Perturbative Supergravity at Higher Orders

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Based on work with
Zvi Bern, John Joseph Carrasco, Wei-Ming Chen,
Henrik Johansson, Mao Zeng

Many uses of gravity and supergravity perturbation theory

Quantum aspects:

- construct loop-level scattering amplitudes of gravitons, superpartners and matter fields
- explore various limits
- explore hidden and on-shell symmetries
- understand the UV behavior

Many uses of gravity and supergravity perturbation theory

Quantum aspects - understand the UV behavior

Extensive work on understanding the UV behavior of (super)gravity

- Supersymmetry constraints Green, Bjornsson, Bossard, Howe, Stelle, Nicolai Elvang, Kiermaier, Ramond, Kallosh, Vanhove, Bern, Davies, Dennen, etc
- Duality constraints Beisert, Elvang, Freedman, Kiermaier, Morales, Stieberger; Kallosh, etc

Consesus: poor UV behavior unless new cancellations between diag's exist that are "not consequences of supersymmetry in any conventional sense" Green, Bjornsson

Such "enhanced cancellations" are known to exist

$\mathcal{N}$=4 SG does not diverge at 3 loops in D=4 Bern et al.
$\mathcal{N}$=5 SG does not diverge at 4 loops in D=4

So… Does $\mathcal{N}$=8 SG diverge at 7 loops in D=4 ?

Does $\mathcal{N}$=8 SG diverge at 5 loops in D=24/5 ?

Suggested that they are related to *SL(L)* reparam. symmetry of *L*-loop integrals
Bern, Enciso, Para-Martinez, Zeng

The UV behavior of supergravity is not a philosophical question but a technical one, whose answer gives nontrivial perspectives on the symmetries of the theory

Many uses of gravity and supergravity perturbation theory

Quantum aspects:
- construct loop-level scattering amplitudes of gravitons, superpartners and other matter fields
- explore various limits
- explore hidden and on-shell symmetries
- understand the UV behavior

"Textbook" approach: - Feynman diagrammatics

- Lots of graphs/terms, e.g.

Loops	**3**	**4**	**5**
# of terms in 4pt amp.	$\mathcal{O}(10^{20})$	$\mathcal{O}(10^{26})$	$\mathcal{O}(10^{31})$

with standard origin:
- non-manifest gauge invariance
- presence of unphysical degrees of freedom

Method(s) of choice: Generalized unitarity, color/kinematics duality, double copy relation between gravity and gauge th's. fancy loop-level integration technology

Many uses of gravity and supergravity perturbation theory

Classical aspects:
- construct (deformations of) solutions of classical eqs. of motion (global features require resummation)

  e.g. gravitational waves, black holes, deformations of AdS corresponding to g.t. deformations, etc
- through AdS/CFT: leading strong coupling term of correlation functions of CFTs

Standard approach:
- identify the small parameter
- expand Einstein's equations
- solve iteratively

Difficulties related to the complexity of Einstein's equations, lack of symmetries, complexity of solution at each order, etc

The novel methods developed for quantum calculations may also have direct applications to this type of problems

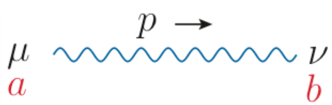
A plan

- An outline of color/kinematics duality and the double-copy, their successes and limitations
- Diff. invariance, double-copy and relaxation of color/kinematics duality
- Higher-loop scattering amplitudes, contact terms and novel methods for their determination
- An application and a word on integration

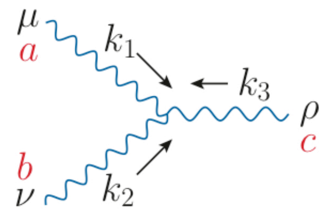
Scattering amplitudes and color/kinematics duality

Textbook approach: scattering amplitudes from Feynman rules

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\, a} + \text{matter} \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \mathrm{g}\, \mathrm{f}^{abc} A^b_\mu A^c_\nu$$



$$\frac{i\eta^{\mu\nu}\delta^{ab}}{p^2 + i\epsilon}$$



$$-g f^{abc}(\eta^{\mu\nu}(k_1 - k_2)^\rho + 2\ \text{more})$$

- General form of an *L*-loop amplitude

$$\mathcal{A}^{L-\text{loop}}_m = i^L\, g^{m-2+2L} \sum_{i \in \mathcal{G}_3} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_i} \frac{n_i C_i}{\prod_{\alpha_i} p^2_{\alpha_i}} \qquad n_i = n_i(p_\alpha \cdot p_\beta, \epsilon \cdot p_\alpha, \dots)$$

$$f^{abc} \sim \mathrm{Tr}[T^a [T^b, T^c]]$$

$$\downarrow$$

- amplitudes = sums of traces of gauge group generators — Bern, Kosower

$$\mathcal{A}^{L-\text{loop}}_m = \sum_{\text{noncyclic}} A(\sigma_1 \dots \sigma_m) \mathrm{Tr}[T^{a_{\sigma_1}} \dots T^{a_{\sigma_m}}] + \text{multi-traces}$$

↑

Color-ordered amplitudes

Example: 4-particle amplitude

Bern, Carrasco, Johansson

$$\mathcal{A}_4^{\text{tree}}(1,2,3,4) = \text{Tr}[T^{a_1}T^{a_2}T^{a_3}T^{a_4}]A_4^{\text{tree}}(1,2,3,4) + \text{Tr}[T^{a_1}T^{a_3}T^{a_4}T^{a_2}]A_4^{\text{tree}}(1,3,4,2) + \text{Tr}[T^{a_1}T^{a_4}T^{a_2}T^{a_3}]A_4^{\text{tree}}(1,4,2,3) + 3 \text{ more}$$

VS.

$$\mathcal{A}_4^{\text{tree}}(1,2,3,4) = g^2\left(\frac{c_s n_s(p,\epsilon)}{s} + \frac{c_t n_t(p,\epsilon)}{t} + \frac{c_u n_u(p,\epsilon)}{u}\right)$$

$$c_s = f^{a_1 a_2 b} f^{b a_3 a_4} \qquad c_t = f^{a_2 a_3 b} f^{b a_1 a_4} \qquad c_u = f^{a_3 a_1 b} f^{b a_2 a_4}$$

$$A_4^{\text{tree}}(1,2,3,4) = \frac{n_s}{s} - \frac{n_t}{t}\ ; \quad A_4^{\text{tree}}(1,3,4,2) = \frac{n_u}{u} - \frac{n_s}{s}\ ; \quad A_4^{\text{tree}}(1,4,2,3) = \frac{n_t}{t} - \frac{n_u}{u}$$

$$tA_4^{\text{tree}}(1,2,3,4) = uA_4^{\text{tree}}(1,3,4,2), etc \longrightarrow n'_s + n'_t + n'_u = 0$$

Color Jacobi relations:

$$c_s + c_t + c_u = 0$$

Invariance of $\mathcal{A}$ →

$$n'_s = n_s + \alpha\ s$$
$$n'_t = n_t + \alpha\ t$$
$$n'_u = n_u + \alpha\ u$$

Generalized gauge symmetry

Example: 4-particle amplitude

Bern, Carrasco, Johansson

$$\mathcal{A}_4^{\text{tree}}(1,2,3,4) = \text{Tr}[T^{a_1}T^{a_2}T^{a_3}T^{a_4}]A_4^{\text{tree}}(1,2,3,4) + \text{Tr}[T^{a_1}T^{a_3}T^{a_4}T^{a_2}]A_4^{\text{tree}}(1,3,4,2)$$
$$+ \text{Tr}[T^{a_1}T^{a_4}T^{a_2}T^{a_3}]A_4^{\text{tree}}(1,4,2,3) + 3 \text{ more}$$

VS.

$$\mathcal{A}_4^{\text{tree}}(1,2,3,4) = g^2\left(\frac{c_s n_s(p,\epsilon)}{s} + \frac{c_t n_t(p,\epsilon)}{t} + \frac{c_u n_u(p,\epsilon)}{u}\right)$$

$$c_s = f^{a_1 a_2 b} f^{b a_3 a_4} \qquad c_t = f^{a_2 a_3 b} f^{b a_1 a_4} \qquad c_u = f^{a_3 a_1 b} f^{b a_2 a_4}$$

$$A_4^{\text{tree}}(1,2,3,4) = \frac{n_s}{s} - \frac{n_t}{t}\ ; \quad A_4^{\text{tree}}(1,3,4,2) = \frac{n_u}{u} - \frac{n_s}{s}\ ; \quad A_4^{\text{tree}}(1,4,2,3) = \frac{n_t}{t} - \frac{n_u}{u}$$

$$n'_s + n'_t + n'_u = 0 \longleftrightarrow \quad tA_4^{\text{tree}}(1,2,3,4) = uA_4^{\text{tree}}(1,3,4,2), etc$$

General *n*-point tree amplitudes: BCJ amplitudes relations

$$\sum_{i=1}^{n} k_m \cdot k_{1\ldots,i}\, A_n^{\text{tree}}(1,\ldots,i,m,i+1,\ldots,n) = 0$$

The general picture/conjecture: a duality between color and kinematics

adjoint rep: Bern, Carrasco, Johansson

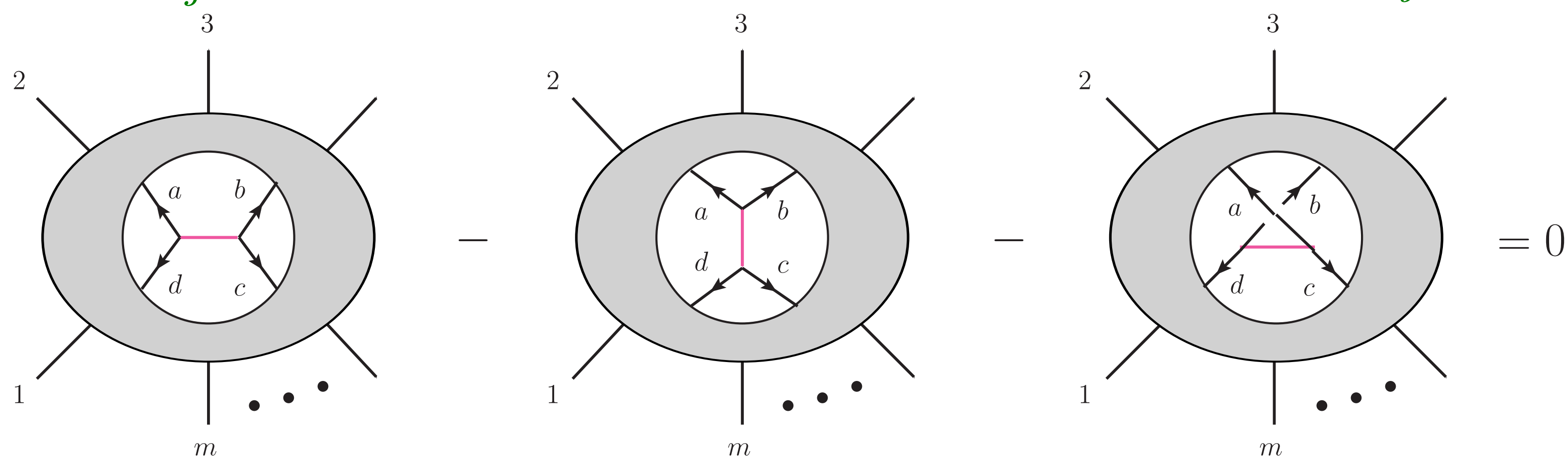
non-adjoint rep: Chiodaroli, Jin, RR; Johansson, Ochirov

Chiodaroli, Gunaydin, Johansson, RR

- For (s)YM theories in any dimension with certain additional matter

$$\mathcal{A}_m^{L-\text{loop}} = i^L g^{m-2+2L} \sum_{i\in\mathcal{G}_3} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_i} \frac{n_i C_i}{\prod_{\alpha_i} p_{\alpha_i}^2} \qquad n_i = n_i(p_\alpha \cdot p_\beta, \epsilon\cdot p_\alpha, \dots)$$

when $C_i + C_j + C_k = 0$ is required by gauge invariance $n_i + n_j + n_k = 0$



- Present in many theories: YM+matter, QCD, Coulomb branch, ϕ^3, *Z*-theory, BLG, ABJM,… as well as certain form factors and correlation fcts.
- Implies nontrivial relations btwn amplitudes (*L*=0) and integrands (*L*>0)

Color/kinematics

Bern, Carrasco, Johansson

$$\mathcal{A}_m^{L-\text{loop}} = i^L g^{m-2+2L} \sum_{i\in\mathcal{G}_3} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_i} \frac{n_i C_i}{\prod_{\alpha_i} p^2_{\alpha_i}}$$

$$n_i = n_i(p_\alpha \cdot p_\beta, \epsilon \cdot p_\alpha, \dots)$$

$$C_i = \dots f^{a_1 bc} f^{c a_2 d} \dots$$

$$C_i + C_j + C_k = 0 \quad \leftrightarrow \quad n_i + n_j + n_k = 0$$

n_i are not gauge-invariants

- 5-point 2-loop all-plus amplitude — Mogull, O'Connell
  remarkably-complicated expression; remarkably bad powercounting
- 2-loop 4-point amplitudes in $\mathcal{N}$=2 SQCD — Johansson, Kaelin, Mogull
- Explicit color/kinematics-satisfying numerators for NLSM — Du, Fu
- Color/kinematics-satisfying Feynman rules from a NLSM action — Cheung, Shen
- Suggestion for a(nother) symmetry behind BCJ amplitudes relations — Brown, Naculich
  momentum-dependent shift of color factors
- Can be defined for form factors of certain operators; — Boels, Kniehl, Tarasov, Yang
  first 5-loop computation – the form factor of the 20' operator in $\mathcal{N}$=4 sYM — Yang

  Can be defined for correlation functions of certain operators — cf. Engelund, RR
- Generalization of BCJ amp. rel's at higher loops — Vanhove, Tourkine; also He, Schlotterer; Stieberger, Hohenegger; Chiodaroli, Gunaydin, Johansson, RR; earlier Boels, Isermann

Color/kinematics

Bern, Carrasco, Johansson

- Generalization of BCJ amp. relations at higher loop — Vanhove, Tourkine; also He, Schlotterer; Stieberger, Hohenegger; Chiodaroli, Gunaydin, Johansson, RR; earlier Boels, Isermann
  - Tree amplitudes relations ⟶ rel's between cuts w/ extra linear numerator factors
    ⟶ expect rel's between loop amp's w/ extra linear numerator factors
  - From loop-level monodromy relations in string theory (issues w/ moduli space integration?)
- ► Loop momentum-dependent relations between amplitudes' integrands up to total derivatives

- Examples in field theory limit at 1 loop: — Vanhove, Tourkine

$$\sum_{i=2}^{p-1} k_1 \cdot k_{2\ldots i} A(2,\ldots,i,1,i+1,\ldots p|p+1,\ldots,n) + \sum_{i=p}^{n} k_1 \cdot k_{p+1\ldots i} A(2,\ldots p|p+1,\ldots,i,1,i+1,\ldots,n)$$

$$= -\sum_{i=p}^{n} A(2,\ldots p|p+1,\ldots,i,1,i+1,\ldots,n)[l\cdot k_1]$$

$$A(1,2\ldots n)[l\cdot k_1] + A(2,1\ldots n)[(l+k_2)\cdot k_1] + \cdots + A(1\ldots n-1,1,n)[(l+k_{23\ldots n-1})\cdot k_1] = 0$$

- To any loop order — Chiodaroli, Gunaydin, Johansson, RR

$$\sum_{\mathrm{cyclic}(2,\ldots,n)} A^{\mathrm{YM},(L)}_{1-\mathrm{trace}}(1,2\ldots,n)[p_1\cdot q_1] = 0$$

Using such relations one may be able to argue for existence of loop-level color/kinematics duality w/o explicit construction of integrand

Color/kinematics and the double copy

Bern, Carrasco, Johansson

Order by order in perturbation theory

$$\mathcal{M}_m^{L-\text{loop}} = i^{L+1} \left(\frac{\kappa}{2}\right)^{m-2+2L} \sum_{i\in\mathcal{G}_3} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_i} \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} p_{\alpha_i}^2}$$

- Property of many pure & YM/Maxwell-Einstein SGs w/ further matter, open string theory, self-dual gravity, $R + R^3$, EYM+SSB,...

- 5-loop double copy of $\mathcal{N}$=4 sYM Sudakov form factor — Gang Yang
  - physical interpretation is under debate; not necessarily a form factor of local op.
  - ½-BPS ⟶ expect worse UV behavior than amplitudes

- 2-loop 4-point amplitudes in $\mathcal{N}$=2 SG + matter — Johansson, Kaelin, Mogull

- New perspective on "enhanced cancellations" — Bern, Enciso, Para-Martinez, Zeng

- Progress in the identification of SG symmetries i.t.o. YM operations — Anastasiou, Borsten, Duff, Hughes, Marrani, Nagy, Zoccali

- First example of 3-point scattering amplitude in curved space from double-copy — Adamo, Casali, Mason, Nekovar

- YM classical solutions ⟶ (S)G classical solutions — O'Connell et al; Goldberger, Ridgeway

- New techniques for SG amplitudes when c/k is expected but not manifest — Bern, Carrasco, Chen, Johansson, RR

A word on classical gravity solutions from YM classical solutions

- Kerr-Schild-type solutions — Monteiro, O'Connell, White

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu} \qquad h_{\mu\nu} = -\frac{\kappa}{2}\phi k_\mu k_\nu \qquad \bar{g}_{\mu\nu}k^\mu k^\nu = 0 \qquad (k\cdot D)k = 0$$

$$A^\mu = g\frac{1}{4\pi r}k^\mu \quad \longrightarrow \quad h^{\mu\nu} = -\frac{M\kappa}{2}\frac{1}{4\pi r}k^\mu k^\nu$$

Schwarzschild ⟷ (Coulomb field of point charge)^2

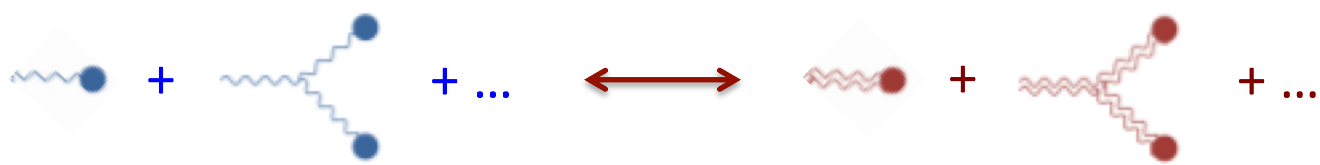
- Other solutions:

Kerr black hole, some higher dimensional black holes, supersymmetric black holes, Taub-NUT spaces, spaces w/ cosmological constant, radiation from accelerating b.h.

Luna, Monteiro, Nicholson, O'Connell, White; Goldberger, Ridgway; Cardoso, Nagy, Nampuri; Ridgway, Wise

- Algorithm for perturbative construction of gravity sol/s i.t.o. gauge th. sol's

Luna, Monteiro, Nicholson, O'Connell, Ochirov, Westerberg, White



- Perturbative gravitational radiation for colliding masses/b.h. from gluon radiation

Goldberger, Ridgway

possible applications to LIGO (in the early stages of a merger)

Color/kinematics and the double-copy

Bern, Carrasco, Johansson

- Many open questions; progress on some of them hinges on several technical issues
    + frustratingly difficult to find manifest c/k-satisfying representations
        - large ansatze ⟶ large linear systems -- $\mathcal{O}(10^6)$ unknowns
    + the result can have unexpectedly high powers of loop mom. Mogull, O'Connell
        - larger ansatze than one might expect
    + going straight for (super)gravity amplitudes only makes it worse
    + classical solution construction slightly different from scattering amp's; c/k duality needs some reanalysis at higher points
- What is needed:
    + keep the idea of the double copy
    + avoid large ansatze ⟷ construct amplitudes one piece at a time
        - may address possible difficulties with construction of classical solutions
    + some kind of structure should be present

Diff inv. from gauge inv. and what to expect w/o manifest c/k duality:

If c/k is manifest, all double-copy theories are diffeomorphism-invariant:

BCJ; JO;BDHK; CGJR

1. Linearized YM gauge transformations: $\epsilon^\mu(p) \mapsto p^\mu$

$$\mathcal{A} = \sum_\Gamma \frac{n_\Gamma(\epsilon_1(p_1), \epsilon_2, \dots)c_\Gamma}{D_\Gamma} \longrightarrow 0 = \sum_\Gamma \frac{n_\Gamma(p_1, \epsilon_2, \dots)c_\Gamma}{D_\Gamma}$$

1. structure of n_Γ 2. Jacobi identities for c_Γ

2. Linearized diffeomorphisms: $\epsilon^{\mu\nu}(p) \mapsto p^{(\mu}q^{\nu)}$

$$\epsilon^{\mu\nu}(p) \equiv \epsilon^{(\mu}(p)\epsilon'^{\nu)}(p) \mapsto p^{(\mu}\epsilon'^{\nu)}(p) + p^{(\nu}\epsilon^{\mu)}(p)$$

- follow from YM linearized gauge symmtry

$$\mathcal{M} = \sum_\Gamma \frac{n_\Gamma(\epsilon_1(p_1), \epsilon_2, \dots)\tilde{n}_\Gamma(\epsilon'_1(p_1), \epsilon'_2, \dots)}{D_\Gamma}$$

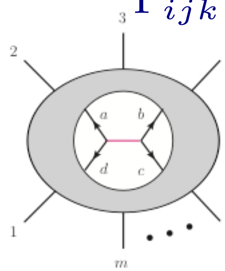
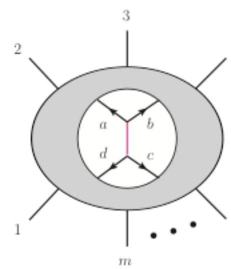
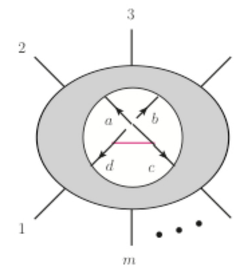
$$\delta\mathcal{M} = \sum_\Gamma \frac{n_\Gamma(p_1, \epsilon_2, \dots)\tilde{n}_\Gamma(\epsilon'_1(p_1), \epsilon'_2, \dots)}{D_\Gamma} + (n \leftrightarrow \tilde{n})$$

$n_\Gamma, \tilde{n}_\Gamma \& c_\Gamma$ have the same properties $\Longrightarrow$ $\delta\mathcal{M} = 0$ for the same reasons as in YM theory

What if c/k is expected but not manifest and yet one naively double copies?

Closest analog: gauge theory in which we formally relax the color Jacobi relations

$$\delta\mathcal{A} \sim \sum_{\Gamma_{ijk}} \frac{f_{\Gamma_{ijk}}(\hat{\epsilon}_1, \epsilon_2, \ldots, p_1, \ldots)(c_{\Gamma_i} + c_{\Gamma_j} + c_{\Gamma_k})}{D_{\Gamma_{ijk}}}$$

Γ_i :  Γ_j :  Γ_k : 

On to gravity:

$$\delta\mathcal{M} \sim \sum_{\Gamma_{ijk}} \frac{f_{\Gamma_{ijk}}(\hat{\epsilon}_1, \epsilon_2, \ldots, p_1, \ldots)(\tilde{n}_{\Gamma_i} + \tilde{n}_{\Gamma_j} + \tilde{n}_{\Gamma_k})}{D_{\Gamma_{ijk}}} + (n \leftrightarrow \tilde{n})$$

$$\sim \sum_{\Gamma} \sum_{\lambda \in \Gamma} \frac{f_{\{\Gamma,\lambda\}}(\hat{\epsilon}_1, \epsilon_2, \ldots, p_1, \ldots)\, \tilde{J}_{\{\Gamma,\lambda\}}}{D_{\Gamma}} + (n \leftrightarrow \tilde{n})$$

Conclusions:

1. Breaking of diff. inv. in naïve double-copy is itself a double copy
2. Correction terms restoring diff. inv. should also be double-copies
3. Relevant factors are $J_{\{\Gamma,\lambda\}}$ and $\tilde{J}_{\{\Gamma,\lambda\}}$ -- violations of the kinematic Jacobi relations in the two gauge theory factors

0. Structure exists ► should be possible to correct a naïve double-copy

Most straightforward test of these ideas is at tree level

↓

Should be equally straightforward to use them to find generalized cuts

KLT: too many terms, too many spurious poles, not organized in terms of graphs

↓

More efficient methods always come in handy

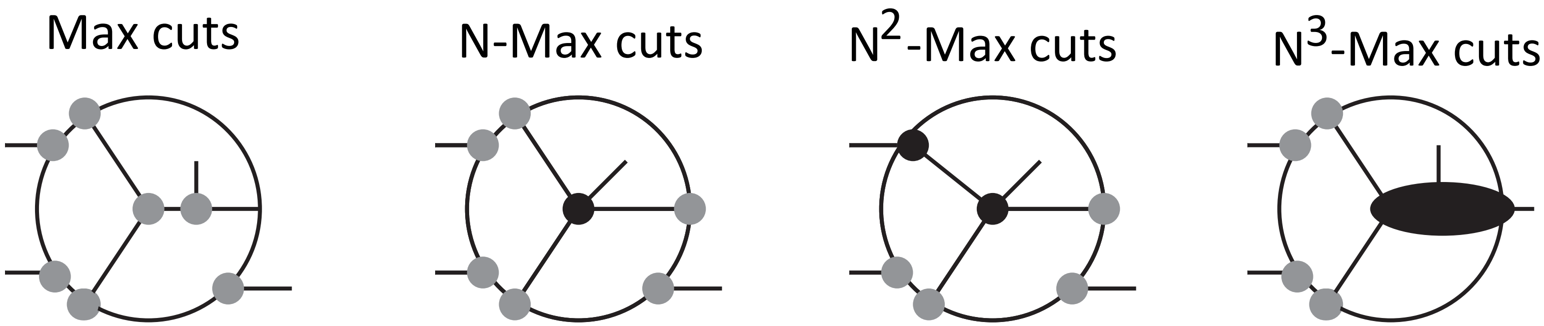
Generalized unitarity/method of maximal cuts:

1. Organize amplitude in terms of graphs of φ^3 theory; each graph gets an ansatz for numerator with some desired properties

$$\mathcal{A}^{YM} = \sum_{\Gamma} \int \frac{n_\Gamma c_\Gamma}{D_\Gamma} \qquad \mathcal{M}^{(S)G} = \sum_{\Gamma} \int \frac{N_\Gamma}{D_\Gamma}$$

2. Fix numerators by fitting them onto cuts

E.g.



► Leads to large linear systems

To avoid this…

Generalized unitarity/the contact term method: Bern, Carrasco, Chen, Johansson, RR

-focus on (super)gravity

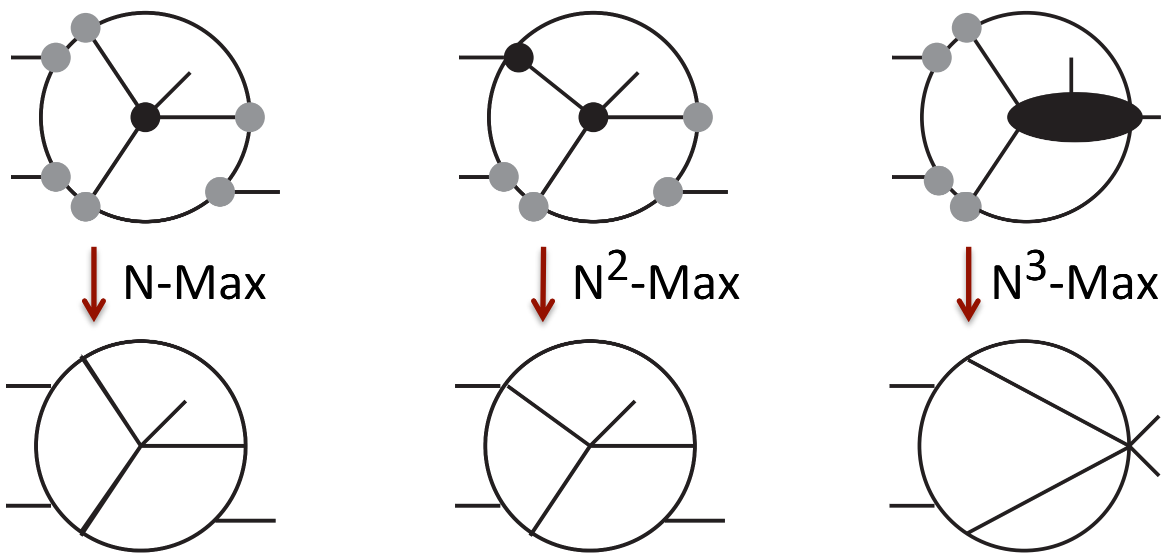
1. Start with some approximation of the supergravity amplitude, organized in terms of the graphs of φ^3 theory, which has the correct maximal cuts, e.g.

a naïve double-copy: $$\mathcal{M}^{(S)G} = \sum_{\Gamma} \int \frac{n_\Gamma \tilde{n}_\Gamma}{D_\Gamma}$$

2. Iteratively correct it w/ graphs w/ higher-pt. vert's to satisfy such that N^k-Max cuts

N^k-contact = N^k-Max cut – (cut of approximation of amplitude)

E.g.



- Each cut gives an independent contrib. to amplitude
- Freedom in choosing each of them
- Lots of cuts
- But a finite number!

- Effectively a tree-level calculation
- Ideal if cuts are organized in terms of cubic tree graphs

Unexpected and welcome features

N^k-contact = N^k-Max cut – (cut of approximation of amplitude)

0. A naïve double-copy has the correct maximal and next-to-maximal cuts
   $M_4^{\rm tr}(1,2,3) = i A_3^{\rm tr}(1,2,3) A_4^{\rm tr}(1,2,3)$ & 4-pt amp's obey c/k duality

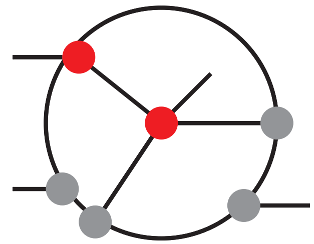
Using KLT to construct SG cuts:

1. Contact terms are much simpler than one has the right to expect
   - In $\mathcal{N}$=8 SG most of them vanish (at least through 5 loops)
2. Four-point double-contact terms factorize; each factor has features resembling gauge theory quantities
3. Higher-contact terms no longer factorize but, in hindsight, can be written as sums of products of factors with features resembling gauge theory quantities
4. These observations match the expected features of the conclusions we drew from the diff. invariance constraints on corrections to a naïve double copy.

   Expect that it should be possible to express cuts and contacts in terms of BCJ discrepancy functions, $J_{\Gamma,\lambda}$ and $\tilde{J}_{\Gamma,\lambda}$, using solely gauge theory information

   Key for using this is the generalized gauge symmetry

All double-4-point cut and contact terms from gauge theory data



Bern, Carrasco, Chen, Johansson, RR

Gauge theory cut:

$$\mathcal{C}^{4\times 4}_{\rm YM} = \sum_{i_1,i_2} \frac{n_{i_1 i_2} c_{i_1 i_2}}{d^{(1)}_{i_1} d^{(2)}_{i_2}}$$

Transformation relating it to c/k-satisfying one:

$$\delta_{i_1 i_2} \equiv n_{i_1 i_2} - n^{\rm BCJ}_{i_1, i_2} = d^{(1)}_{i_1} k^{(2)}(i_2) + d^{(2)}_{i_2} k^{(1)}(i_1)$$

Properties of gauge parameters:

$$\sum_{i_1,i_2} \frac{\delta_{i_1 i_2} c_{i_1 i_2}}{d^{(1)}_{i_1} d^{(2)}_{i_2}} = 0 = \sum_{i_1,i_2} \frac{\delta_{i_1 i_2} n^{\rm BCJ}_{i_1 i_2}}{d^{(1)}_{i_1} d^{(2)}_{i_2}}$$

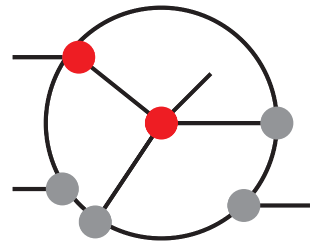
BCJ discrepancy functions:

$$J_{\bullet, i_2} \equiv \sum_{i_1} n_{i_1 i_2} = d^{(2)}_{i_2} \sum_{i_1} k^{(1)}(i_1) \qquad J_{i_1, \bullet} \equiv \sum_{i_2} n_{i_1 i_2} = d^{(1)}_{i_1} \sum_{i_2} k^{(2)}(i_2)$$

Supergravity cut (there are several equivalent variants):

$$\mathcal{C}^{4\times 4}_{\rm SG} = \sum_{i_1,i_2} \frac{n^{\rm BCJ}_{i_1 i_2} \tilde{n}^{\rm BCJ}_{i_1 i_2}}{d^{(1)}_{i_1} d^{(2)}_{i_2}} = \sum_{i_1,i_2} \frac{n_{i_1 i_2} \tilde{n}_{i_1 i_2}}{d^{(1)}_{i_1} d^{(2)}_{i_2}} - \frac{1}{d^{(1)}_{1} d^{(2)}_{1}} \left(J_{\bullet,1} \tilde{J}_{1,\bullet} + J_{1,\bullet} \tilde{J}_{\bullet,1} \right)$$

All double-4-point cut and contact terms from gauge theory data



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Gauge theory cut:

$$\mathcal{C}^{4\times 4}_{\rm YM} = \sum_{i_1,i_2} \frac{n_{i_1 i_2} c_{i_1 i_2}}{d^{(1)}_{i_1} d^{(2)}_{i_2}}$$

Transformation relating it to c/k-satisfying one:

$$\delta_{i_1 i_2} \equiv n_{i_1 i_2} - n^{\rm BCJ}_{i_1,i_2} = d^{(1)}_{i_1} k^{(2)}(i_2) + d^{(2)}_{i_2} k^{(1)}(i_1)$$

Properties of gauge parameters:

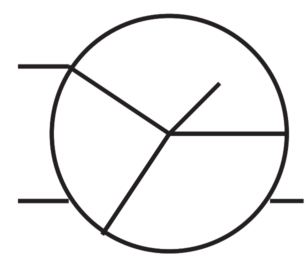
$$\sum_{i_1,i_2} \frac{\delta_{i_1 i_2} c_{i_1 i_2}}{d^{(1)}_{i_1} d^{(2)}_{i_2}} = 0 = \sum_{i_1,i_2} \frac{\delta_{i_1 i_2} n^{\rm BCJ}_{i_1 i_2}}{d^{(1)}_{i_1} d^{(2)}_{i_2}}$$

BCJ discrepancy functions:

$$J_{\bullet, i_2} \equiv \sum_{i_1} n_{i_1 i_2} = d^{(2)}_{i_2} \sum_{i_1} k^{(1)}(i_1) \qquad J_{i_1,\bullet} \equiv \sum_{i_2} n_{i_1 i_2} = d^{(1)}_{i_1} \sum_{i_2} k^{(2)}(i_2)$$

Supergravity cut (there are several equivalent variants):

$$\mathcal{C}^{4\times 4}_{\rm SG} = \sum_{i_1,i_2} \frac{n_{i_1 i_2} \tilde{n}_{i_1 i_2}}{d^{(1)}_{i_1} d^{(2)}_{i_2}} \boxed{- \frac{1}{d^{(1)}_{1} d^{(2)}_{1}} \left(J_{\bullet,1} \tilde{J}_{1,\bullet} + J_{1,\bullet} \tilde{J}_{\bullet,1} \right)}$$



Valid in any double-copy (super)gravity

Generalization: cuts have (fairly) closed-form structured expressions i.t.o. cubic graphs

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$$\mathcal{C}_{\text{SG}}^{4\times 4\times 4} = \sum_{i_1,i_2,i_3} \frac{n_{i_1 i_2 i_3} \tilde{n}_{i_1 i_2 i_3}}{d_{i_1}^{(1)} d_{i_2}^{(2)} d_{i_3}^{(3)}} - T_1 - T_2$$

$$T_1 = \sum_{i_3} \frac{J_{\bullet,1,i_3} \tilde{J}_{1,\bullet,i_3} + J_{1,\bullet,i_3} \tilde{J}_{\bullet,1,i_3}}{d_1^{(1)} d_1^{(2)} d_{i_3}^{(3)}} + \sum_{i_2} \frac{J_{\bullet,i_2,1} \tilde{J}_{1,i_2,\bullet} + J_{1,i_2,\bullet} \tilde{J}_{\bullet,i_2,1}}{d_1^{(1)} d_{i_2}^{(2)} d_1^{(3)}}$$
$$+ \sum_{i_1} \frac{J_{i_1,\bullet,1} \tilde{J}_{i_1,1,\bullet} + J_{i_1,1,\bullet} \tilde{J}_{i_1,\bullet,1}}{d_{i_1}^{(1)} d_1^{(2)} d_1^{(3)}}$$

$$T_2 = -\frac{J_{\bullet,1,1} \tilde{J}_{1,\bullet,\bullet} + J_{1,\bullet,\bullet} \tilde{J}_{\bullet,1,1}}{d_1^{(1)} d_1^{(2)} d_1^{(3)}} - \frac{J_{1,\bullet,1} \tilde{J}_{\bullet,1,\bullet} + J_{\bullet,1,\bullet} J'_{1,\bullet,1}}{d_1^{(1)} d_1^{(2)} d_1^{(3)}}$$
$$- \frac{J_{1,1,\bullet} \tilde{J}_{\bullet,\bullet,1} + J_{\bullet,\bullet,1} \tilde{J}_{1,1,\bullet}}{d_1^{(1)} d_1^{(2)} d_1^{(3)} d_1^{(2)} d_1^{(3)}}$$

$$\mathcal{C}_{\text{SG}}^{4\times\cdots\times 4} = \ldots$$

- Subtraction of the cuts of the approximate amplitude is straightforward
- Built-in verification: difference must be local

Many generalized cuts have (fairly) closed-form structured expressions

$$\mathcal{C}^{5}_{\rm SG} = \sum_{i=1}^{15} \frac{n_i \tilde{n}_i}{d^{(1)}_{i,1} d^{(1)}_{i,2}} - \frac{1}{6} \sum_{i=1}^{15} \frac{J_{\{i,1\}} \tilde{J}_{\{i,2\}} + J_{\{i,2\}} J'_{\{i,1\}}}{d^{(1)}_{i,1} d^{(1)}_{i,2}}$$

$$\mathcal{C}^{5\times 4}_{\rm SG} = \sum_{i} \frac{n_i \tilde{n}_i}{d^{(1)}_{i,1} d^{(1)}_{i,2} d^{(2)}_{i,1}} + \text{more complicated}$$

$$\mathcal{C}^{5\times 4 \times \cdots \times 4}_{\rm SG} = \ldots$$

Others, e.g. $\mathcal{C}^{6}_{\rm SG}$ have currently a… less pleasant appearance

- These formulae hold in any double-copy (super) gravity

- The 5-point formula is similar (though prettier) to a known 5-point tree formula, written in a basis of discrepancy functions Bjerrum-Bohr, Damgaard, Sondergaad, Vanhove

Some features:

- Starting point can be any graph-based representation of amplitudes, including Feynman diagrams

- Novel way to find gravity tree-level amplitudes adapted to cubic graphs

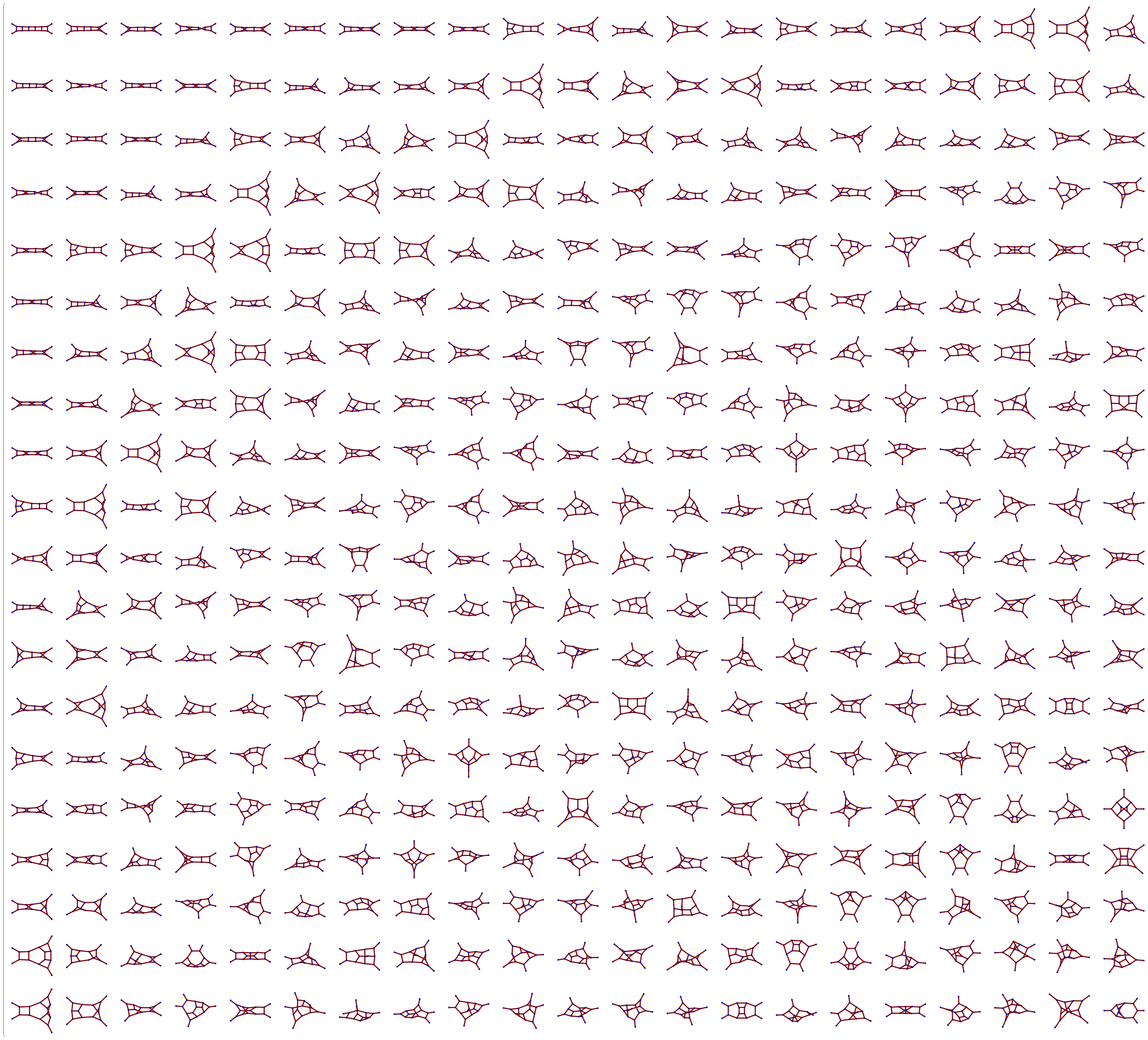
- Cuts are naturally in a cubic graph-based form; identification of the new contact term is straightforward

- Highest contact terms depend on the power counting of the theory; top levels are very simple – linear in momentum invariants. Numerical approach – rather than analytic simplification – may be more efficient

- But the proof is in the pudding…

Allowed us to construct the 4-point 5-loop integrand of $\mathcal{N}$=8 supergravity

To appear - Bern, Carrasco, Chen, Johansson, RR, Zeng



together with
2-, 3-, 4-, 5-, and
6-collapsed
propagator graphs:

N2: 9159
N3: 17935
N4: 23996
N5: 24198
N6: 17110

about 20% of which
are nonzero

Explicit power ct is poor
because of poor rep.
of $\mathcal{N}$=4 sYM amplitude

On integration

Chetyrkin, Kataev, Tkachov; Laporta; A.V. Smirnov; V. A. Smirnov; Vladimirov; Marcus, Sagnotti; Czakon; Laporta; Kosower; Larsen, Zhang; Zeng, etc

- General structure of the amplitude:

$$\mathcal{M}_4^{(5)} \sim (stu\mathcal{M}_4^{(0)})\; s^2 \int d^{5D}l \sum_{k=0}^{6} \frac{\mathcal{N}_{6-k}(p^2, l\cdot p, l_i\cdot l_j)}{((l+p)^2)^{16-k}}$$

$$\sim (stu\mathcal{M}_4^{(0)})\; s^2 \int d^{5D}l \left[F_{-10}(l_i\cdot l_j) + sF_{-11}(l_i\cdot l_j) + s^2F_{-12}(l_i\cdot l_j) + \ldots\right]$$

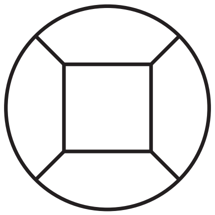
Critical dimension: 4 22/5 24/5

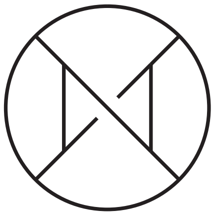
- 5-loop vacuum integrals are state of the art in QCD

- QCD beta function: need to expand to second order in external momenta; Here second order (6 external momenta) checks convergence in D=22/5
  - constrained by supersymmetry
  - checks our construction of the integrand

Observations: 1. All linear relations among integrands are IBPs ($\sim$ SL(L) symmetry)
2. Lower loops suggest that integrals with maximal cuts have highest transcendentality Kosower, Larsen; Abreu, Britto, Duhr, Gardi; Bosma, Sogaard, Zhang; Schabinger

Two such integrals; through IBPs, they receive contributions from many terms





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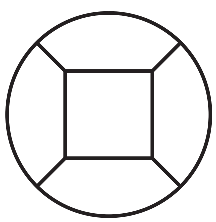

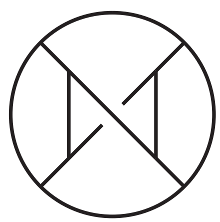

coefficients vanish, as expected

On integration

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  - checks our construction of the integrand
  - further strong indication (but no proof) that integrand is correct

- Enhanced cancellations probed at fourth order -- $\mathcal{O}(10^8)$ terms in $F_{-12}(l_i\cdot l_j)$

Stay tuned!

Bern, Carrasco, Chen, Johansson, RR, Zeng – in progress

An outlook

- Reviewed recent developments and illustrated some of them
    - Focused on color/kinematics and double-copy
- Many open questions, some computational, some conceptual
- New method for constructing supergravity amplitudes:
  can convert any representation of gauge theory amp's into supergravity amp's
    - Takes over when c/k duality is for some reason impractical; algorithmic construction of amplitudes' contact terms in terms of the breaking of kinematic Jacobi relations
    - Terms in amplitudes are constructed one by one
    - Allows the construction of the 5-loop 4-graviton integrand of $\mathcal{N}$=8 SG
      checked cuts through N^8-Max; indications for susy cancellations
    - May have applications to construction of classical solutions of SG eom
    - Full potential to be explored, as is the physics of the 5-loop $\mathcal{N}$=8 amplitude