtained by deforming the topological theory.

$$\mathring{\mathrm{g}}_{IJ}(z) \;\to\; \mathring{\mathrm{g}}_{IJ}(z) + \bar{\theta}\,\theta\; \mathsf{h}_{IJ}(\sigma)$$

DISSIPATIVE HYDRODYNAMIC ACTIONS

- Working in superspace the symmetries suffice to constrain the terms that can appear in the worldvolume sigma model

$$S_{\rm wv} \equiv \int d^d\sigma\, \mathcal{L}_{\rm wv}\,, \qquad \mathcal{L}_{\rm wv} = \int d\theta\, d\bar\theta\, \frac{\sqrt{-\mathring{g}}}{1+\boldsymbol{\beta}^e \mathring{\mathcal{A}}_e} \left(\mathring{\mathcal{L}} - \frac{i}{4}\, \mathring{\boldsymbol{\eta}}^{(ab)(cd)}\, \mathring{\mathcal{D}}_\theta \mathring{g}_{ab}\, \mathring{\mathcal{D}}_{\bar\theta} \mathring{g}_{cd} \right)$$

- In ordinary space we get back the adiabatic lagrangian + dissipation

$$\mathcal{L}_{\rm wv} = \frac{\sqrt{-g}}{1+\boldsymbol{\beta}^e \mathcal{A}_e} \left\{ \frac{1}{2} \left[\mathbf{T}^{ab}_{\mathcal{L}} - \frac{i}{2} \boldsymbol{\eta}^{(ab)(cd)}\, \left(\mathcal{F}_{\theta\bar\theta}, g_{cd}\right)_{\boldsymbol{\beta}} \right] \tilde{g}_{ab} - \mathbf{N}^a_{\mathcal{L}} \tilde{\mathcal{A}}_a \right.$$
$$\left. + \frac{i}{8} \left(\boldsymbol{\eta}^{(ab)(cd)} + \boldsymbol{\eta}^{(cd)(ab)} \right) \tilde{g}_{ab}\, \tilde{g}_{cd} + \dots \right\},$$

Class LT Lagrangian

Noise fluctuations

Kovtun, Moore, Romatschke 2014
Crossley, Glorioso, Liu 2015

- Again dissipative dynamics spontaneously breaks CPT, KMS field strength picks up a vev and ghost condenses.

HOLOGRAPHIC FLUIDS

- Known second order transport of holographic fluids follows from:

$$\mathcal{L}_{\text{wv}} = c_{\text{eff}} \int d\theta\, d\bar{\theta}\, \frac{\sqrt{-\mathring{\mathsf{g}}}}{1+\boldsymbol{\beta}^e \mathring{\mathcal{A}}_e} \Bigg\{ \left(\frac{4\pi\, \mathring{\mathsf{T}}}{d}\right)^d \left(1 - \frac{i\, d}{8\pi}\, \mathring{\mathsf{P}}^{c\langle a} \mathring{\mathsf{P}}^{b\rangle d}\, \mathring{\mathcal{D}}_\theta \mathring{\mathsf{g}}_{ab}\, \mathring{\mathcal{D}}_{\bar{\theta}} \mathring{\mathsf{g}}_{cd}\right) - \left(\frac{4\pi\, \mathring{\mathsf{T}}}{d}\right)^{d-2} \left[\frac{{}^{\mathcal{W}}\mathring{\mathsf{R}}}{d-2} + \frac{1}{d}\, \text{Harmonic}\left(\frac{2}{d}-1\right) \mathring{\sigma}^2 + \frac{1}{2}\mathring{\omega}^2\right] \Bigg\}$$

- How does the bulk gravity theory realize this effective action?
- Recent attempts get the ideal fluid part correct, but no clear story beyond…

Nickel, Son 2010 (ideal)

Crossley, Glorioso, Liu, Wang 2015 (incomplete at second order)
deBoer, Heller, Pinzani-Fokeeva 2015 (ideal)

FLUCTUATION-DISSIPATION & JARZYNSKI

- Presence of a gauge symmetry which couples to entropy current appears to be manifestly contradicting second law.
- Claim: entropy flows into the physical sector from the ghost sector. Appears to work in superspace cleanly…
- The MMO argument goes through in the hydrodynamic effective action leading to a derivation of the Jarzynski relation which them implies the 2nd law using convexity of the exponential function.

$$\langle e^{-\frac{W}{\mathsf{T}}} \rangle = e^{-\frac{1}{\mathsf{T}}(G_f - G_i)} \qquad \langle W \rangle \geq G_f - G_i$$

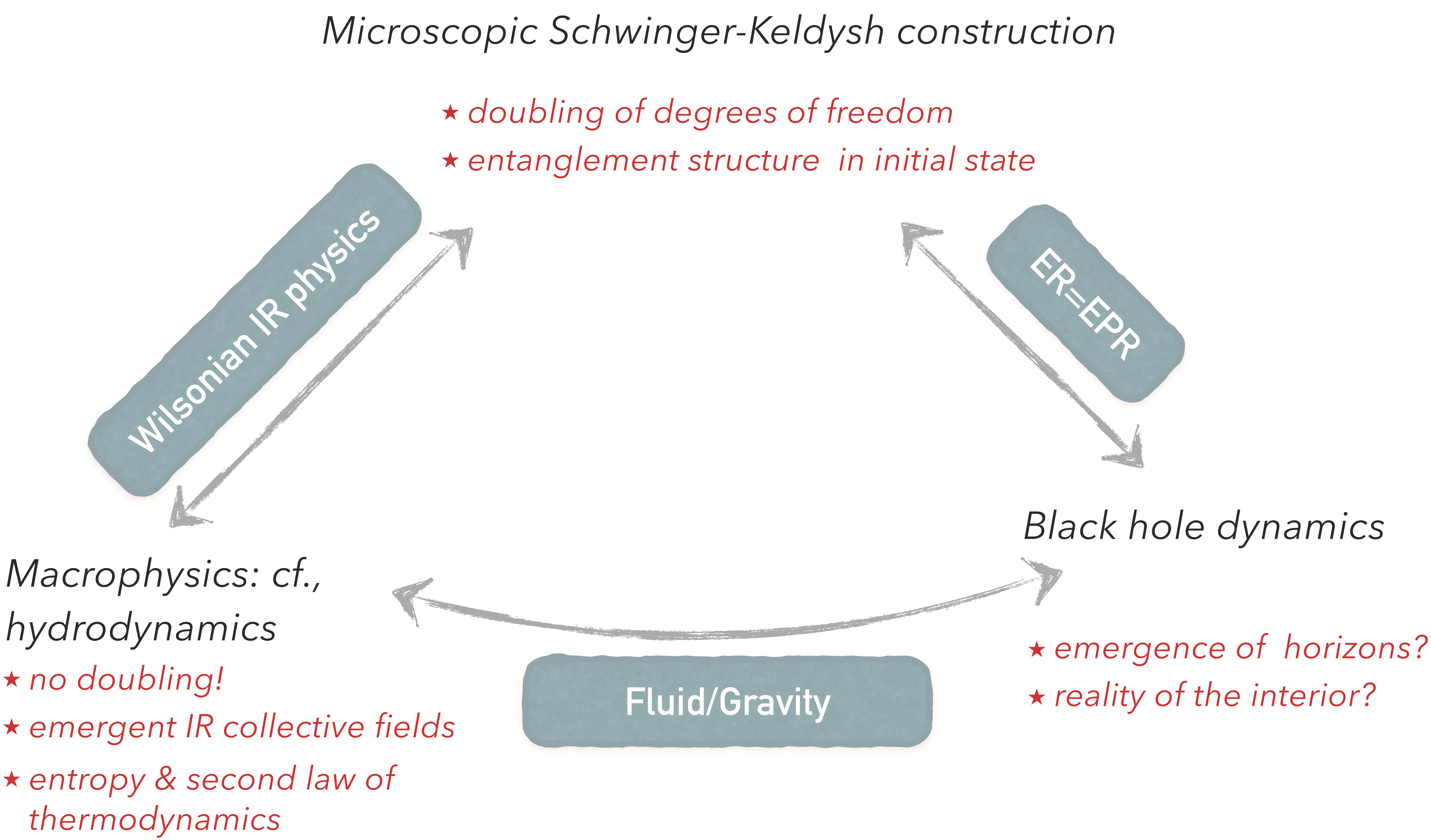
- Note only stochastic fluctuations accounted for thus far. Requires understanding of full KMS structure for quantum effects.

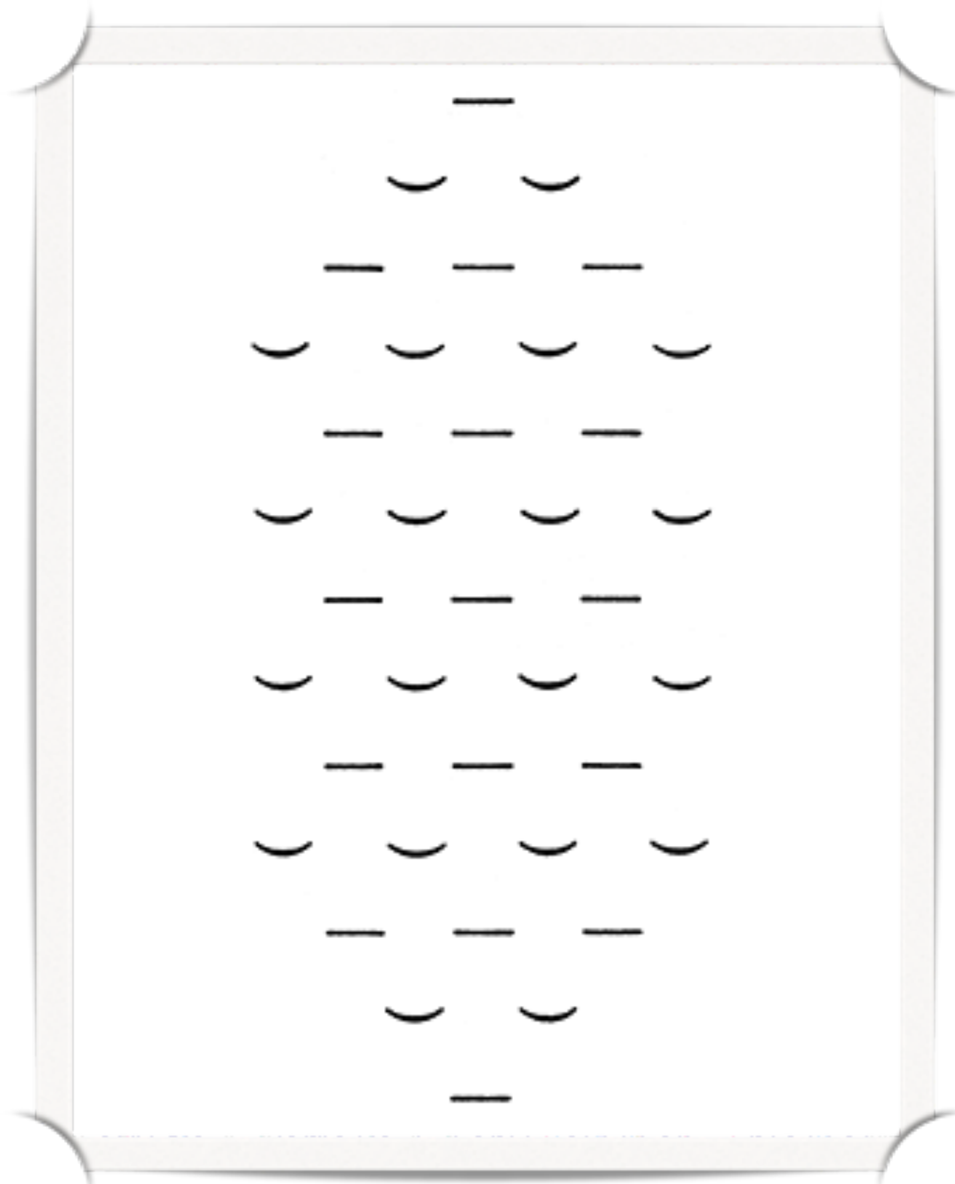
Epilogue: Of that which is yet to be...

LOOKING AHEAD…

- Near-equilibrium dynamics appears to be under control (should however write down the eightfold topological sigma model). What about non-equilibrium?
- Open quantum systems & renormalization *Avinash, Jana, Loganayagam, Rudra 2017*
- How does thermal equivariance extend to include non-stochastic fluctuations? Deformation quantization? *Basart, Flato, Lichnerowicz, Sternheimer 1984*
- Microscopic unitary which enforces fluctuation-dissipation etc., is upheld thanks to the ghost couplings. Lessons for gravity?
- What is the analogous story for higher out-of-time-order correlators?
- Are the similar statements for modular evolutions (equivalent in some contexts), and if so what does it imply for geometry = entanglement?

A ROADMAP FOR THE FUTURE….





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